

## Series 3 : Binary relations

**Exercise 1 :** Let  $R$  be a relation defined on  $\mathbb{R}$  by :

$$xRy \iff x^2 - y^2 = x - y$$

- 1) Prove that  $R$  is an equivalence relation.
- 2) Determine the equivalence classe  $[x]$  for all  $x \in \mathbb{R}$ .
- 3) Determine the quotient set  $\mathbb{R}/R$ .

**Exercise 2 :** Let  $R$  be a relation defined on  $\mathbb{Z} \times \mathbb{N}^*$  by :

$$(a, b)R(a', b') \iff ab' = a'b$$

- 1) Prove that  $R$  is an equivalence relation.
- 2) Let  $(p, q) \in \mathbb{Z} \times \mathbb{N}^*$  with  $p \wedge q = 1$ . Write its equivalence classe  $[(p, q)]$ .

**Exercise 3 :** Let  $<$  be a relation defined on  $\mathbb{N}^2$  by :

$$(a, b) < (a', b') \iff \begin{cases} a < a' \\ or \\ a = a' \text{ and } b \leq b' \end{cases}$$

Prove that  $<$  is an order relation. Is it total or partial?

**Exercise 4 :** On  $\mathbb{N}^*$ , we define a relation  $\ll$  by assuming that for all  $(k, l) \in \mathbb{N}^* \times \mathbb{N}^*$  :

$$k \ll l \iff \text{There exists } n \in \mathbb{N}^* \text{ such that } l = k^n$$

- 1) Prove that  $\ll$  is a partial order relation.
- 2) We consider in the rest of the exercise that  $\mathbb{N}^*$  is ordered by the relation  $\ll$ . Let  $A = \{2, 4, 16\}$ , determine the greatest element and the smallest element of  $A$ .

**Exercise 5 :** Let  $E$  and  $F$  two sets and  $f : E \longrightarrow F$  a map. we define a relation  $R$  on  $E$  by assuming that for all  $(x, x') \in E^2$  :

$$xRx' \iff f(x) = f(x')$$

- 1) Prove that  $R$  is an equivalence relation.
- 2) Determine the equivalence classe  $[x]$  for all  $x \in E$ .
- 3) Why the map :

$$\begin{array}{ccc} E/R & \longrightarrow & F \\ [x] & \longmapsto & f(x) \end{array}$$

is well defined? Show that it is injective.