CHAPTER 2

SETS AND APPLICATIONS

This chapter introduces fundamental concepts of sets and applications (Maps). It begins with the definition of a set, inclusion, and equality of sets. Key operations on sets are defined: difference, union, intersection, and symmetric difference, along with their properties (commutativity, associativity, distributivity, idempotence, and De Morgan's laws). The Cartesian product of two sets is also presented.

The second part focuses on applications, defined as mappings that assign each element of a starting set to a unique element in an arrival set. The concepts of direct image and inverse image of a subset under a map are explained. Finally, the chapter defines and provides examples of two crucial types of maps: surjective (onto) and injective (one-to-one) applications.

2.1 Notion of a Set and Properties

2.1.1 Set.

Definition 2.1 *A set is a collection of mathematical objects (elements) gathered according to one or more common properties. These properties are sufficient to affirm that an object belongs or does not belong to a set.*

Example 18 *(1) E: the set of students at USTO university. (2) We denote by* $\mathbb{N}$ *the set of natural numbers* $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$. *(3) The set of even numbers is denoted:* $P = \{x \in \mathbb{N} / 2 \text{ divides } x\}$. *(4) The empty set is denoted:* $\emptyset$ *which contains no elements.*

2.1.2 Inclusion.

We say that the set A is included in a set B when all the elements of A belong to B and we note $A \subset B$,

$$A \subset B \Leftrightarrow (\forall x, (x \in A \Rightarrow x \in B)).$$

The negation:

$$A \not\subset B \Leftrightarrow (\exists x, (x \in A \land x \notin B)).$$

Example 19 *(1) We denote $\mathbb{R}$ the set of real numbers, we have: $\mathbb{N} \subset \mathbb{R}$.*
(2) We denote $\mathbb{Z}$ the set of relative integers, $\mathbb{Q}$ the set of rationals, we have: $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$.

2.1.3 Equality of Two Sets:

Let A, B be two sets. Knowing $A = B$ means that:

$$A = B \Leftrightarrow ((A \subset B) \text{ and } (B \subset A)).$$

2.1.4 Difference of Two Sets:

The difference of two sets A, B is the set of elements of A that are not in B, denoted $A - B$.

$$A - B = \{x / x \in A \land x \notin B\}.$$

If $A \subset B$ then $B - A$ is also called the complement of A in B, it is denoted C_B^A, A^c

$$C_B^A = \{x / x \in B \land x \notin A\}.$$

2.1.5 Cardinal of a Set.

We call cardinal of a set A the number of its elements, denoted by $Card(A)$, if the cardinal of A is finite, we say that A is finite, otherwise we say that it is infinite.

2.1.6 Operations on Sets.

Union.

The union of two sets A and B is the set of elements that belong to A or B, we write $A \cup B$.

$$x \in A \cup B \Leftrightarrow (x \in A \lor x \in B).$$

The negation:

$$x \notin A \cup B \Leftrightarrow (x \notin A \land x \notin B).$$

Intersection.

The intersection of two sets A, B is the set of elements that belong to A and B, denoted $A \cap B$.

$$x \in A \cap B \Leftrightarrow (x \in A \wedge x \in B).$$

The negation:

$$x \notin A \cap B \Leftrightarrow (x \notin A \vee x \notin B).$$

Remark 3 *(1) If A, B have no elements in common, they are said to be disjoint, then $A \cap B = \emptyset$. (2) $B = C_E^A \Leftrightarrow A \cup B = E$ and $A \cap B = \emptyset$. (3) $A - B = A \cap B^c$.*

Symmetric Difference.

Let E be a non-empty set and $A, B \subset E$, the symmetric difference between two sets A, B is the set of elements that belong to $A - B$ or $B - A$, denoted $A \Delta B$.

$$\begin{aligned} A\Delta B &= (A - B) \cup (B - A) = \left(A \cap C_E^B\right) \cup \left(B \cap C_E^A\right) = (A \cup B) - (A \cap B). \\ x &\in A\Delta B \Leftrightarrow \{x / x \in (A - B) \vee x \in (B - A)\}. \end{aligned}$$

2.1.7 Properties of Operations on Sets.

Commutativity.

For any two sets A, B:

$$A \cap B = B \cap A, A \cup B = B \cup A.$$

Associativity.

For any three sets A, B, C:

$$A \cap (B \cap C) = (A \cap B) \cap C, A \cup (B \cup C) = (A \cup B) \cup C.$$

Distributivity.

For any three sets A, B, C:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C), A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

Idempotence.

$$A \cup A = A, A \cap A = A.$$

De Morgan's Laws.

$$\text{a) } (A \cup B)^c = A^c \cap B^c.$$

$$\text{b) } (A \cap B)^c = A^c \cup B^c.$$

Proof. Let us show that $(A \cup B)^c \subset A^c \cap B^c$ and $A^c \cap B^c \subset (A \cup B)^c$,
$(A \cup B)^c \subset A^c \cap B^c$:
Let $x \in (A \cup B)^c \Rightarrow x \notin (A \cup B) \Rightarrow x \notin A \wedge x \notin B \Rightarrow x \in A^c \wedge x \in B^c$
thus $x \in (A \cup B)^c \Rightarrow x \in (A^c \cap B^c)$, hence $(A \cup B)^c \subset (A^c \cap B^c)$.
$A^c \cap B^c \subset (A \cup B)^c$:
Let $x \in (A^c \cap B^c) \Rightarrow x \in A^c \wedge x \in B^c \Rightarrow x \notin A \wedge x \notin B \Rightarrow x \notin (A \cup B)$, hence $A^c \cap B^c \subset (A \cup B)^c$, thus $(A \cup B)^c = A^c \cap B^c$. We follow the same reasoning for the second relation. □

2.1.8 Power Set $\mathcal{P}(A)$.

Let A be a set of $card(A) = n$, the power set of A is the set denoted by $\mathcal{P}(A)$ which contains the empty set and the set A itself and the all parts possible of A. We write

$$\mathcal{P}(A) = \{\emptyset, X;\ X \subseteq A\},$$

and we have: $Card(\mathcal{P}(A)) = 2^n$.

2.1.9 Cartesian Product.

Let A, B be two sets, $a \in A, b \in B$. We note $A \times B = \{(a, b), a \in A, b \in B\}$. The set $A \times B$ is the set of ordered pairs (a, b); it is called the Cartesian product of the sets A and B.

Proposition 2.1 *If A and B are finite sets and if we denote by:*
Card(A): the number of elements of A.
Card(B): the number of elements of B. we will have:

$$Card(A \times B) = Card(A) \times Card(B).$$

Example 20 *a) Let* $E = \{1, 2, 3, 4, 5, 6, 7, 8\}, A = \{1, 2, 3, 4, 5, 6\}, B = \{2, 4, 6, 8\}$
(1) $A \subset E, B \subset E$. *A is not included in B because* $1 \in A \wedge 1 \notin B$.
B is not included in A because $8 \in B \wedge 8 \notin A$.
(2) $A \cap B = \{2, 4, 6\}, A \cup B = \{1, 2, 3, 4, 5, 6, 8\}$.
(3) $A - B = \{1, 3, 5\}, B - A = \{8\}$.
(4) $A \Delta B = \{1, 3, 5, 8\}$
(5) $\mathcal{P}(B) = \{\emptyset, B, \{2\}, \{4\}, \{6\}, \{8\}, \{2, 4\}, \{2, 6\}, \{2, 8\}, \{4, 6\}, \{4, 8\}, \{6, 8\}, \{2, 4, 6\}, \{2, 6, 8\},$

$\{4,6,8\},\{2,4,8\}\}$.
We have Card($\mathcal{P}(B)$)=$2^4 = 16$.
b) $A = \{1,2\}, B = \{1,2,3\}$

$$A \times B = \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3)\}$$
$$B \times A = \{(1,1),(1,2),(2,1),(2,2),(3,1),(3,2)\}$$

$A \times B \neq B \times A$, *because* $(3,2) \in B \times A$, *and* $(3,2) \notin A \times B$.

2.2 Applications (Maps)

2.2.1 Application (Map).

Definition 2.2 *We call an application (or map) from a set E to a set F a law of correspondence (or a relation of correspondence) allowing us to associate to every $x \in E$ a unique element $y \in F$. E is the starting set and F is the arrival set. The element y associated to x is the image of x by f, we note $x \longmapsto y/y = f(x)$.*

Example 21 *Consider the following application:*
(1) $f_1 : \mathbb{N} \longmapsto \mathbb{N}$, $n \longmapsto 4n+2$.
(2) $f_2 : \mathbb{R} \longmapsto \mathbb{R}$, $x \longmapsto 5x+3$.

2.2.2 Direct Image and Inverse Image.

a) Direct Image. Let $f : E \longmapsto F$ and $A \subset E$, we call the image of A by f a subset of F, denoted $f(A)$, such that

$$f(A) = \{f(x) \in F/x \in A\},$$

knowing that $f(A) \subset F$, and that $A, f(A)$ are sets.
b) Inverse Image. Let $f : E \longmapsto F$ and $B \subset F$, we call the inverse image of B by f, the part of E denoted $f^{-1}(B)$, such that

$$f^{-1}(B) = \{x \in E/f(x) \in B\},$$

knowing that $f^{-1}(B) \subset E$, and that $B, f^{-1}(B)$ are sets.

Example 22 *(1) Let f be the application defined by:*

$$f : [0,3] \longmapsto [0,4] x \longmapsto f(x) = 2x+1$$

Find $f([0,1])$?

$$f([0,1]) = \{f(x)/x \in [0,1]\} = \{2x+1/0 \leq x \leq 1\},$$

we have: $0 \leq x \leq 1 \Rightarrow 0 \leq 2x \leq 2 \Rightarrow 1 \leq 2x+1 \leq 3$, *then* $f([0,1]) = [1,3] \subset [0,4]$.

(2) Let f be the application defined by:

$$g : [0,2] \longmapsto [0,4] x \longmapsto f(x) = (2x-1)^2$$

Calculate $f^{-1}(\{0\}), f^{-1}((0,1))$.

$$\begin{aligned}
f^{-1}(\{0\}) &= \{x \in [0,2]/f(x) \in \{0\}\} = \{x \in [0,2]/f(x) = 0\} \\
&= \{x \in [0,2]/(2x-1)^2 = 0\} = \left\{\frac{1}{2}\right\}. \\
f^{-1}((0,1)) &= \{x \in [0,2]/f(x) \in (0,1)\} = \left\{x \in [0,2]/0 < (2x-1)^2 < 1\right\}, \\
\textit{We have} &: (2x-1)^2 > 0 \textit{ is verified } \forall x \in \mathbb{R} - \left\{\frac{1}{2}\right\}, x \in [0,2]. \textit{ On the other hand} \\
(2x-1)^2 &< 1 \Rightarrow |2x-1| < 1 \Rightarrow -1 < 2x-1 < 1 \Rightarrow 0 < x < 1, \\
\textit{and therefore } x &\in (0,1), \textit{ by combining the two inequalities, we obtain} \\
f^{-1}((0,1)) &= \left(\left[0,\frac{1}{2}\right) \cup \left(\frac{1}{2},2\right]\right) \cap (0,1) = \left(0,\frac{1}{2}\right) \cup \left(\frac{1}{2},1\right).
\end{aligned}$$

2.2.3 Restriction of an applications.

Let $f : E \to F$ a map, the restriction of f is a new map denoted by f/A, obtained by choosing a smaller domain $(A \subset E)$ for the original map f . The map f is then said to extend f/A.

Example 23 *Let* $f : \mathbb{R} \longmapsto \mathbb{R}$ *defined by* $x \longmapsto f(x) = 5x+3$. *Then* $f/[-6,8] : [-6,8] \longmapsto \mathbb{R}$, $x \longmapsto f/[-6,8](x) = f(x) = 5x+3$.

2.2.4 Surjection.

Definition 2.3 *The image $f(E)$ of E by f is a subset of F. If every element of F is the image by f of at least one element of E, we say that f is a surjective application from E into F, we have: $f(E) = F$. f is surjective* $\Leftrightarrow \forall y \in F, \exists x \in E/f(x) = y$.

Example 24 *Are the following applications surjective?*
(1) $f_1 : \mathbb{N} \longmapsto \mathbb{N}$, $n \longmapsto 4n+2$.

f_1 is not surjective; indeed, if we assume it is surjective, that is to say

$$\forall y \in \mathbb{N}, \exists n \in \mathbb{N}/4n+2 = y \Longrightarrow n = \frac{y-2}{4},$$

but $n = \frac{y-2}{4} \notin \mathbb{N}$ for many y (e.g., y=1), contradiction. f_1 is not surjective.

(2) $f_2 : \mathbb{R} \longmapsto \mathbb{R}$ $x \longmapsto 5x+3$.

f_2 is surjective because: $\forall y \in \mathbb{R}, \exists x \in \mathbb{R}/5x+3 = y \Longrightarrow x = \frac{y-3}{5} \in \mathbb{R}$.

2.2.5 Injection.

Definition 2.4 *When two distinct elements of E correspond via f to two different images in F, f is said to be an injective application, we then have:*

$$(f \text{ is injective }) \Leftrightarrow (\forall x_1, x_2 \in E, x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)),$$

or

$$(f \text{ is injective }) \Leftrightarrow (\forall x_1, x_2 \in E, f(x_1) = f(x_2) \Rightarrow x_1 = x_2).$$

Example 25 *Are the following applications injective?*

(1) $f : \mathbb{N} \longmapsto \mathbb{N}$, $n \longmapsto 4n + 2$.
f is injective because: $\forall n_1, n_2 \in \mathbb{N}, f(n_1) = f(n_2) \Rightarrow 4n_1 + 2 = 4n_2 + 2 \Rightarrow 4n_1 = 4n_2 \Rightarrow n_1 = n_2$.

2.2.6 Bijective

Definition 2.5 *A map $f : E \to F$ is called **bijective** if it is both injective and surjective. This definition combines the two previous properties:*
Injective: *Every element in F is the image of **at most one** element in E. (No two different elements in E have the same image in F).*
Surjective: *Every element in F is the image of **at least one** element in E. (The map "covers" all of F).*
Therefore, a map is bijective if and only if:

$$\forall y \in F, \quad \exists!\, x \in E \text{ such that } f(x) = y.$$

This means that for every element $y \in F$, there exists **one and only one** element $x \in E$ such that $f(x) = y$. A bijective map establishes a perfect "pairing" or "one-to-one correspondence" between the elements of the set E and the elements of the set F.

Example 26 *Determine if the following functions are bijective.*

1. $f_1 : \mathbb{R} \to \mathbb{R}$ defined by $f_1(x) = 2x + 3$.

   Solution:

   - *Injective?* Suppose $f_1(x_1) = f_1(x_2)$. Then $2x_1+3 = 2x_2+3$, which implies $2x_1 = 2x_2$, so $x_1 = x_2$. Yes, it is injective.
   - *Surjective?* For any $y \in \mathbb{R}$, we solve $y = 2x + 3$ for x: $x = \frac{y-3}{2} \in \mathbb{R}$. Yes, it is surjective.

Since f_1 is both injective and surjective, it is **bijective**.

2. $f_2 : \mathbb{N} \to \mathbb{N}$ defined by $f_2(n) = n + 1$.

   Solution:

   - *Injective?* Suppose $f_2(n_1) = f_2(n_2)$. Then $n_1 + 1 = n_2 + 1$, so $n_1 = n_2$. Yes, it is injective.
   - *Surjective?* Let $y = 0 \in \mathbb{N}$. Is there an $n \in \mathbb{N}$ such that $n + 1 = 0$? No, because $n = -1 \notin \mathbb{N}$. Therefore, f_2 is **not surjective**, and hence not bijective.

3. $f_3 : \mathbb{R} \to \mathbb{R}$ defined by $f_3(x) = x^2$.

   Solution:

   - *Injective?* We have $f_3(1) = 1$ and $f_3(-1) = 1$. Different inputs give the same output. So, it is **not injective**.
   - *Surjective?* Let $y = -1 \in \mathbb{R}$. Is there an $x \in \mathbb{R}$ such that $x^2 = -1$? No. So, it is also **not surjective**. Therefore, f_3 is not bijective.

4. $f_4 : \mathbb{R}^+ \to \mathbb{R}^+$ defined by $f_4(x) = x^2$, where $\mathbb{R}^+$ is the set of non-negative real numbers.

   Solution:

   - *Injective?* For $x_1, x_2 \geq 0$, if $x_1^2 = x_2^2$, then $x_1 = x_2$. Yes, it is injective.
   - *Surjective?* For any $y \geq 0$, we have $x = \sqrt{y} \geq 0$ and $f_4(\sqrt{y}) = y$. Yes, it is surjective.

   Therefore, f_4 is **bijective**. This example shows how restricting the domain and codomain can change the properties of a function.

2.2.7 Composition of Maps

Definition 2.6 *Let E, F, G be three sets. Let $f : E \to F$ and $g : F \to G$ be two maps. The **composition** of f and g is the map denoted $g \circ f$ (read "g round f") from E to G, defined by:*

$$\forall x \in E, \quad (g \circ f)(x) = g(f(x)).$$

The element x is first mapped to $f(x)$ in F, and then $f(x)$ is mapped to $g(f(x))$ in G.

Remark 4 *The composition $g \circ f$ is only defined if the arrival set of f is the same as the starting set of g. The order of operations is important: $g \circ f$ means "apply f first, then g".*

Example 1.8 Let $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x + 1$, and $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x^2$.
- $(g \circ f)(x) = g(f(x)) = g(x + 1) = (x + 1)^2$. - $(f \circ g)(x) = f(g(x)) = f(x^2) = x^2 + 1$.
We see that in general, $g \circ f \neq f \circ g$. Composition of maps is **not commutative**.

Proposition 2.2 ***(Associativity of Composition)*** *Let $f : E \to F$, $g : F \to G$, and $h : G \to H$ be maps. Then:*

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

This map from E to H is simply denoted $h \circ g \circ f$.

Proof. For any $x \in E$:

$$(h \circ (g \circ f))(x) = h((g \circ f)(x)) = h(g(f(x)))$$

$$((h \circ g) \circ f)(x) = (h \circ g)(f(x)) = h(g(f(x)))$$

The results are identical, so the maps are equal. □

2.2.8 Reciprocal Map (Inverse Map)

Definition 2.7 *A map $f : E \to F$ is called **bijective** if it is both injective and surjective. This means that for every element $y \in F$, there exists **exactly one** element $x \in E$ such that $f(x) = y$.*

Definition 2.8 *Let $f : E \to F$ be a **bijective** map. The **reciprocal** (or inverse) map of f, denoted $f^{-1} : F \to E$, is the map that associates to each $y \in F$ the unique element $x \in E$ for which $f(x) = y$. In other words:*

$$\forall y \in F, \quad f^{-1}(y) = x \iff f(x) = y.$$

Warning: The notation f^{-1} in this context refers to the *inverse map*, which is different from the *inverse image* of a set $f^{-1}(B)$. The inverse map only exists when f is bijective.

Proposition 2.3 ***(Characterization of the Inverse)*** *Let $f : E \to F$ be a bijective map. Its inverse $f^{-1} : F \to E$ satisfies the following properties:*
1. $f^{-1} \circ f = Id_E$, where Id_E is the identity map on E ($Id_E(x) = x$).
2. $f \circ f^{-1} = Id_F$, where Id_F is the identity map on F ($Id_F(y) = y$).
Conversely, if there exists a map $g : F \to E$ such that $g \circ f = Id_E$ and $f \circ g = Id_F$, then f is bijective and $g = f^{-1}$.

1. Let $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 3x - 5$. This function is bijective. Let's find its inverse. Let $y = 3x - 5$. We solve for x: $x = \frac{y+5}{3}$. Therefore, the inverse function is $f^{-1}(y) = \frac{y+5}{3}$. We verify: $(f^{-1} \circ f)(x) = f^{-1}(3x - 5) = \frac{(3x-5)+5}{3} = x$.

2. The function $g : \mathbb{R}^+ \to \mathbb{R}^+$ defined by $g(x) = x^2$ is bijective (from the non-negative reals to the non-negative reals). Its inverse is the square root function: $g^{-1}(x) = \sqrt{x}$.

Proposition 2.4 ***(Inverse of a Composition)*** *Let $f : E \to F$ and $g : F \to G$ be bijective maps. Then the composition $g \circ f$ is bijective, and its inverse is given by:*

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}.$$

Proof. We check the characteristic property using associativity:

$$(f^{-1} \circ g^{-1}) \circ (g \circ f) = f^{-1} \circ (g^{-1} \circ g) \circ f = f^{-1} \circ Id_F \circ f = f^{-1} \circ f = Id_E.$$

Similarly, $(g \circ f) \circ (f^{-1} \circ g^{-1}) = Id_G$. Therefore, $f^{-1} \circ g^{-1}$ is the inverse of $g \circ f$. □

2.3 Exercises

Exercise 1: Assuming the set $A = \{w, x, y, z\}$, $B = \{x, y\}$, $C = \{x, y, z\}$ and $D = \{x, z\}$ three parts of A. Identify the elements in each set: $B^c, C^c, B\backslash C, B\backslash D, B \cap C, B \cap D, B \cap (C \cup D), (B \cap C) \cup D, B\backslash D, D\backslash B, B \times C, C \times B, B \times D, P(B)$ and $P(C)$.

Solution. Given:

$$A = \{w, x, y, z\}, \quad B = \{x, y\}, \quad C = \{x, y, z\}, \quad D = \{x, z\}$$

- $B^c = A \setminus B = \{w, z\}$
- $C^c = A \setminus C = \{w\}$
- $B \setminus C = \emptyset$
- $B \setminus D = \{y\}$
- $B \cap C = \{x, y\}$
- $B \cap D = \{x\}$
- $B \cap (C \cup D) = B \cap \{x, y, z\} = \{x, y\}$
- $(B \cap C) \cup D = \{x, y\} \cup \{x, z\} = \{x, y, z\}$
- $D \setminus B = \{z\}$
- $B \times C = \{(x, x), (x, y), (x, z), (y, x), (y, y), (y, z)\}$

- $C \times B = \{(x,x),(x,y),(y,x),(y,y),(z,x),(z,y)\}$
- $B \times D = \{(x,x),(x,z),(y,x),(y,z)\}$
- $\mathcal{P}(B) = \{\emptyset,\{x\},\{y\},\{x,y\}\}$
- $\mathcal{P}(C) = \{\emptyset,\{x\},\{y\},\{z\},\{x,y\},\{x,z\},\{y,z\},\{x,y,z\}\}$

□

Exercise 2: Let A, B and C be three parts of a set E. Prove that:
1) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
2) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Solution. Let A, B and C be three subsets of a set E.

1. Prove that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

   Let $x \in A \cup (B \cap C)$. Then $x \in A$ or $x \in B \cap C$.

   - If $x \in A$, then $x \in A \cup B$ and $x \in A \cup C$, so $x \in (A \cup B) \cap (A \cup C)$.
   - If $x \in B \cap C$, then $x \in B$ and $x \in C$, so $x \in A \cup B$ and $x \in A \cup C$, hence $x \in (A \cup B) \cap (A \cup C)$.

   Thus $A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C)$.

   Now let $x \in (A \cup B) \cap (A \cup C)$. Then $x \in A \cup B$ and $x \in A \cup C$.

   - If $x \in A$, then $x \in A \cup (B \cap C)$.
   - If $x \notin A$, then from $x \in A \cup B$ we get $x \in B$, and from $x \in A \cup C$ we get $x \in C$. So $x \in B \cap C$, hence $x \in A \cup (B \cap C)$.

   Thus $(A \cup B) \cap (A \cup C) \subset A \cup (B \cap C)$.

   Therefore, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

2. Prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

   Let $x \in A \cap (B \cup C)$. Then $x \in A$ and $x \in B \cup C$.

   - If $x \in B$, then $x \in A \cap B$.
   - If $x \in C$, then $x \in A \cap C$.

   So $x \in (A \cap B) \cup (A \cap C)$. Now let $x \in (A \cap B) \cup (A \cap C)$.

   - If $x \in A \cap B$, then $x \in A$ and $x \in B \subset B \cup C$, so $x \in A \cap (B \cup C)$.

- If $x \in A \cap C$, then $x \in A$ and $x \in C \subset B \cup C$, so $x \in A \cap (B \cup C)$.

Therefore, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

□

Exercise 3: Let E be a set and A and B two parts of E. Assume that:

$$A \cap B \neq \emptyset,\ A \cup B \neq E,\ A \not\subseteq B \text{ and } B \not\subseteq A.$$

Suppose that: $A_1 = A \cap B$, $A_2 = A \cap B^c$, $A_3 = B \cap A^c$, $A_4 = (A \cup B)^c$.
1) Prove that A_1, A_2, A_3 and A_4 are not emply.
2) Prove that A_1, A_2, A_3 and A_4 are two by two disjoint.
3) Prove that $A_1 \cup A_2 \cup A_3 \cup A_4 = E$.

Solution. Let E be a set and A and B two subsets of E such that:

$$A \cap B \neq \emptyset, \quad A \cup B \neq E, \quad A \not\subseteq B \text{ and } B \not\subseteq A.$$

Define: $A_1 = A \cap B, \quad A_2 = A \cap B^c, \quad A_3 = B \cap A^c, \quad A_4 = (A \cup B)^c.$

1. Prove that A_1, A_2, A_3 and A_4 are not empty.

   - $A_1 = A \cap B \neq \emptyset$ by hypothesis.
   - $A_2 = A \cap B^c \neq \emptyset$ because $A \not\subseteq B$.
   - $A_3 = B \cap A^c \neq \emptyset$ because $B \not\subseteq A$.
   - $A_4 = (A \cup B)^c \neq \emptyset$ because $A \cup B \neq E$.

2. Prove that A_1, A_2, A_3 and A_4 are pairwise disjoint.

   - $A_1 \cap A_2 = (A \cap B) \cap (A \cap B^c) = A \cap B \cap B^c = \emptyset$
   - $A_1 \cap A_3 = (A \cap B) \cap (B \cap A^c) = A \cap A^c \cap B = \emptyset$
   - $A_1 \cap A_4 = (A \cap B) \cap (A \cup B)^c = (A \cap B) \cap (A^c \cap B^c) = \emptyset$
   - $A_2 \cap A_3 = (A \cap B^c) \cap (B \cap A^c) = \emptyset$
   - $A_2 \cap A_4 = (A \cap B^c) \cap (A \cup B)^c = (A \cap B^c) \cap (A^c \cap B^c) = \emptyset$
   - $A_3 \cap A_4 = (B \cap A^c) \cap (A \cup B)^c = (B \cap A^c) \cap (A^c \cap B^c) = \emptyset$

3. Prove that $A_1 \cup A_2 \cup A_3 \cup A_4 = E$.

   Let $x \in E$. There are four mutually exclusive cases:

   - $x \in A$ and $x \in B$: then $x \in A_1$

- $x \in A$ and $x \notin B$: then $x \in A_2$
- $x \notin A$ and $x \in B$: then $x \in A_3$
- $x \notin A$ and $x \notin B$: then $x \in A_4$

Thus every element of E belongs to exactly one of the four sets.

□

Exercise 4: Let A, B and C be three parts of a set E.
1) What do you think about the implication: $(A \cup B \not\subset C) \Longrightarrow (A \not\subset C$ or $B \not\subset C)$?
2) Suppose that we have $A \cup B \subset A \cup C$ and $A \cap B \subset A \cap C$. Prove that $B \subset C$.

Solution. Let A, B and C be three subsets of a set E.

1. The implication $(A \cup B \not\subset C) \implies (A \not\subset C$ or $B \not\subset C)$ is true.

   If $A \cup B \not\subset C$, then there exists $x \in A \cup B$ such that $x \notin C$. Since $x \in A \cup B$, we have $x \in A$ or $x \in B$.

   - If $x \in A$, then $A \not\subset C$.
   - If $x \in B$, then $B \not\subset C$.

   So in either case, $A \not\subset C$ or $B \not\subset C$.

2. Suppose $A \cup B \subset A \cup C$ and $A \cap B \subset A \cap C$. Prove that $B \subset C$.

   Let $x \in B$. We consider two cases:

   - If $x \in A$, then $x \in A \cap B \subset A \cap C$, so $x \in C$.
   - If $x \notin A$, then $x \in B \subset A \cup B \subset A \cup C$. Since $x \notin A$, we must have $x \in C$.

   In both cases, $x \in C$. Therefore, $B \subset C$.

□

Exercise 5: Let E a set and A and B two parts of E. Demonstrate that:
1) $F \subset G \Longleftrightarrow F \cup G = G$.
2) $F \subset G \Longleftrightarrow F \cap G^c = \emptyset$.

Solution. Let E be a set and F and G two subsets of E.

1. Prove that $F \subset G \iff F \cup G = G$.

   $(\Rightarrow)$ If $F \subset G$, then $F \cup G = G$.

   $(\Leftarrow)$ If $F \cup G = G$, then for any $x \in F$, we have $x \in F \cup G = G$, so $F \subset G$.

2. Prove that $F \subset G \iff F \cap G^c = \emptyset$.

   ($\Rightarrow$) If $F \subset G$, then no element of F is in G^c, so $F \cap G^c = \emptyset$.

   ($\Leftarrow$) If $F \cap G^c = \emptyset$, then no element of F is outside G, so $F \subset G$.

□

Exercise 6: Let $f : I \longrightarrow J$ the function defined by $f(x) = x^2$.
1) Give sets I and J such that f will be injective but not surjective.
2) Give sets I and J such that f will be surjective but not injective.
3) Give sets I and J such that f will be neither injective nor surjective.
4) Give sets I and J such that f will be injective and surjective.

Solution. Let $f : I \to J$ be defined by $f(x) = x^2$.

1. Injective but not surjective: $I = \mathbb{N}, J = \mathbb{N}$
2. Surjective but not injective: $I = \mathbb{R}, J = \mathbb{R}^+$
3. Neither injective nor surjective: $I = \mathbb{R}, J = \mathbb{R}$
4. Bijective: $I = \mathbb{R}^+, J = \mathbb{R}^+$

□

Exercise 7: We consider the map $f : \mathbb{N} \longrightarrow \mathbb{N}$ defined by: for all $n \in \mathbb{N}$, $f(n) = n^2$.
1) Is it exist a map $g : \mathbb{N} \longrightarrow \mathbb{N}$ such that $f \circ g = Id_{\mathbb{N}}$?
2) Is it exist a map $h : \mathbb{N} \longrightarrow \mathbb{N}$ such that $h \circ f = Id_{\mathbb{N}}$?

Solution. Consider $f : \mathbb{N} \to \mathbb{N}$ defined by $f(n) = n^2$.

1. Does there exist $g : \mathbb{N} \to \mathbb{N}$ such that $f \circ g = Id_{\mathbb{N}}$?

   No, because f is not surjective (e.g., 2 has no preimage).

2. Does there exist $h : \mathbb{N} \to \mathbb{N}$ such that $h \circ f = Id_{\mathbb{N}}$?

   Yes, define $h(n) = \sqrt{n}$ if n is a perfect square, and $h(n) = 0$ otherwise.

□

Exercise 8: Let E and F two sets and a map $f : E \longrightarrow F$. Let A and B two parts of E. Demonstrate that:
1) $f(A \cup B) = f(A) \cup f(B)$.
2) $f(A \cap B) \subset f(A) \cap f(B)$.
Give an example for the second property. Then prove that f is injective iff for any parts A and B of E, we have $f(A \cap B) = f(A) \cap f(B)$.

Solution. Let $f : E \to F$, and $A, B \subset E$.

1. Prove that $f(A \cup B) = f(A) \cup f(B)$.

   ($\subset$) If $y \in f(A \cup B)$, then $\exists x \in A \cup B$ with $f(x) = y$. If $x \in A$, then $y \in f(A)$; if $x \in B$, then $y \in f(B)$. So $y \in f(A) \cup f(B)$.

   ($\supset$) If $y \in f(A) \cup f(B)$, then $y \in f(A)$ or $y \in f(B)$. If $y \in f(A)$, then $\exists x \in A$ with $f(x) = y$, so $x \in A \cup B$ and $y \in f(A \cup B)$. Similarly if $y \in f(B)$.

2. Prove that $f(A \cap B) \subset f(A) \cap f(B)$.

   If $y \in f(A \cap B)$, then $\exists x \in A \cap B$ with $f(x) = y$. Since $x \in A$ and $x \in B$, we have $y \in f(A)$ and $y \in f(B)$, so $y \in f(A) \cap f(B)$.

   Example where equality fails: Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$, $A = [-1, 0]$, $B = [0, 1]$. Then $A \cap B = \{0\}$, so $f(A \cap B) = \{0\}$, but $f(A) = [0, 1]$, $f(B) = [0, 1]$, so $f(A) \cap f(B) = [0, 1]$.

3. Prove that f is injective $\iff$ for any $A, B \subset E$, $f(A \cap B) = f(A) \cap f(B)$.

   ($\Rightarrow$) If f is injective, we already have $f(A \cap B) \subset f(A) \cap f(B)$. For the reverse inclusion: if $y \in f(A) \cap f(B)$, then $\exists a \in A$ with $f(a) = y$ and $\exists b \in B$ with $f(b) = y$. Since f is injective, $a = b$, so $a \in A \cap B$ and $y \in f(A \cap B)$.

   ($\Leftarrow$) Suppose $f(A \cap B) = f(A) \cap f(B)$ for all $A, B \subset E$. Let $x_1, x_2 \in E$ with $f(x_1) = f(x_2)$. Take $A = \{x_1\}$, $B = \{x_2\}$. Then $f(A \cap B) = f(A) \cap f(B)$. If $x_1 \neq x_2$, then $A \cap B = \emptyset$, so $f(A \cap B) = \emptyset$, but $f(A) \cap f(B) = \{f(x_1)\} \neq \emptyset$. Contradiction. So $x_1 = x_2$, and f is injective.

$\square$

Exercise 9: 1) Let f the map of $\{1, 2, 3, 4\}$ in it self defined by: $f(1) = 4$, $f(2) = 1$, $f(3) = 2$ and $f(4) = 2$.
Determine $f^{-1}(A)$ when $A = \{2\}$, $A = \{1, 2\}$ and $A = \{3\}$.
2) Let f the map of $\mathbb{R}$ in $\mathbb{R}$ defined by: $f(x) = x^2$. Determine $f^{-1}(A)$ when $A = \{1\}$ and $A = [1, 2]$.

Solution.

1. $f : \{1, 2, 3, 4\} \to \{1, 2, 3, 4\}$ with $f(1) = 4, f(2) = 1, f(3) = 2, f(4) = 2$

   - $f^{-1}(\{2\}) = \{3, 4\}$
   - $f^{-1}(\{1, 2\}) = \{2, 3, 4\}$
   - $f^{-1}(\{3\}) = \emptyset$

2. $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x^2$

   - $f^{-1}(\{1\}) = \{-1, 1\}$

- $f^{-1}([1,2]) = [-\sqrt{2}, -1] \cup [1, \sqrt{2}]$

$\square$

Exercise 10: 1) Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by: $f(x,y) = x$. Determine $f([0,1] \times [0,1])$ and $f^{-1}([-1,1])$.

2) Let $g : \mathbb{R} \longrightarrow [-1,1]$ defined by: $g(x) = \cos(\pi x)$. Determine $g(\mathbb{N})$, $g(2\mathbb{N})$ and $g^{-1}(\{-1,1\})$.

Solution.

1. $f : \mathbb{R}^2 \to \mathbb{R}$ with $f(x,y) = x$
   - $f([0,1] \times [0,1]) = [0,1]$
   - $f^{-1}([-1,1]) = [-1,1] \times \mathbb{R}$
2. $g : \mathbb{R} \to [-1,1]$ with $g(x) = \cos(\pi x)$
   - $g(\mathbb{N}) = \{\cos(\pi n) : n \in \mathbb{N}\} = \{(-1)^n : n \in \mathbb{N}\} = \{-1, 1\}$
   - $g(2\mathbb{N}) = \{\cos(2\pi n) : n \in \mathbb{N}\} = \{1\}$
   - $g^{-1}((-1,1)) = \{x \in \mathbb{R} : \cos(\pi x) \in (-1,1)\} = \mathbb{R} \setminus \mathbb{Z}$

$\square$

Exercise 11: Let E and F two sets and a map $f : E \longrightarrow F$. Let C and D two parts not emply of F. Demonstrate that:

1) $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$.

2) $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$.

Solution. Let $f : E \to F$, and $C, D \subset F$.

1. Prove that $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$.

   $x \in f^{-1}(C \cup D) \iff f(x) \in C \cup D \iff f(x) \in C$ or $f(x) \in D$
   $\iff x \in f^{-1}(C)$ or $x \in f^{-1}(D) \iff x \in f^{-1}(C) \cup f^{-1}(D)$.

2. Prove that $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$.

   $x \in f^{-1}(C \cap D) \iff f(x) \in C \cap D \iff f(x) \in C$ and $f(x) \in D$
   $\iff x \in f^{-1}(C)$ and $x \in f^{-1}(D) \iff x \in f^{-1}(C) \cap f^{-1}(D)$.

$\square$

Exercise 12: Let E and F two sets and a map $f : E \longrightarrow F$.

1) Prove that for any part A of E we have : $A \subset f^{-1}(f(A))$.

2) Prove that for any parts B of E we have : $f(f^{-1}(B)) \subset B$.

3) Prove that f is injective iff for any part A of E we have : $A = f^{-1}(f(A))$.
4) Prove that f is surjective iff for any part B of E we have : $f(f^{-1}(B)) = B$.

Solution. Let $f : E \to F$.

1. Prove that for any $A \subset E$, $A \subset f^{-1}(f(A))$.

   If $x \in A$, then $f(x) \in f(A)$, so $x \in f^{-1}(f(A))$.

2. Prove that for any $B \subset F$, $f(f^{-1}(B)) \subset B$.

   If $y \in f(f^{-1}(B))$, then $\exists x \in f^{-1}(B)$ with $f(x) = y$. Since $x \in f^{-1}(B)$, we have $f(x) \in B$, so $y \in B$.

3. Prove that f is injective $\iff$ for any $A \subset E$, $A = f^{-1}(f(A))$.

   ($\Rightarrow$) If f is injective, we already have $A \subset f^{-1}(f(A))$. For the reverse: if $x \in f^{-1}(f(A))$, then $f(x) \in f(A)$, so $\exists a \in A$ with $f(a) = f(x)$. Since f is injective, $x = a \in A$.

   ($\Leftarrow$) Suppose $A = f^{-1}(f(A))$ for all $A \subset E$. Let $x_1, x_2 \in E$ with $f(x_1) = f(x_2)$. Take $A = \{x_1\}$. Then $f^{-1}(f(A)) = f^{-1}(\{f(x_1)\})$. Since $f(x_2) = f(x_1)$, we have $x_2 \in f^{-1}(f(A)) = A = \{x_1\}$, so $x_2 = x_1$. Thus f is injective.

4. Prove that f is surjective $\iff$ for any $B \subset F$, $f(f^{-1}(B)) = B$.

   ($\Rightarrow$) If f is surjective, we already have $f(f^{-1}(B)) \subset B$. For the reverse: if $y \in B$, then since f is surjective, $\exists x \in E$ with $f(x) = y$. Then $x \in f^{-1}(B)$ and $y = f(x) \in f(f^{-1}(B))$.

   ($\Leftarrow$) Suppose $f(f^{-1}(B)) = B$ for all $B \subset F$. Take $B = F$. Then $f(f^{-1}(F)) = F$. But $f^{-1}(F) = E$, so $f(E) = F$, meaning f is surjective.

□

Exercise 13: 1) Let $q_1 \in \mathbb{N}_{-\{0,1\}}$ and $q_2 \in \mathbb{N}_{-\{0,1\}}$. Prove that:

$$-\frac{1}{2} < \frac{1}{q_1} - \frac{1}{q_2} < \frac{1}{2}$$

2) Let $f : \mathbb{Z} \times \mathbb{N}_{-\{0,1\}} \longrightarrow \mathbb{Q}$ the map defined by: $f(p, q) = p + \frac{1}{q}$.
a) Prove that f is injective.
b) Is f surjective?

Solution.

1. Let $q_1, q_2 \in \mathbb{N} \setminus \{0, 1\}$. Prove that:

$$-\frac{1}{2} < \frac{1}{q_1} - \frac{1}{q_2} < \frac{1}{2}$$

   Since $q_1, q_2 \geq 2$, we have $0 < \frac{1}{q_1}, \frac{1}{q_2} \leq \frac{1}{2}$. The maximum of $\frac{1}{q_1} - \frac{1}{q_2}$ occurs when $q_1 = 2$ and $q_2 \to \infty$, giving $\frac{1}{2} - 0 = \frac{1}{2}$. The minimum occurs when $q_1 \to \infty$ and $q_2 = 2$, giving $0 - \frac{1}{2} = -\frac{1}{2}$. Since q_1, q_2 are integers ≥ 2, the difference cannot actually equal $\pm\frac{1}{2}$.

2. Let $f : \mathbb{Z} \times (\mathbb{N} \setminus \{0, 1\}) \to \mathbb{Q}$ defined by $f(p, q) = p + \frac{1}{q}$.

   (a) Prove that f is injective.

   Suppose $f(p_1, q_1) = f(p_2, q_2)$, i.e., $p_1 + \frac{1}{q_1} = p_2 + \frac{1}{q_2}$. Then $p_1 - p_2 = \frac{1}{q_2} - \frac{1}{q_1}$. The left side is an integer, and from part (1), the right side satisfies $-\frac{1}{2} < \frac{1}{q_2} - \frac{1}{q_1} < \frac{1}{2}$. The only integer in $(-\frac{1}{2}, \frac{1}{2})$ is 0, so $p_1 = p_2$ and $\frac{1}{q_1} = \frac{1}{q_2}$, hence $q_1 = q_2$.

   (b) Is f surjective?

   No, f is not surjective. For example, $\frac{1}{2}$ is not in the image because: If $p + \frac{1}{q} = \frac{1}{2}$, then $p = \frac{1}{2} - \frac{1}{q}$. For $q \geq 2$, $\frac{1}{2} - \frac{1}{q}$ is not an integer.

□