

Series 3 : Binary relations

Exercise 1 : Let R be a relation defined on \mathbb{R} by :

$$xRy \iff x^2 - y^2 = x - y$$

- 1) Prove that R is an equivalence relation.
- 2) Determine the equivalence classe $[x]$ for all $x \in \mathbb{R}$.
- 3) Determine the quotient set \mathbb{R}/R .

Exercise 2 : Let R be a relation defined on $\mathbb{Z} \times \mathbb{N}^*$ by :

$$(a, b)R(a', b') \iff ab' = a'b$$

- 1) Prove that R is an equivalence relation.
- 2) Let $(p, q) \in \mathbb{Z} \times \mathbb{N}^*$ with $p \wedge q = 1$. Write its equivalence classe $[(p, q)]$.

Exercise 3 : Let $<$ be a relation defined on \mathbb{N}^2 by :

$$(a, b) < (a', b') \iff \begin{cases} a < a' \\ \text{or} \\ a = a' \text{ and } b \leq b' \end{cases}$$

Prove that $<$ is an order relation. Is it total or partial ?

Exercise 4 : On \mathbb{N}^* , we define a relation \ll by assuming that for all $(k, l) \in \mathbb{N}^* \times \mathbb{N}^*$:

$$k \ll l \iff \text{There exists } n \in \mathbb{N}^* \text{ such that } l = k^n$$

- 1) Prove that \ll is a partial order relation.
- 2) We consider in the rest of the exercise that \mathbb{N}^* is ordered by the relation \ll . Let $A = \{2, 4, 16\}$, determine the greatest element and the smallest element of A .

Exercise 5 : Let E and F two sets and $f : E \rightarrow F$ a map. we define a relation R on E by assuming that for all $(x, x') \in E^2$:

$$xRx' \iff f(x) = f(x')$$

- 1) Prove that R is an equivalence relation.
- 2) Determine the equivalence classe $[x]$ for all $x \in E$.
- 3) Why the map :

$$\begin{array}{ccc} E/R & \longrightarrow & F \\ [x] & \longmapsto & f(x) \end{array}$$

is well defined ? Show that it is injective.