CHAPTER 3

BINARY RELATIONS

This chapter introduces the fundamental concept of **binary relations** on sets, covering their definition as subsets of Cartesian products and various representation methods (set, matrix, and graph representations). It explores key properties including reflexivity, symmetry, antisymmetry, and transitivity, leading to the study of two major types of relations: **equivalence relations** and **order relations**.

For equivalence relations, the chapter develops the crucial concepts of **equivalence classes**, **quotient sets**, and the **canonical projection**, demonstrating how equivalence relations partition sets into disjoint classes.

For order relations, it covers **partial orders**, **total orders**, and their visual representation through **Hasse diagrams**. The chapter also examines special elements in partially ordered sets (maximal/minimal elements, maximum/minimum) and establishes important results about their existence and uniqueness in finite posets.

The theoretical framework is complemented by numerous examples and exercises that illustrate applications across different mathematical contexts, providing a comprehensive foundation for understanding relational structures in algebra.

3.1 Definitions and Basic Properties

3.1.1 Definition and Representations

Definition 3.1 *Let E be a set. A **binary relation** $\mathcal{R}$ on E is a correspondence between elements of E. We write $x\mathcal{R}y$ to mean that element x is related to element y.*

Mathematical Definition: Formally, a binary relation $\mathcal{R}$ on E is a subset of the Cartesian product $E \times E$. Thus:

$$\mathcal{R} \subseteq E \times E$$

We say $x\mathcal{R}y$ if and only if $(x, y) \in \mathcal{R}$.

1. Let $E = \{1, 2, 3, 4\}$. The relation "is less than" is:

$$\mathcal{R} = \{(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)\}$$

2. On $\mathbb{Z}$, the relation "is congruent to modulo 3" is:

$$\mathcal{R} = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid x \equiv y \pmod 3\}$$

3.1.2 Types of Representations

1. Set Representation: As shown above, by explicitly listing the pairs.

2. Matrix Representation: For finite sets, we can use a matrix $M = (m_{ij})$ where:

$$m_{ij} = \begin{cases} 1 & \text{if } x_i\mathcal{R}x_j \\ 0 & \text{otherwise} \end{cases}$$

3. Graph Representation: Represent elements as vertices and draw arrows from x to y when $x\mathcal{R}y$.

Example 27 *Let $E = \{a, b, c\}$ with $\mathcal{R} = \{(a, b), (b, c), (c, a)\}$. Matrix representation (ordering: a,b,c):*

$$M = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Graph representation: A triangle with vertices a, b, c and arrows a?b, b?c, c?a.

3.2 Properties of Binary Relations

3.2.1 Fundamental Properties

Definition 3.2 *Let $\mathcal{R}$ be a binary relation on a set E.*

*$\mathcal{R}$ is **reflexive** if: $\forall x \in E, x\mathcal{R}x$*

*$\mathcal{R}$ is **symmetric** if: $\forall x, y \in E, x\mathcal{R}y \Rightarrow y\mathcal{R}x$*

*$\mathcal{R}$ is **antisymmetric** if: $\forall x, y \in E, (x\mathcal{R}y \wedge y\mathcal{R}x) \Rightarrow x = y$*

*$\mathcal{R}$ is **transitive** if: $\forall x, y, z \in E, (x\mathcal{R}y \wedge y\mathcal{R}z) \Rightarrow x\mathcal{R}z$*

Example 28 *1. On $\mathbb{R}$, "=" is reflexive, symmetric, antisymmetric, and transitive.*

2. On $\mathbb{R}$, "<" is transitive and antisymmetric, but neither reflexive nor symmetric.

3. On any set, the empty relation is symmetric, antisymmetric, and transitive, but not reflexive.

3.2.2 Special Types of Relations

Definition 3.3 ∘ *An **equivalence relation** is a relation that is reflexive, symmetric, and transitive.*
∘ *A **partial order** is a relation that is reflexive, antisymmetric, and transitive.*
∘ *A **total order** is a partial order where every two elements are comparable:* $\forall x, y \in E, x\mathcal{R}y \vee y\mathcal{R}x$

Example 29 *1. On* $\mathbb{Z}$*, congruence modulo* n *is an equivalence relation.*
2. On $\mathbb{R}$*, "=" is a total order.*
3. On $P(E)$ *(power set), "?" is a partial order but not necessarily total.*

3.3 Equivalence Relations

3.3.1 Equivalence Classes

Definition 3.4 *Let* $\mathcal{R}$ *be an equivalence relation on* E*. For* $x \in E$*, the **equivalence class** of* x *is:*

$$[x] = \{y \in E \mid x\mathcal{R}y\}$$

*The set of all equivalence classes is called the **quotient set** and is denoted* $E/\mathcal{R}$*.*

Proposition 3.1 *Let* $\mathcal{R}$ *be an equivalence relation on* E*. Then:*
1. $\forall x \in E, x \in [x]$ *(classes are non-empty)*
2. $[x] = [y] \iff x\mathcal{R}y$
3. $[x] \cap [y] \neq \emptyset \iff [x] = [y]$
4. The equivalence classes form a partition of E

Proof. Let $\mathcal{R}$ be an equivalence relation on E. We prove the four properties.

1. **Classes are non-empty:** $\forall x \in E, x \in [x]$.

   Since $\mathcal{R}$ is reflexive, $\forall x \in E, x\mathcal{R}x$. By the definition of an equivalence class, $x\mathcal{R}x$ means precisely that $x \in [x]$. Therefore, every equivalence class contains at least its representative, and is thus non-empty.

2. **Equal classes condition:** $[x] = [y] \iff x\mathcal{R}y$.

   ($\Rightarrow$) Suppose $[x] = [y]$. From part (1), we know $y \in [y]$. Therefore, $y \in [x]$, which by definition means $x\mathcal{R}y$.

   ($\Leftarrow$) Suppose $x\mathcal{R}y$. We will show $[x] \subseteq [y]$ and $[y] \subseteq [x]$.

- Let $z \in [x]$. This means $x\mathcal{R}z$. We have $x\mathcal{R}y$ (by assumption) and, by symmetry, $y\mathcal{R}x$. Using transitivity on $y\mathcal{R}x$ and $x\mathcal{R}z$, we get $y\mathcal{R}z$. Therefore, $z \in [y]$. This proves $[x] \subseteq [y]$.
- Let $z \in [y]$. This means $y\mathcal{R}z$. We have $x\mathcal{R}y$ (by assumption). Using transitivity on $x\mathcal{R}y$ and $y\mathcal{R}z$, we get $x\mathcal{R}z$. Therefore, $z \in [x]$. This proves $[y] \subseteq [x]$.

Since we have shown mutual inclusion, $[x] = [y]$.

3. **Non-empty intersection condition:** $[x] \cap [y] \neq \emptyset \iff [x] = [y]$.

   ($\Rightarrow$) Suppose $[x] \cap [y] \neq \emptyset$. Then there exists some $z \in E$ such that $z \in [x]$ and $z \in [y]$. By definition, this means $x\mathcal{R}z$ and $y\mathcal{R}z$. By symmetry of $\mathcal{R}$, $y\mathcal{R}z$ implies $z\mathcal{R}y$. Now, by transitivity of $\mathcal{R}$ on $x\mathcal{R}z$ and $z\mathcal{R}y$, we get $x\mathcal{R}y$. From part (2), this implies $[x] = [y]$.

   ($\Leftarrow$) Suppose $[x] = [y]$. From part (1), $x \in [x]$, so $x \in [y]$. Therefore, x is an element in $[x] \cap [y]$, so the intersection is non-empty.

4. **Equivalence classes form a partition of E.**

   A partition of E is a collection of non-empty, pairwise disjoint subsets whose union is E. We verify these properties for the set of equivalence classes $E/\mathcal{R} = \{[x] \mid x \in E\}$.

   - **Non-empty:** Each $[x]$ is non-empty by part (1).
   - **Union is E:** Every element $x \in E$ belongs to its own class $[x]$, so $\bigcup_{[x]\in E/\mathcal{R}}[x] = E$.
   - **Pairwise disjoint:** Let $[x]$ and $[y]$ be two equivalence classes. Suppose they are not disjoint, i.e., $[x] \cap [y] \neq \emptyset$. Then by part (3), $[x] = [y]$. Therefore, any two distinct classes (that are not equal) must be disjoint.

   This completes the proof that the equivalence classes form a partition of E.

$\square$

Example 30 *On $\mathbb{Z}$, consider congruence modulo 3:*
$[0] = \{\dots, -6, -3, 0, 3, 6, \dots\}$
$[1] = \{\dots, -5, -2, 1, 4, 7, \dots\}$
$[2] = \{\dots, -4, -1, 2, 5, 8, \dots\}$
These three classes partition $\mathbb{Z}$, and $\mathbb{Z}/\equiv_3 = \{[0], [1], [2]\}$.

3.3.2 Canonical Projection

Definition 3.5 *Let $\mathcal{R}$ be an equivalence relation on E. The **canonical projection** is the map:*

$$\pi : E \to E/\mathcal{R}, \quad \pi(x) = [x]$$

This map is surjective and each element of the quotient set is the image of all elements in its equivalence class.

3.4 Order Relations

3.4.1 Partial Orders

Definition 3.6 *A **partially ordered set** (poset) is a pair $(E, \preceq)$ where $\preceq$ is a partial order on E.*

Example 31 *1. $(\mathbb{R}, \leq)$ is a totally ordered set.*
2. $(P(E), \subseteq)$ is a poset but not necessarily totally ordered.
3. $(\mathbb{N}, |)$ (divisibility) is a poset.

3.4.2 Hasse Diagrams

For finite posets, we can represent them using **Hasse diagrams**:

- Draw elements as points
- If $x \preceq y$ and there's no z with $x \preceq z \preceq y$ (except x and y), draw y above x with a line segment
- Omit arrows (direction is implied by vertical position)
- Omit reflexive and transitive edges

Example 32 *For $E = \{1, 2, 3\}$ with divisibility relation:*
1 divides 2, 3
No relation between 2 and 3
The Hasse diagram has 1 at the bottom, with lines to 2 and 3 above it.

3.4.3 Special Elements in Posets

Definition 3.7 *Let $(E, \preceq)$ be a poset and $A \subseteq E$.*
*$m \in E$ is a **maximal element** if: $\forall x \in E, m \preceq x \Rightarrow m = x$.*
*$m \in E$ is a **maximum** (or greatest element) if: $\forall x \in E, x \preceq m$.*

*$m \in E$ is a **minimal element** if: $\forall x \in E, x \preceq m \Rightarrow x = m$.*
*$m \in E$ is a **minimum** (or least element) if: $\forall x \in E, m \preceq x$.*

Proposition 3.2 *In a finite poset, maximal (minimal) elements always exist. A maximum (minimum), if it exists, is unique.*

Proof. Let $(E, \preceq)$ be a finite poset. We prove the two statements.

1. **In a finite poset, maximal (minimal) elements always exist.**

   We prove the statement for maximal elements; the proof for minimal elements is analogous.

   Let $E = \{x_1, x_2, \ldots, x_n\}$ since E is finite. We construct a sequence to find a maximal element.

   Start with $m_1 = x_1$. Consider x_2. If $m_1 \preceq x_2$ and $m_1 \neq x_2$, then set $m_2 = x_2$. Otherwise, keep $m_2 = m_1$. Continue this process: for each x_k, if $m_{k-1} \preceq x_k$ and $m_{k-1} \neq x_k$, then set $m_k = x_k$; otherwise, set $m_k = m_{k-1}$.

   After processing all elements, we obtain an element $m = m_n$. We claim m is maximal.

   Suppose, for contradiction, that m is not maximal. Then there exists some $y \in E$ such that $m \preceq y$ and $m \neq y$. But y is one of the elements x_k in our finite list. When we processed $x_k = y$ in our algorithm, we would have found that $m_{k-1} \preceq y$ and $m_{k-1} \neq y$, so we would have set $m_k = y$. Since $m \preceq y$ and $y \preceq m_n = m$ (by the construction, as we only move to "greater" elements), by antisymmetry we get $m = y$, contradicting $m \neq y$.

   Therefore, our assumption was false, and m must be maximal.

2. **A maximum (minimum), if it exists, is unique.**

   We prove the statement for the maximum; the proof for the minimum is analogous.

   Suppose m_1 and m_2 are both maximum elements of E. By definition of maximum:

   - Since m_1 is a maximum, $\forall x \in E, x \preceq m_1$. In particular, $m_2 \preceq m_1$.
   - Since m_2 is a maximum, $\forall x \in E, x \preceq m_2$. In particular, $m_1 \preceq m_2$.

   Now we have $m_1 \preceq m_2$ and $m_2 \preceq m_1$. By the antisymmetry property of the partial order $\preceq$, this implies $m_1 = m_2$.

   Therefore, the maximum element, if it exists, is unique.

□

3.5 Exercises

Exercise 1: Let R be a relation defined on $\mathbb{R}$ by:

$$xRy \iff x^2 - y^2 = x - y$$

1) Prove that R is an equivalence relation.
2) Determine the equivalence classe $[x]$ for all $x \in \mathbb{R}$.
3) Determine the quotient set $\mathbb{R}/R$.

Solution. Let R be a relation defined on $\mathbb{R}$ by:

$$xRy \iff x^2 - y^2 = x - y$$

1. **Proof that R is an equivalence relation:**

   - **Reflexivity:** For all $x \in \mathbb{R}$,

     $$x^2 - x^2 = 0 = x - x \Rightarrow xRx$$

     So R is reflexive.

   - **Symmetry:** Suppose xRy, i.e., $x^2 - y^2 = x - y$. Then:

     $$y^2 - x^2 = -(x^2 - y^2) = -(x - y) = y - x \Rightarrow yRx$$

     So R is symmetric.

   - **Transitivity:** Suppose xRy and yRz. Then:

     $$x^2 - y^2 = x - y \quad (1)$$
     $$y^2 - z^2 = y - z \quad (2)$$

     Adding (1) and (2):

     $$x^2 - z^2 = (x - y) + (y - z) = x - z \Rightarrow xRz$$

     So R is transitive.

   Since R is reflexive, symmetric, and transitive, it is an equivalence relation.

2. **Equivalence class $[x]$:**

   We have:

   $$xRy \iff x^2 - y^2 = x - y \iff (x - y)(x + y) = (x - y)$$
   $$\iff (x - y)(x + y - 1) = 0 \iff x = y \quad \text{or} \quad y = 1 - x$$

   Therefore, the equivalence class of x is:

   $$[x] = \{x, 1 - x\}$$

3. **Quotient set $\mathbb{R}/R$:**

   The quotient set consists of all equivalence classes:

$$\mathbb{R}/R = \{\{x, 1-x\} \mid x \in \mathbb{R}\}$$

   Note that $[x] = [1-x]$ for all $x \in \mathbb{R}$, so each pair $\{x, 1-x\}$ appears only once.

□

Exercise 2: Let R be a relation defined on $\mathbb{Z} \times \mathbb{N}^*$ by:

$$(a,b)R(a',b') \iff ab' = a'b$$

1) Prove that R is an equivalence relation.
2) Let $(p,q) \in \mathbb{Z} \times \mathbb{N}^*$ with $p \wedge q = 1$. Write its equivalence classe $[(p,q)]$.

Solution. Let R be a relation defined on $\mathbb{Z} \times \mathbb{N}^*$ by:

$$(a,b)R(a',b') \iff ab' = a'b$$

1. **Proof that R is an equivalence relation:**

   - **Reflexivity:** For all $(a,b) \in \mathbb{Z} \times \mathbb{N}^*$,

$$ab = ab \Rightarrow (a,b)R(a,b)$$

     So R is reflexive.

   - **Symmetry:** Suppose $(a,b)R(a',b')$, i.e., $ab' = a'b$. Then:

$$a'b = ab' \Rightarrow (a',b')R(a,b)$$

     So R is symmetric.

   - **Transitivity:** Suppose $(a,b)R(a',b')$ and $(a',b')R(a'',b'')$. Then:

$$ab' = a'b \quad (1)$$
$$a'b'' = a''b' \quad (2)$$

     Multiply (1) by b'' and (2) by b:

$$ab'b'' = a'bb''$$
$$a'b''b = a''b'b$$

     Since $b' \neq 0$, we can equate:

$$ab'' = a''b \Rightarrow (a,b)R(a'',b'')$$

     So R is transitive.

Therefore, R is an equivalence relation.

2. **Equivalence class** $[(p,q)]$ **with** $p \wedge q = 1$**:**

   We have:
   $$(a,b)R(p,q) \iff aq = pb$$
   Since $p \wedge q = 1$ (coprime), q divides b and p divides a. Let $b = kq$, $a = kp$ for some $k \in \mathbb{N}^*$.

   Therefore:
   $$[(p,q)] = \{(kp,kq) \mid k \in \mathbb{N}^*\}$$
   This represents all fractions equivalent to $\frac{p}{q}$ in lowest terms.

□

Exercise 3: Let $<$ be a relation defined on $\mathbb{N}^2$ by:

$$(a,b) < (a',b') \iff \begin{cases} a < a' \\ \textit{or} \\ a = a' \text{ and } b \leq b' \end{cases}$$

Prove that $<$ is an order relation. Is it total or partial?

Solution. Let $<$ be a relation defined on $\mathbb{N}^2$ by:

$$(a,b) < (a',b') \iff \begin{cases} a < a' \\ \text{or} \\ a = a' \text{ and } b \leq b' \end{cases}$$

Proof that $<$ is an order relation:

- **Reflexivity:** For all $(a,b) \in \mathbb{N}^2$, $a = a$ and $b \leq b$, so $(a,b) < (a,b)$.
- **Antisymmetry:** Suppose $(a,b) < (a',b')$ and $(a',b') < (a,b)$.

  If $a < a'$, then we cannot have $a' < a$ or $a' = a$, contradiction. So we must have $a = a'$. Then $b \leq b'$ and $b' \leq b$, so $b = b'$. Thus $(a,b) = (a',b')$.
- **Transitivity:** Suppose $(a,b) < (a',b')$ and $(a',b') < (a'',b'')$.
  - If $a < a'$, then:
    - If $a' < a''$, then $a < a'' \Rightarrow (a,b) < (a'',b'')$
    - If $a' = a''$ and $b' \leq b''$, then $a < a'' \Rightarrow (a,b) < (a'',b'')$
  - If $a = a'$ and $b \leq b'$, then:

- $*$ If $a' < a''$, then $a < a'' \Rightarrow (a,b) < (a'',b'')$
- $*$ If $a' = a''$ and $b' \leq b''$, then $a = a''$ and $b \leq b'' \Rightarrow (a,b) < (a'',b'')$

So $<$ is transitive.

Therefore, $<$ is an order relation (lexicographic order).
Is it total or partial?
It is total: for any $(a,b),(a',b') \in \mathbb{N}^2$, either $a < a'$, or $a > a'$, or $a = a'$ and then either $b \leq b'$ or $b > b'$. □

Exercise 4: On $\mathbb{N}^*$, we define a relation $\ll$ by assuming that for all $(k,l) \in \mathbb{N}^* \times \mathbb{N}^*$:

$$k \ll l \Longleftrightarrow \text{There exists } n \in \mathbb{N}^* \text{ such that } l = k^n$$

1) Prove that $\ll$ is a partial order relation.
2) We consider in the rest of the exercise that $\mathbb{N}^*$ is ordered by the relation $\ll$. Let $A = \{2,4,16\}$, determine the greatest element and the smallest element of A.

Solution. On $\mathbb{N}^*$, we define a relation $\ll$ by:

$$k \ll l \Longleftrightarrow \text{There exists } n \in \mathbb{N}^* \text{ such that } l = k^n$$

1. **Proof that $\ll$ is a partial order relation:**

   - **Reflexivity:** For all $k \in \mathbb{N}^*$, $k = k^1 \Rightarrow k \ll k$.
   - **Antisymmetry:** Suppose $k \ll l$ and $l \ll k$. Then:

     $$l = k^m \quad \text{and} \quad k = l^n \quad \text{for some } m,n \in \mathbb{N}^*$$

     Substituting: $k = (k^m)^n = k^{mn} \Rightarrow k^{mn-1} = 1 \Rightarrow k = 1$ or $mn = 1 \Rightarrow m = n = 1$.
     If $k = 1$, then $l = 1^m = 1 = k$. If $m = n = 1$, then $l = k^1 = k$. So in both cases, $k = l$.
   - **Transitivity:** Suppose $k \ll l$ and $l \ll m$. Then:

     $$l = k^p \quad \text{and} \quad m = l^q \quad \text{for some } p,q \in \mathbb{N}^*$$

     Then $m = (k^p)^q = k^{pq} \Rightarrow k \ll m$.

   Therefore, $\ll$ is a partial order relation.

2. **For $A = \{2,4,16\}$:**

   - $2 \ll 4$ since $4 = 2^2$

- $2 \ll 16$ since $16 = 2^4$
- $4 \ll 16$ since $16 = 4^2$

So all elements are comparable.

Greatest element: 16 (since $2 \ll 16$, $4 \ll 16$, and $16 \ll 16$)

Smallest element: 2 (since $2 \ll 2$, $2 \ll 4$, $2 \ll 16$)

□

Exercise 5: Let E and F two sets and $f : E \longrightarrow F$ a map. we define a relation R on E by assuming that for all $(x, x') \in E^2$:

$$xRx' \Longleftrightarrow f(x) = f(x')$$

1) Prove that R is an equivalence relation.
2) Determine the equivalence classe $[x]$ for all $x \in E$.
3) Why the map:

$$\begin{array}{ccc} E/R & \longrightarrow & F \\ [x] & \longmapsto & f(x) \end{array}$$

is well defined? Show that it is injective.

Solution. Let E and F be two sets and $f : E \longrightarrow F$ a map. Define a relation R on E by:

$$xRx' \Longleftrightarrow f(x) = f(x')$$

1. **Proof that R is an equivalence relation:**

   - **Reflexivity:** For all $x \in E$, $f(x) = f(x) \Rightarrow xRx$
   - **Symmetry:** If xRx', then $f(x) = f(x') \Rightarrow f(x') = f(x) \Rightarrow x'Rx$
   - **Transitivity:** If xRx' and $x'Rx''$, then $f(x) = f(x')$ and $f(x') = f(x'') \Rightarrow f(x) = f(x'') \Rightarrow xRx''$

   So R is an equivalence relation.

2. **Equivalence class $[x]$:**

   $$[x] = \{y \in E \mid f(y) = f(x)\} = f^{-1}(\{f(x)\})$$

   The class of x is the fiber of f over $f(x)$.

3. **The map $\varphi : E/R \longrightarrow F, [x] \mapsto f(x)$ is well defined and injective:**

- **Well defined:** If $[x] = [y]$, then $xRy \Rightarrow f(x) = f(y)$, so $\varphi([x]) = f(x) = f(y) = \varphi([y])$. The image does not depend on the representative.
- **Injective:** Suppose $\varphi([x]) = \varphi([y])$. Then $f(x) = f(y) \Rightarrow xRy \Rightarrow [x] = [y]$.

Therefore, φ is a well-defined injective map.

□