CHAPTER 1

LOGIC AND METHODS OF REASONING

This chapter provides a formal introduction to the principles of mathematical logic and reasoning. It begins by defining propositions and their truth values, then explores core logical connectors: conjunction, disjunction, implication, and equivalence. Key concepts include the contrapositive, converse, and negation of implications, governed by foundational algebraic properties and De Morgan's Laws. The scope of propositions is extended through universal and existential quantifiers ($\forall$, $\exists$), alongside the rules for their negation. Finally, the chapter details essential proof techniques, including direct proof, proof by contraposition, proof by contradiction, and mathematical induction, establishing the rigorous framework necessary for mathematical argumentation.

1.1 Rules of Formal Logic

Definition 1.1 *A proposition is an expression to which the truth value true or false can be assigned.*

Example 1 *(1) ≪ Every prime number is even ≫, This proposition is false.*

(2) $\sqrt{2}$ is an irrational number, this proposition is true.
(3) 2 is less than 4, this proposition is true.

Definition 1.2 *Any proposition demonstrated to be true is called a theorem (for example the theorem of Pythagoras, Thales...)*

1.2 Negation: $notP, \bar{P}$ or $\neg P$:

Definition 1.3 *Let P be a proposition, the negation of P is a proposition designating the opposite which we note (not P) or $\overline{P}$ or $\neg P$. Here is its truth table, we denote by 1 if the proposition is true and 0 if it is false.*

P	$\bar{P}$
1	0
0	1

Example 2 *(1) Let $E \neq \emptyset, P : (a \in E)$, then $\bar{P} : (a \notin E)$.*

(2) P : The function f is positive, then $\bar{P}$: The function f is not positive.
(3) $P : x + 2 = 0$, then (not P) : $x + 2 \neq 0$.

1.3 Logical Connectors.

Let P, Q be two propositions

1. The conjunction ≪**and**≫, $\ll \wedge \gg$

Definition 1.4 *the conjunction is the logical connective ≪ and ≫, $\ll \wedge \gg$, the proposition (P and Q) or $(P \wedge Q)$ is the conjunction of the two propositions P, Q.*

- $(P \wedge Q)$ is true if both P and Q are true.
- $(P \wedge Q)$ is false in other cases. We summarize all this in the following truth table.

P	Q	$P \wedge Q$
1	1	1
1	0	0
0	1	0
0	0	0

Example 3 *(1) 2 is an even number and 3 is a prime number, this proposition is true*

(2) $3 \leq 2$ and $4 \geq 2$, This proposition is false.
2. The disjunction $\ll$ **or** $\gg, \ll \vee \gg$

Definition 1.5 *Disjunction is a logical connective $\ll$ or $\gg, \ll \vee \gg$ we note the disjunction between P, Q by $(P$ or $Q),(P \vee Q)$. $P \vee Q$ is false if P and Q are both false, otherwise $(P \vee Q)$ is true.*

We summarize all of this in the following truth table.

P	Q	$P \vee Q$
1	1	1
1	0	1
0	1	1
0	0	0

Example 4 *(1) 2 is an even number or 3 is a prime number. True.*

(2) $3 \leq 2$ or $2 \geq 4$. False

1.4 Implication

Definition 1.6 *The implication of two propositions P, Q is noted $P \Rightarrow Q$ we say P implies Q or if P then Q. $P \Rightarrow Q$ is false if P is true and Q is false, otherwise $(P \Rightarrow Q)$ is true in the other cases.*

P	Q	$P \Rightarrow Q$
1	*1*	*1*
1	*0*	*0*
0	*1*	*1*
0	*0*	*1*

Example 5 *(1)* $0 \leq x \leq 9 \Rightarrow \sqrt{x} \leq 3$. *True.*

(2) It's raining, so I take my umbrella. It's true, it's a consequence.
(3) Omar won the lottery $\Rightarrow$ Omar played the lottery. True, it's a consequence.

1.5 Converse of an Implication

Definition 1.7 *The converse of the implication $(P \Rightarrow Q)$ is the implication $Q \Rightarrow P$.*

Example 6 *1) The converse of:* $0 \le x \le 9 \Rightarrow \sqrt{x} \le 3$*, is:* $\sqrt{x} \le 3 \Rightarrow 0 \le x \le 9$.

2) The converse of: (It's raining, so I take my umbrella), is: (I take my umbrella, so it's raining).

3) The converse of: (Omar won the lottery $\Rightarrow$ *Omar played the lottery), is: (Omar played the lottery* $\Rightarrow$ *Omar won the lottery).*

4) The contrapositive of implication. Let P, Q *be two propositions, the contrapositive of* $(P \Rightarrow Q)$ *is* $(\bar{Q} \Rightarrow \bar{P})$*, we have*

$$(P \Rightarrow Q) \Longleftrightarrow (\bar{Q} \Rightarrow \bar{P})$$

Remark 1 $(P \Rightarrow Q)$ *and* $(\bar{Q} \Rightarrow \bar{P})$ *have the same truth table, i.e., the same truth value.*

Example 7 *(1) The contrapositive of: (It's raining, so I take my umbrella) is: (I don't take my umbrella, so it doesn't rain).*

(2) The contrapositive of: (Omar won the lottery $\Rightarrow$ *Omar played the lottery) is: (Omar didn't play the lottery* $\Rightarrow$ *Omar didn't win the lottery).*

1.6 Equivalence

Definition 1.8 *The equivalence of two propositions* P, Q *is noted* $P \Leftrightarrow Q$*, we can also write* $(P \Rightarrow Q)$ *and* $(Q \Rightarrow P)$*. We say that* $P \Leftrightarrow Q$ *if* P *and* Q *have the same truth value, otherwise* $(P \Leftrightarrow Q)$ *is false.*

P	Q	$P \Leftrightarrow Q$
1	1	1
1	0	0
0	1	0
0	0	1

Remark 2 *(1)* $P \not\Leftrightarrow Q$ *that is to say* P *is not equivalent to* Q *when* $P \not\Rightarrow Q$ *or* $Q \not\Leftarrow P$.
(2) $P \Leftrightarrow Q$ *can be read* P *if and only if* Q.

Example 8 *(1)* $x + 2 = 0 \Leftrightarrow x = -2$.
(2) Omar won the lottery $\Leftrightarrow$ *Amar played the lottery.*

1.7 Negation of an Implication

Let P, Q be two propositions we have

$$\overline{(P \Rightarrow Q)} \Leftrightarrow (P \wedge \bar{Q}).$$

Example 9 *(1) The negation of: (it's raining, so I take my umbrella) is: (it's raining and I don't take my umbrella).*

(2) The negation of: (Omar won the lottery $\Rightarrow$ Omar played the lottery) is: (Omar won the lottery and Omar did not play the lottery).

(3) $(x \in [0,1] \Rightarrow x \geq 0)$ its negation is : $(x \in [0,1] \wedge x < 0)$.

1.8 Conclusion

(1) The negation of $(P \Rightarrow Q)$ is $(P \wedge \bar{Q})$.

(2) The contraposition of $(P \Rightarrow Q)$ is $(\bar{Q} \Rightarrow \bar{P})$.

(3) The converse of $(P \Rightarrow Q)$ is $(Q \Rightarrow P)$.

Proposition 1.1 *We have:* $(P \Rightarrow Q) \Leftrightarrow (\bar{P} \vee Q)$.

Proof. It suffices to show that $(P \Rightarrow Q)$ has the same truth value as $(\bar{P} \vee Q)$, we can see this clearly in the following truth table:

P	Q	$\bar{P}$	$P \Rightarrow Q$	$\bar{P} \vee Q$
1	1	0	1	1
1	0	0	0	0
0	1	1	1	1
0	0	1	1	1

□

Theorem 1 *Let P, Q be two propositions, we have:*

$$(P \Leftrightarrow Q) \Leftrightarrow (P \Rightarrow Q) \wedge (Q \Rightarrow P).$$

Proof. We have

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$	$(P \Rightarrow Q) \wedge (Q \Rightarrow P)$	$(P \Leftrightarrow Q)$
1	1	1	1	1	1
1	0	0	1	0	0
0	1	1	0	0	0
0	0	1	1	1	1

□

1.9 Properties of Logical Connectors

Proposition 1.2 *Whatever the truth value of the propositions P, Q, R the following properties are always true.*

(1) $\bar{P} \vee P$ *(Tautology)*
(2) $\overline{\overline{P}} \Leftrightarrow P$ *(Double Negation)*
(3) $P \wedge P \Leftrightarrow P$ *(Idempotence of $\wedge$)* (4) $P \wedge Q \Leftrightarrow Q \wedge P$. *Commutativity of $\wedge$*
(5) $P \vee Q \Leftrightarrow Q \vee P$. *Commutativity of $\vee$*
(6) $((P \wedge Q) \wedge R) \Leftrightarrow (P \wedge (Q \wedge R))$ *Associativity of $\wedge$*
(7) $((P \vee Q) \vee R) \Leftrightarrow (P \vee (Q \vee R))$ *Associativity of $\vee$*
(8) $P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$ *Distributivity of $\wedge$ over $\vee$*
(9) $P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$ *Distributivity of $\vee$ over $\wedge$*
(10) $\overline{P \wedge Q} \Leftrightarrow (\bar{P} \vee \bar{Q})$ *(De Morgan's Law)*
(11) $\overline{P \vee Q} \Leftrightarrow (\bar{P} \wedge \bar{Q})$ *(De Morgan's Law)*
(12) $(P \Rightarrow Q) \Leftrightarrow (\bar{P} \vee Q)$
(13) $\overline{P \Rightarrow Q} \Leftrightarrow (P \wedge \bar{Q})$
(14) $(P \Leftrightarrow Q) \Leftrightarrow (P \Rightarrow Q) \wedge (Q \Rightarrow P)$
(15) $\overline{P \Leftrightarrow Q} \Leftrightarrow (P \Leftrightarrow \bar{Q}) \Leftrightarrow (\bar{P} \Leftrightarrow Q)$
(16) $(P \Rightarrow Q) \Leftrightarrow (\bar{Q} \Rightarrow \bar{P})$ *(Contraposition)*
(17) $(P \Rightarrow Q) \Leftrightarrow (P \wedge \bar{Q} \Rightarrow \bar{P})$
(18) $(P \Rightarrow Q) \Leftrightarrow (P \wedge \bar{Q} \Rightarrow Q)$
(19) $(P \Rightarrow Q) \Leftrightarrow (P \wedge \bar{Q} \Rightarrow R \wedge \bar{R})$
(20) $(P \Rightarrow Q \vee R) \Leftrightarrow (P \wedge \bar{Q} \Rightarrow R)$
(21) $(P \wedge Q \Rightarrow R) \Leftrightarrow (P \Rightarrow (Q \Rightarrow R))$
(22) $(P \Rightarrow Q) \wedge (Q \Rightarrow R) \Rightarrow (P \Rightarrow R)$ *(Transitivity of implication)*
(23) $(P \Leftrightarrow Q) \wedge (Q \Leftrightarrow R) \Rightarrow (P \Leftrightarrow R)$ *(Transitivity of equivalence)*
(24) $(P \Rightarrow Q) \wedge (R \Rightarrow S) \Rightarrow (P \wedge R \Rightarrow Q \wedge S)$
(25) $(P \Rightarrow Q) \wedge (R \Rightarrow S) \Rightarrow (P \vee R \Rightarrow Q \vee S)$

1.10 Quantifiers

1.10.1 Universal quantifier $\forall$

Definition 1.9 *Let $P(x)$ be a proposition that depends on x and E a set, we have:*

$$(\forall x \in E, P(x)) \Leftrightarrow (x \in E \Rightarrow P(x)).$$

Example 10 *(1)* $\forall x \in \mathbb{R}, x^2 \geq 0$ *(True)*

(2) $\forall x \in \mathbb{R}, x^2 > 0$ (False, because $0^2 = 0$)

1.10.2 Existential Quantifier $\exists$

Definition 1.10 *Let $P(x)$ be a proposition that depends on x and E a set, we have:*

$$(\exists x \in E, P(x)) \Leftrightarrow (x \in E \wedge P(x)).$$

Example 11 *(1) $\exists x \in \mathbb{R}, x^2 = 2$ (True, $x = \sqrt{2}$)*

(2) $\exists x \in \mathbb{R}, x^2 = -1$ (False)

1.10.3 Uniqueness Quantifier $\exists!$

Definition 1.11 *Let $P(x)$ be a proposition that depends on x and E a set, we have:*

$$(\exists! x \in E, P(x)) \Leftrightarrow \begin{cases} \exists x \in E, P(x) \\ \forall x, y \in E, (P(x) \wedge P(y) \Rightarrow x = y) \end{cases}$$

Example 12 *(1) $\exists! x \in \mathbb{R}, x^2 = 0$ (True, $x = 0$)*

(2) $\exists! x \in \mathbb{R}, x^2 = 1$ (False, $x = 1$ and $x = -1$)

1.10.4 Negation of Quantifiers:

We have:
(1) $\overline{(\forall x \in E, P(x))} \Leftrightarrow (\exists x \in E, \bar{P}(x))$
(2) $\overline{(\exists x \in E, P(x))} \Leftrightarrow (\forall x \in E, \bar{P}(x))$
(3) $\overline{(\exists! x \in E, P(x))} \Leftrightarrow [(\forall x \in E, \bar{P}(x)) \vee (\exists x, y \in E, x \neq y, P(x) \wedge P(y))]$

Example 13 *(1) $\overline{(\forall x \in \mathbb{R}, x^2 \geq 0)} \Leftrightarrow (\exists x \in \mathbb{R}, x^2 < 0)$*

(2) $\overline{(\exists x \in \mathbb{R}, x^2 = 2)} \Leftrightarrow (\forall x \in \mathbb{R}, x^2 \neq 2)$
(3) $\overline{(\exists! x \in \mathbb{R}, x^2 = 0)} \Leftrightarrow (\forall x \in \mathbb{R}, x^2 \neq 0) \vee (\exists x, y \in \mathbb{R}, x \neq y, x^2 = 0, y^2 = 0)$

1.11 Methods of Reasoning

1.11.1 Direct Reasoning

Theorem 2 *To prove that $P \Rightarrow Q$ is true, we assume that P is true and we show that Q is true.*

Example 14 *Show that: $\forall n \in \mathbb{N}, n$ even $\Rightarrow n^2$ even.*

Proof. Let $n \in \mathbb{N}$ even, so $\exists k \in \mathbb{N}, n = 2k$, then $n^2 = 4k^2 = 2(2k^2)$ so n^2 is even. □

1.11.2 Reasoning by Contraposition

Theorem 3 *To prove that $P \Rightarrow Q$ is true, we show that $\bar{Q} \Rightarrow \bar{P}$ is true.*

Example 15 *Show that: $\forall n \in \mathbb{N}, n^2$ even $\Rightarrow n$ even.*

Proof. We show by contraposition: n odd $\Rightarrow n^2$ odd.

Let $n \in \mathbb{N}$ odd, so $\exists k \in \mathbb{N}, n = 2k+1$, then $n^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$ so n^2 is odd. □

1.11.3 Reasoning by Contradiction (Absurdity)

Theorem 4 *To prove that P is true, we assume that $\bar{P}$ is true and we show that this leads to a contradiction (absurdity).*

Example 16 *Show that: $\sqrt{2}$ is irrational.*

Proof. Assume that $\sqrt{2}$ is rational, so $\exists p, q \in \mathbb{N}^*$, with $p \wedge q = 1$ such that $\sqrt{2} = \frac{p}{q}$, then $p^2 = 2q^2$, so p^2 is even, so p is even, so $\exists k \in \mathbb{N}^*, p = 2k$, then $4k^2 = 2q^2 \Rightarrow q^2 = 2k^2$, so q^2 is even, so q is even, so $p \wedge q \neq 1$, contradiction. □

1.11.4 Reasoning by Recurrence

Theorem 5 *Let $P(n)$ be a proposition that depends on $n \in \mathbb{N}$.*

To prove that $\forall n \in \mathbb{N}, P(n)$ is true, it suffices to show:

1) Initialization: $P(0)$ is true.

2) Heredity: $\forall n \in \mathbb{N}, P(n) \Rightarrow P(n+1)$ is true.

Example 17 *Show that: $\forall n \in \mathbb{N}, \sum_{k=0}^{n} k = \frac{n(n+1)}{2}$.*

Proof. 1) For $n = 0, \sum_{k=0}^{0} k = 0$ and $\frac{0(0+1)}{2} = 0$, so $P(0)$ is true.

2) Let $n \in \mathbb{N}$, assume that $P(n)$ is true, i.e., $\sum_{k=0}^{n} k = \frac{n(n+1)}{2}$, let's show that $P(n+1)$ is true.

We have

$$\sum_{k=0}^{n+1} k = \sum_{k=0}^{n} k + (n+1) = \frac{n(n+1)}{2} + (n+1) = \frac{n(n+1) + 2(n+1)}{2} = \frac{(n+1)(n+2)}{2},$$

so $P(n+1)$ is true.

By recurrence, $\forall n \in \mathbb{N}, P(n)$ is true. □

1.12 Exercises

Exercise 1 ***Truth Tables***

Let P, Q and L be three logical propositions. Construct the truth tables of the following formulas:

$$(P \Longrightarrow Q) \Longrightarrow L, \quad (P \vee Q) \Longrightarrow (L \vee Q), \quad ((P \vee Q) \wedge L) \Longrightarrow ((\neg P \wedge Q) \vee (Q \wedge L))$$

Solution. We will use 1 for True and 0 for False.

1. **Formula:** $(P \Longrightarrow Q) \Longrightarrow L$

P	Q	L	$P \Rightarrow Q$	$(P \Rightarrow Q) \Rightarrow L$
1	1	1	1	1
1	1	0	1	0
1	0	1	0	1
1	0	0	0	1
0	1	1	1	1
0	1	0	1	0
0	0	1	1	1
0	0	0	1	0

2. **Formula:** $(P \vee Q) \Longrightarrow (L \vee Q)$

P	Q	L	$P \vee Q$	$L \vee Q$	$(P \vee Q) \Rightarrow (L \vee Q)$
1	1	1	1	1	1
1	1	0	1	1	1
1	0	1	1	1	1
1	0	0	1	0	0
0	1	1	1	1	1
0	1	0	1	1	1
0	0	1	0	1	1
0	0	0	0	0	1

3. **Formula:** $((P \vee Q) \wedge L) \Longrightarrow ((\neg P \wedge Q) \vee (Q \wedge L))$

P	Q	L	$P \vee Q$	$(P \vee Q) \wedge L$	$\neg P$	$\neg P \wedge Q$	$Q \wedge L$	$(\neg P \wedge Q) \vee (Q \wedge L)$	Formula $\Rightarrow$
1	1	1	1	1	0	0	1	1	1
1	1	0	1	0	0	0	0	0	1
1	0	1	1	1	0	0	0	0	0
1	0	0	1	0	0	0	0	0	1
0	1	1	1	1	1	1	1	1	1
0	1	0	1	0	1	1	0	1	1
0	0	1	0	0	1	0	0	0	1
0	0	0	0	0	1	0	0	0	1

□

Exercise 2 ***Implications and Negation***

Let P and Q be two logical propositions. 1) Is the proposition $(P \wedge Q) \Longrightarrow (\neg P \vee Q)$ true? 2) Give the negation of $P \Longrightarrow Q$ and the negation of $(P \Longrightarrow Q) \Longrightarrow Q$.

Solution. 1) The proposition $(P \wedge Q) \Longrightarrow (\neg P \vee Q)$ is a tautology (always true). Justification: $\neg P \vee Q$ is logically equivalent to $P \Rightarrow Q$. The proposition becomes $(P \wedge Q) \Rightarrow (P \Rightarrow Q)$. If $P \wedge Q$ is true, then P and Q are true. If Q is true, then $P \Rightarrow Q$ is true. The implication is therefore always true.

2) Negations:

- Negation of $P \Longrightarrow Q$: $\neg(P \Rightarrow Q) \equiv P \wedge \neg Q$
- Negation of $(P \Longrightarrow Q) \Longrightarrow Q$:

$$\begin{aligned}\neg((P \Rightarrow Q) \Rightarrow Q) &\equiv (P \Rightarrow Q) \wedge \neg Q \\ &\equiv (\neg P \vee Q) \wedge \neg Q \\ &\equiv (\neg P \wedge \neg Q) \vee (Q \wedge \neg Q) \\ &\equiv (\neg P \wedge \neg Q) \vee \text{False} \\ &\equiv \neg P \wedge \neg Q\end{aligned}$$

□

Exercise 3 ***Quantifiers***

Let f and g be two functions from R to R. Translate the following expressions into terms of quantifiers: 1) f is increasing, bounded, even, odd.
2) f is never zero.
3) f is periodic.
4) f is increasing, decreasing.
5) f is not the zero function.
6) f never takes the same value at two distinct points.
7) f reaches all values in N.
8) f is less than g, f is not less than g.

Solution.

1.
   - Increasing: $\forall x, y \in R, (x < y \Longrightarrow f(x) < f(y))$
   - Bounded: $\exists M > 0, \forall x \in R, |f(x)| \leq M$
   - Even: $\forall x \in R, f(-x) = f(x)$

- Odd: $\forall x \in R, f(-x) = -f(x)$

2. $\forall x \in R, f(x) \neq 0$ or $\nexists x \in R, f(x) = 0$

3. $\exists T > 0, \forall x \in R, f(x+T) = f(x)$

4. - Increasing: $\forall x, y \in R, (x < y \Longrightarrow f(x) \leq f(y))$
   - Decreasing: $\forall x, y \in R, (x < y \Longrightarrow f(x) \geq f(y))$

5. $\exists x \in R, f(x) \neq 0$

6. $\forall x, y \in R, (x \neq y \Longrightarrow f(x) \neq f(y))$ or $\forall x, y \in R, (f(x) = f(y) \Longrightarrow x = y)$

7. $\forall n \in N, \exists x \in R, f(x) = n$

8. - f is less than g: $\forall x \in R, f(x) < g(x)$
   - f is not less than g: $\exists x \in R, f(x) \geq g(x)$

□

Exercise 4 *Truth Value and Negation*

Consider the following assertions:

1. $\exists x \in R, \forall y \in R, x + y > 0$

2. $\forall x \in R, \exists y \in R, x + y > 0$

3. $\forall x \in R, \forall y \in R, x + y > 0$

4. $\exists x \in R, \forall y \in R, y^2 > x$

5. $\forall \epsilon \in R^{+*}, \exists \alpha \in R^{+*}, |x| < \alpha \Longrightarrow |x^2| < \epsilon$

Are these assertions true or false? Give their negations.

Solution.

1. False. For a fixed x, choose $y = -x - 1$. Then $x + y = -1 > 0$ is false.
   Negation: $\forall x \in R, \exists y \in R, x + y \leq 0$

2. True. For any x, choose $y = |x| + 1$. Then $x + y = x + |x| + 1 \geq 1 > 0$.
   Negation: $\exists x \in R, \forall y \in R, x + y \leq 0$

3. False. Counterexample: $x = -1, y = -1 \Rightarrow x + y = -2 > 0$ is false.
   Negation: $\exists x \in R, \exists y \in R, x + y \leq 0$

4. True. Choose $x = -1$. For all $y \in R$, $y^2 \geq 0 > -1$.
   Negation: $\forall x \in R, \exists y \in R, y^2 \leq x$

5. True. This defines $\lim_{x\to 0} x^2 = 0$. For any $\epsilon > 0$, choose $\alpha = \sqrt{\epsilon}$. If $|x| < \sqrt{\epsilon}$, then $|x^2| < \epsilon$.
   Negation: $\exists \epsilon \in R^{+*}, \forall \alpha \in R^{+*}, (|x| < \alpha) \wedge (|x^2| \geq \epsilon)$

□

Exercise 5 ***Equivalence of Propositions***
Let P and Q be two polynomials. Are the following propositions equivalent?

1. $\forall x \in R, (P(x) = 0 \textit{ and } Q(x) = 0)$ *and* $[(\forall x \in R, P(x) = 0)$ *and* $(\forall x \in R, Q(x) = 0)]$

2. $\forall x \in R, (P(x) = 0 \textit{ or } Q(x) = 0)$ *and* $[(\forall x \in R, P(x) = 0)$ *or* $(\forall x \in R, Q(x) = 0)]$

Solution.

1. These statements are equivalent. The left part, $\forall x(P(x) = 0 \wedge Q(x) = 0)$, means both polynomials are zero at every point x, which is the definition of both being the zero polynomial: $(\forall x, P(x) = 0) \wedge (\forall x, Q(x) = 0)$.

2. These statements are not equivalent.

   - The left part, $\forall x(P(x) = 0 \vee Q(x) = 0)$, means every real number is a root of at least one polynomial (e.g., $P(x) = x$, $Q(x) = x - 1$).
   - The right part, $(\forall x, P(x) = 0) \vee (\forall x, Q(x) = 0)$, means at least one polynomial is identically zero.
   - The left can be true without the right being true (see example).

□

Exercise 6 ***Negation and the Empty Set***
Let A be a subset of R.
1) Let P be the proposition "For any real $x \in A, x^2 \geq 12$". Negate P.
2) Assume that $A = \emptyset$. Is the negation of P true or false? Is P true or false?

Solution. 1) Negation of P: $\exists x \in A, x^2 < 12$
2) If $A = \emptyset$:

- P: "For all x in the empty set, $x^2 \geq 12$". This is a vacuously true statement. There are no elements in A to violate the condition.

- Negation of P ($\exists x \in A, x^2 < 12$): This claims an element exists in the empty set. This is false.

□

Exercise 7 ***Proof Techniques***

1) Prove by contraposition that for any natural number n, if n^2 is even then n is even.
2) Let x be a positive or zero real. Prove that if for every positive real $y, x \leq y$, then $x = 0$.
3) Let $n \in N^*$. Prove by contradiction that $n^2 + 1$ is not a perfect square.

Solution. 1) Proof by contraposition: The contrapositive is: If n is not even (i.e., odd), then n^2 is not even (i.e., odd). Assume n is odd. Then $n = 2k + 1$ for some integer k. Then $n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$. Since $2k^2 + 2k$ is an integer, n^2 is odd. QED.

2) Proof by contraposition: We prove: if $x \neq 0$, then $\exists y > 0$ such that $x > y$ (i.e., the hypothesis is false). Assume $x \neq 0$. Since $x \geq 0$, we have $x > 0$. Choose $y = \frac{x}{2}$. Clearly $y > 0$. And $x > \frac{x}{2} = y$. We have found a $y > 0$ such that $x > y$, contradicting $\forall y > 0, x \leq y$. Therefore, the original implication is true.

3) Proof by contradiction: Assume, for contradiction, that $\exists n \in N^*$ such that $n^2 + 1$ is a perfect square, say m^2. So, $n^2 + 1 = m^2$, which implies $1 = m^2 - n^2 = (m-n)(m+n)$. Since $m, n \in N^*$ and $m^2 = n^2 + 1 > n^2$, we have $m > n$, so $(m-n)$ and $(m+n)$ are positive integers. The only factorizations of 1 are 1×1. Therefore: $m - n = 1$ and $m + n = 1$. Solving this system: Adding the equations gives $2m = 2 \Rightarrow m = 1$. Substituting back: $1 - n = 1 \Rightarrow n = 0$. But $n = 0 \notin N^*$. This is a contradiction. Therefore, our initial assumption was false, and $n^2 + 1$ is not a perfect square for $n \geq 1$. □

Exercise 8 ***Limit Proof***

Prove that

$$\forall \epsilon > 0, \exists N \in \mathbb{N} : (n \geq N) \Longrightarrow (2 - \epsilon < \frac{2n+1}{n+2} < 2 + \epsilon).$$

Solution. First, simplify the expression:

$$\frac{2n+1}{n+2} = \frac{2n+4-3}{n+2} = 2 - \frac{3}{n+2}$$

We want to find N such that for all $n \geq N$:

$$\left|\frac{2n+1}{n+2} - 2\right| < \epsilon \quad \Rightarrow \quad \left|-\frac{3}{n+2}\right| < \epsilon \quad \Rightarrow \quad \frac{3}{n+2} < \epsilon$$

Solving for n:

$$\frac{3}{n+2} < \epsilon \Longrightarrow n + 2 > \frac{3}{\epsilon} \Longrightarrow n > \frac{3}{\epsilon} - 2$$

Proof: Let $\epsilon > 0$ be given. Choose $N \in N$ such that $N > \frac{3}{\epsilon} - 2$. (Such an N exists by the Archimedean Property). Now, for all $n \geq N$, we have:

$$n \geq N > \frac{3}{\epsilon} - 2 \implies n + 2 > \frac{3}{\epsilon} \implies \frac{3}{n+2} < \epsilon$$

Since $\frac{3}{n+2} > 0$, this means $\left|\frac{2n+1}{n+2} - 2\right| = \frac{3}{n+2} < \epsilon$. This is equivalent to $2 - \epsilon < \frac{2n+1}{n+2} < 2 + \epsilon$. This completes the proof. □

Exercise 9 ***Mathematical Induction***

For $n \in N$, define two properties: $P_n : 3 \mid (4^n - 1)$ and $Q_n : 3 \mid (4^n + 1)$. 1) Prove that for any $n \in N, P_n \implies P_{n+1}$ and $Q_n \implies Q_{n+1}$. 2) Prove that P_n is true for any $n \in N$. 3) What can be concluded about the assertion: $\exists n_0 \in N, \forall n \in N, n_0 \geq n \implies Q_n$?

Solution. 1) Inductive Steps:

- For $P_n \implies P_{n+1}$: Assume P_n is true: $4^n - 1 = 3k$.

$$\begin{aligned} 4^{n+1} - 1 &= 4 \cdot 4^n - 1 \\ &= 4(3k + 1) - 1 \\ &= 12k + 4 - 1 \\ &= 12k + 3 \\ &= 3(4k + 1) \end{aligned}$$

  So P_{n+1} is true.

- For $Q_n \implies Q_{n+1}$: Assume Q_n is true: $4^n + 1 = 3m$.

$$\begin{aligned} 4^{n+1} + 1 &= 4 \cdot 4^n + 1 \\ &= 4(3m - 1) + 1 \\ &= 12m - 4 + 1 \\ &= 12m - 3 \\ &= 3(4m - 1) \end{aligned}$$

  So Q_{n+1} is true.

2) Proof that P_n is true for all $n \in N$:

- Base Case (n=0): $4^0 - 1 = 1 - 1 = 0$. Since $3 \mid 0$, P_0 is true.
- Inductive Step: Shown in part (1).

- By mathematical induction, P_n is true for all $n \in N$.

3) The assertion $\exists n_0 \in N, \forall n \in N, n_0 \geq n \Longrightarrow Q_n$ is false.

- Check small values: $Q_0 : 4^0 + 1 = 2$ (not divisible by 3), $Q_1 : 4 + 1 = 5$ (not divisible by 3), $Q_2 : 16 + 1 = 17$ (not divisible by 3).
- The base case Q_0 is false. The implication $Q_n \Longrightarrow Q_{n+1}$ is valid, but it only propagates truth. Since the foundation (Q_0) is false, Q_n is false for all $n \in N$.
- Therefore, there does not exist any n_0 for which Q_n is true for all $n \leq n_0$.

□

Exercise 10 *Let P, Q, R be three propositions. Simplify the following propositions: 1)* $(P \wedge Q) \vee (P \wedge \bar{Q})$
2) $(P \wedge Q) \vee (P \wedge \bar{Q} \wedge R)$
3) $(P \Rightarrow Q) \wedge (P \Rightarrow \bar{Q})$
4) $(P \Rightarrow Q) \wedge (\bar{P} \Rightarrow Q)$
5) $(P \Rightarrow Q) \vee (P \Rightarrow \bar{Q})$
6) $(P \Rightarrow Q) \vee (\bar{P} \Rightarrow Q)$

Solution.
1) $(P \wedge Q) \vee (P \wedge \bar{Q}) = P \wedge (Q \vee \bar{Q}) = P \wedge 1 = P$
2) $(P \wedge Q) \vee (P \wedge \bar{Q} \wedge R) \Longleftrightarrow P \wedge (Q \vee (\bar{Q} \wedge R)) \Longleftrightarrow P \wedge ((Q \vee \bar{Q}) \wedge (Q \vee R))$
$\Longleftrightarrow P \wedge (1 \wedge (Q \vee R)) \Longleftrightarrow P \wedge (Q \vee R)$
3) $(P \Rightarrow Q) \wedge (P \Rightarrow \bar{Q}) \Longleftrightarrow (P \Rightarrow Q \wedge \bar{Q}) \Longleftrightarrow (P \Rightarrow 0) = \bar{P}$
4) $(P \Rightarrow Q) \wedge (\bar{P} \Rightarrow Q) \Longleftrightarrow (\bar{P} \vee Q) \wedge (P \vee Q) \Longleftrightarrow (\bar{P} \wedge P) \vee Q \Longleftrightarrow 0 \vee Q = Q$
5) $(P \Rightarrow Q) \vee (P \Rightarrow \bar{Q}) \Longleftrightarrow (\bar{P} \vee Q) \vee (\bar{P} \vee \bar{Q}) \Longleftrightarrow \bar{P} \vee (Q \vee \bar{Q}) \Longleftrightarrow \bar{P} \vee 1 = 1$
6) $(P \Rightarrow Q) \vee (\bar{P} \Rightarrow Q) \Longleftrightarrow (\bar{P} \vee Q) \vee (P \vee Q) \Longleftrightarrow (\bar{P} \vee P) \vee (Q \vee Q) \Longleftrightarrow 1 \vee Q \Longleftrightarrow 1$ □

Exercise 11 *Let P, Q be two propositions. Show that:*
1) $(P \Rightarrow Q) \Leftrightarrow (P \wedge \bar{Q} \Rightarrow \bar{P})$
2) $(P \Rightarrow Q) \Leftrightarrow (P \wedge \bar{Q} \Rightarrow Q)$
3) $(P \Rightarrow Q) \Leftrightarrow (P \wedge \bar{Q} \Rightarrow R \wedge \bar{R})$
4) $(P \Rightarrow Q \vee R) \Leftrightarrow (P \wedge \bar{Q} \Rightarrow R)$
5) $(P \wedge Q \Rightarrow R) \Leftrightarrow (P \Rightarrow (Q \Rightarrow R))$

Solution.
1) $(P \wedge \bar{Q} \Rightarrow \bar{P}) \Longleftrightarrow \overline{(P \wedge \bar{Q})} \vee \bar{P} \Longleftrightarrow (\bar{P} \vee Q) \vee \bar{P} \Longleftrightarrow \bar{P} \vee Q \Longleftrightarrow P \Rightarrow Q$
2) $(P \wedge \bar{Q} \Rightarrow Q) \Longleftrightarrow \overline{(P \wedge \bar{Q})} \vee Q \Longleftrightarrow (\bar{P} \vee Q) \vee Q \Longleftrightarrow \bar{P} \vee Q \Longleftrightarrow P \Rightarrow Q$
3) $(P \wedge \bar{Q} \Rightarrow R \wedge \bar{R}) \Longleftrightarrow \overline{(P \wedge \bar{Q})} \vee (R \wedge \bar{R}) \Longleftrightarrow (\bar{P} \vee Q) \vee 0 \Longleftrightarrow \bar{P} \vee Q \Longleftrightarrow P \Rightarrow Q$
4) $(P \wedge \bar{Q} \Rightarrow R) \Longleftrightarrow \overline{(P \wedge \bar{Q})} \vee R \Longleftrightarrow (\bar{P} \vee Q) \vee R \Longleftrightarrow \bar{P} \vee (Q \vee R) \Longleftrightarrow P \Rightarrow (Q \vee R)$
5) $(P \Rightarrow (Q \Rightarrow R)) \Longleftrightarrow \bar{P} \vee (\bar{Q} \vee R) \Longleftrightarrow (\bar{P} \vee \bar{Q}) \vee R \Longleftrightarrow \overline{(P \wedge Q)} \vee R \Longleftrightarrow (P \wedge Q) \Rightarrow R$ □

Exercise 12 *Write the negation of the following propositions:*

1) $\forall x \in R, \exists y \in R, x + y = 0$

2) $\exists M \in R, \forall x \in R, f(x) \leq M$

3) $\forall \varepsilon > 0, \exists \eta > 0, \forall x \in R, |x - a| < \eta \Rightarrow |f(x) - f(a)| < \varepsilon$

4) $\forall x \in R, \forall y \in R, (x \leq y \Rightarrow f(x) \leq f(y))$

5) $\forall x \in R, \forall y \in R, x \neq y \Rightarrow f(x) \neq f(y)$

Solution.

1) $\exists x \in R, \forall y \in R, x + y \neq 0$

2) $\forall M \in R, \exists x \in R, f(x) > M$

3) $\exists \varepsilon > 0, \forall \eta > 0, \exists x \in R, |x - a| < \eta \wedge |f(x) - f(a)| \geq \varepsilon$

4) $\exists x \in R, \exists y \in R, (x \leq y \wedge f(x) > f(y))$

5) $\exists x \in R, \exists y \in R, x \neq y \wedge f(x) = f(y)$ □

Exercise 13 *Show by recurrence that:*

1) $\forall n \in N^*, \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

2) $\forall n \in N^*, \sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$

3) $\forall n \in N, 3^{2n+1} + 2^{4n+2}$ *is divisible by 7*

4) $\forall n \in N, 10^n + 3 \times 4^{n+2} + 5$ *is divisible by 9*

Solution.

1) For $n = 1, \sum_{k=1}^{1} k^2 = 1$ and $\frac{1(1+1)(2+1)}{6} = \frac{6}{6} = 1$, so $P(1)$ is true.

Let $n \in N^*$, assume that $P(n)$ is true, i.e., $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$, let's show that $P(n+1)$ is true.

We have

$$\begin{aligned}
\sum_{k=1}^{n+1} k^2 &= \sum_{k=1}^{n} k^2 + (n+1)^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \\
&= \frac{n(n+1)(2n+1) + 6(n+1)^2}{6} = \frac{(n+1)(n(2n+1) + 6(n+1))}{6} \\
&= \frac{(n+1)(2n^2 + 7n + 6)}{6} = \frac{(n+1)(n+2)(2n+3)}{6},
\end{aligned}$$

so $P(n+1)$ is true.

By recurrence, $\forall n \in N^*, P(n)$ is true.

2) For $n = 1, \sum_{k=1}^{1} k^3 = 1$ and $\left(\frac{1(1+1)}{2}\right)^2 = 1$, so $P(1)$ is true.

Let $n \in N^*$, assume that $P(n)$ is true, i.e., $\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$, let's show that $P(n+1)$ is true.

We have

$$\begin{aligned}
\sum_{k=1}^{n+1} k^3 &= \sum_{k=1}^{n} k^3 + (n+1)^3 = \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3 \\
&= \frac{n^2(n+1)^2 + 4(n+1)^3}{4} = \frac{(n+1)^2(n^2+4n+4)}{4} \\
&= \frac{(n+1)^2(n+2)^2}{4} = \left(\frac{(n+1)(n+2)}{2}\right)^2,
\end{aligned}$$

so $P(n+1)$ is true.

By recurrence, $\forall n \in N^*, P(n)$ is true.

3) For $n = 0, 3^1 + 2^2 = 3 + 4 = 7$ divisible by 7, so $P(0)$ is true.

Let $n \in N$, assume that $P(n)$ is true, i.e., $7 \mid 3^{2n+1} + 2^{4n+2}$, let's show that $P(n+1)$ is true.

We have

$$\begin{aligned}
3^{2(n+1)+1} + 2^{4(n+1)+2} &= 3^{2n+3} + 2^{4n+6} = 9 \times 3^{2n+1} + 16 \times 2^{4n+2} \\
&= 9(3^{2n+1} + 2^{4n+2}) + 7 \times 2^{4n+2},
\end{aligned}$$

so $7 \mid 3^{2n+3} + 2^{4n+6}$, so $P(n+1)$ is true.

By recurrence, $\forall n \in N, P(n)$ is true.

4) For $n = 0, 10^0 + 3 \times 4^2 + 5 = 1 + 48 + 5 = 54$ divisible by 9, so $P(0)$ is true.

Let $n \in N$, assume that $P(n)$ is true, i.e., $9 \mid 10^n + 3 \times 4^{n+2} + 5$, let's show that $P(n+1)$ is true.

We have 1

$$\begin{aligned}
0^{n+1} + 3 \times 4^{n+3} + 5 &= 10 \times 10^n + 12 \times 4^{n+2} + 5 \\
&= 10(10^n + 3 \times 4^{n+2} + 5) - 18 \times 4^{n+2} - 45 \\
&= 10(10^n + 3 \times 4^{n+2} + 5) - 9(2 \times 4^{n+2} + 5),
\end{aligned}$$

so $9 \mid 10^{n+1} + 3 \times 4^{n+3} + 5$, so $P(n+1)$ is true.

By recurrence, $\forall n \in N, P(n)$ is true. □