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Course
Mathématiques 2
ST-SM

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For

Level L_1

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Chapitre 1

Integrals

1.1 Indefinite integrals

1.1.1 Primitive of a continuous function

Example 1.1 *(Introductory example)*
Let us consider the funnctions f, F *defined on* $]0\ +\infty[$ *by* $F(x) = \ln(x) + \frac{1}{2}x^2$ *and* $f(x) = \frac{1}{x} + x$. *It is easily seen that the two functions are continuous. On the other hand, ,* F *is derivable on* $]0\ +\infty[$ *and* $\forall x \in]0\ +\infty[\,, F'(x) = f(x)$. *Such function* F *is called a primitive of* f *on* I.

Definition 1.1 *Let* $f, F : I \subset \mathbb{R} \longrightarrow \mathbb{R}$, *be two continuous functions on* I. *We say that* F *is a primitive of* f *if* F *is derivable on* I *and if* $F'(x) = f(x)$ *for all* $x \in I$.

Example 1.2 *The function* $F :\ x \longmapsto F(x) = \frac{1}{2}e^{2x+1}$ *is a primitive of the fonction* $f :$ $x \longmapsto f(x) = e^{2x+1}$ *since* $F'(x) = e^{2x+1}$.

Proposition 1.1 *Let* f *be a continuous function on* $I \subset \mathbb{R}$.

1. *If* F *is a primitive of* f *then* $F + k$ *is a primitive of* f, *where* k *is a constant fonction on* I.
2. *If* F, G *are two primitives of* f *then* $F - G$ *is constant. In other words, if* F *is a primitive of* f *then any other primitive* G *of* f *is of the forme* $G = F + k$ *where* k *is a constant.*
3. *If* F *is a primitive of* f *and* $x_0 \in I$, *then it exists a unique primitive* G *of* f *which satisfies* $G(x_0) = k$

Definition 1.2 *The set of all the primitives of a continuous function* f *on an interval* $I \subset \mathbb{R}$ *is called indefinite integral of* f *on* I. *We denote this set by* $\int f(x)\,dx$.

Remarks 1.1
1. *In general, if* $F'(x) = f(x)$ *then* $\int f(x)\,dx = F(x) + k,\ \ k \in \mathbb{R}$.
2. *The rôle of* dx *is to indicate that we look for primitives with respect the variable* x.

3. *The integration or the search for the primitives of f is, in fact, the inverse operation of differentiation.*

Example 1.3 *According to the previous examples, we have*

$$\int e^{2x+1}\,dx = \frac{1}{2}e^{2x+1} + k, \quad k \in \mathbb{R}.$$

$$\int \frac{1}{x}\,dx = \ln(|x|) + k, \qquad k \in \mathbb{R}.$$

1.1.2 Some immediate indefinite integrals

$\int 0\,dx = k, \quad k \in \mathbb{R}$	$\int \frac{1}{\cos^2(x)}\,dx = \tan(x) + k$
$\int a\,dx = ax + k, \quad k \in \mathbb{R}$	$\int \frac{1}{\sin^2(x)}\,dx = -\cot(x) + k$
$\int x^m\,dx = \frac{x^{m+1}}{m+1} + k, \quad k \in \mathbb{R}, m \neq -1$	$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin(x) + k$
$\int \frac{1}{x}\,dx = \ln(\|x\|) + k, \quad k \in \mathbb{R}$	$\int \frac{-1}{\sqrt{1-x^2}}\,dx = \arccos(x) + k$
$\int e^x\,dx = e^x + k, \quad k \in \mathbb{R}$	$\int \frac{1}{1+x^2}\,dx = \arctan(x) + k$
$\int \sin(x)\,dx = -\cos(x) + k, \quad k \in \mathbb{R}$	$\int \frac{-1}{1+x^2}\,dx = \text{arccotan}(x) + k$
$\int \cos(x)\,dx = \sin(x) + k, \quad k \in \mathbb{R}$	$k \in \mathbb{R}$

Properties of indefinite integrals

Let f, g be two continuous functions on an interval I and F, G respectively there primitives .

- $\int (\lambda f(x))\,dx = \lambda \int f(x)\,dx, \quad \lambda \in \mathbb{R}$

  Example 1.4 $\int (-5x^2)\,dx = -5\int x^2\,dx$
- $\int (f(x) + g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$

  Example 1.5 $\int (-5x^2 + \frac{1}{x})\,dx = \int -5x^2\,dx + \int \frac{1}{x}\,dx$

Remark 1.1 *In the general case*

$$\int (f(x) \times g(x))\,dx \neq \int f(x)\,dx \times \int g(x)\,dx$$

The question we ask is : how to calculate the primitive of a product of two functions

1.1.3 Integration by parts

It is a technique that stems from the differentiation of a product of two functions. It is used to calculate the primitives of certain functions that are expressed in the form of a product of functions..

Proposition 1.2 *Let f, g be Two functions that are continuous and differentiable on an interval I of $\mathbb{R}$ such that $f^{'}$ and $g^{'}$ are continuous on$I$. Then*

$$\int \left(f(x) \times g^{'}(x)\right) dx = (f(x)g(x)) - \int \left(f^{'}(x) \times g(x)\right) dx$$

Example 1.6 *Using an integration by parts, find* $\int xe^x \, dx$
We set $u = x$, which implies that $u'(x) = 1$,
and by setting $v^{'}(x) = e^x$, we have $v(x) = e^x$ (we take the primitive of $x \to e^x$ for $k = 0$)
$\int xe^x \, dx = (xe^x) - \int e^x \, dx = xe^x - e^x + k = (x-1)\, e^x + k, \quad k \in \mathbb{R}.$

While integration by parts is an important tool for calculating primitives of function products, it does not allow for the calculation of all types of function products. Another widely used integration technique is integration by a change of integration variable.

1.1.4 Integration by change of variable(substitution)

The following integration formulas immediately follow from the rules of differentiation using a simple change of variable.
We set $y = f(x)$, it follows that $\dfrac{dy}{dx} = f^{'}(x)$, hence $f^{'}(x)\, dx = dy$.

$$\int (f(x))^{n+1} f^{'}(x)\, dx = \frac{(f(x))^{n+1}}{n+1} + k, \quad k \in \mathbb{R}.$$
$$\int f^{'}(x) e^{f(x)} dx = e^{f(x)} + k, \quad k \in \mathbb{R}.$$
$$\int \frac{f^{'}(x)}{f(x)} = \ln|f(x)| + k, \quad k \in \mathbb{R}.$$
$$\int f^{'}(x) \sin(f(x))\, dx = -\cos(f(x)) + k, \quad k \in \mathbb{R}.$$
$$\int f^{'}(x) \cos(f(x))\, dx = -\sin(f(x)) + k, \quad k \in \mathbb{R}.$$
$$\int \frac{f^{'}(x)}{1 + f^2(x)} = \arctan(f(x)) + k, \quad k \in \mathbb{R}.$$
$$\int \frac{f^{'}(x)}{\sqrt{1 - f^2(x)}} = \arcsin(f(x)) + k, \quad k \in \mathbb{R}.$$
$$\int \frac{-f^{'}(x)}{\sqrt{1 - f^2(x)}} = \arccos(f(x)) + k, \quad k \in \mathbb{R}.$$

Exercise 1.1 *Prove the results in the above table*

Examples 1.1 *1)* $\int (2x+5)^3 \, dx = \frac{1}{2}\int 2(2x+5)^3 \, dx = \frac{1}{8}(2x+5)^4 + k, \quad k \in \mathbb{R}.$
2) $\int e^{3x+1} \, dx = \frac{1}{3}\int 3e^{3x+1} \, dx = \frac{1}{3}e^{3x+1} + k, \quad k \in \mathbb{R}.$
3) $\int \frac{2x}{1 + (x)^2} = \arctan(x^2) + k, \quad k \in \mathbb{R}.$
4) $\int \frac{2x}{\sqrt{1 - (x)^2}} = \arcsin(x^2) + k, \quad k \in \mathbb{R}.$

We can generalize this result in the following way

Proposition 1.3 *Let g be a continuous function on $I \subset \mathbb{R}$ and G its primitive on I. and let f be a differentiable function on I and f' its derivative.*

$$\int f'(x)g(f(x)) = G(f(x)) + k, \quad k \in \mathbb{R}.$$

Preuve. We set $u = f(x)$ then $\dfrac{du}{dx} = f'(x)$, and $du = f'(x)dx$.
This implies that

$$\int f'(x)g(f(x))\,dx = \int g(u)\,du = G(u) + k = G(f(x)) + k, \quad k \in \mathbb{R}.$$

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Example 1.7 *Calculate*

$$\int x.(3x+4)^5\,dx$$

Set $u = 3x + 4$ then $\dfrac{du}{dx} = 3$ then $dx = \dfrac{1}{2}du$.
On the other hand, $x = \dfrac{u-4}{3} = \dfrac{1}{3}u - \dfrac{4}{3}$.
Then, we obtain :

$$\begin{aligned}\int x.(3x+4)^5\,dx &= \frac{1}{3}\int (\frac{1}{3}u - \frac{4}{3})u^5\,du \\ &= \frac{1}{9}u^6 - \frac{4}{9}u^5 + k, \quad k \in \mathbb{R}.\end{aligned}$$

Unfortunately, the previous rules are not always directly applicable. In some cases, one must simplify the function before calculating its primitive. This is the case, for example, with rational functions.

1.1.5 Integration of the function $x \longmapsto \dfrac{1}{ax^2+bx+c}$

We seek to calculate the primitives of the function $x \mapsto \dfrac{1}{ax^2+bx+c}$, $a, b, c \in \mathbb{R}$.
First case : if $a = 0, b \neq 0, c \in \mathbb{R}$.

$$\int \frac{1}{bx+c}\,dx = \frac{1}{b}\int \frac{b}{bx+c}\,dx = \frac{1}{b}\ln|bx+c| + k, \quad k \in \mathbb{R}.$$

Example 1.8 $\displaystyle\int \frac{1}{2x+1}\,dx = \frac{1}{2}\ln|2x+1| + k, \quad k \in \mathbb{R}.$

Second case : if $a \neq 0, b, c \in \mathbb{R}$
$ax^2 + bx + c = a\left[\left(x + \dfrac{b}{2a}\right)^2 - \dfrac{\Delta}{4a^2}\right]$.
The integral to be calculated depends on Δ, and we distinguish three situations.

1. if $\Delta < 0$ on pose $\Delta = -\rho,\ \rho > 0$. Then we have :

$$ax^2 + bx + c = \frac{\rho}{4a}\left[\left(\frac{2a}{\sqrt{\rho}}x + \frac{b}{\sqrt{\rho}}\right)^2 + 1\right]$$

hence,

$$\frac{1}{ax^2 + bx + c} = \frac{4a}{\rho}\frac{1}{\left(\frac{2a}{\sqrt{\rho}}x + \frac{b}{\sqrt{\rho}}\right)^2 + 1}.$$

By using the change of variable $u = \frac{2a}{\sqrt{\rho}}x + \frac{b}{\sqrt{\rho}}$,

we have , $dx = \frac{\sqrt{\rho}}{2a}\,du$. As a result,

$$\begin{aligned}\int \frac{1}{ax^2 + bx + c}\,dx &= \frac{2}{\sqrt{\rho}}\int \frac{1}{u^2 + 1}\,du \\ &= \frac{2}{\sqrt{\rho}}\arctan(u) + k, \quad k \in \mathbb{R}, \\ &= \frac{2}{\sqrt{\rho}}\arctan\left(\frac{2a}{\sqrt{\rho}}x + \frac{b}{\sqrt{\rho}}\right) + k, \quad k \in \mathbb{R}.\end{aligned}$$

Example 1.9 *Calculate* $\int \frac{1}{2x^2 + x + 5}\,dx.$

$\Delta = -39, \rho = 39, a = 2, u = \frac{4}{\sqrt{39}}x + \frac{1}{\sqrt{39}}$

$$\begin{aligned}\int \frac{1}{2x^2 + x + 5}\,dx &= \frac{2}{\sqrt{39}}\int \frac{1}{u^2 + 1}\,du, \\ &= \frac{2}{\sqrt{39}}\arctan(u) + k, \quad k \in \mathbb{R}, \\ &= \frac{2}{\sqrt{39}}\arctan\left(\frac{4}{\sqrt{39}}x + \frac{1}{\sqrt{39}}\right) + k, \quad k \in \mathbb{R}.\end{aligned}$$

2. If $\Delta = 0$, we have, $ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2$.

Using variable change $u = x + \frac{b}{2a}$, $du = dx$, we will have

$$\begin{aligned}
\int \frac{1}{ax^2+bx+c}\,dx &= \int \frac{1}{a\left(x+\frac{b}{2a}\right)^2}\,dx,\\
&= \frac{1}{a}\int \frac{1}{u^2}\,du,\\
&= \frac{1}{a}\int u^{-2}\\
&= \frac{1}{a}\frac{u^{-2+1}}{-2+1}+k,\quad k\in\mathbb{R},\\
&= -\frac{1}{a}\frac{1}{u}+k,\quad k\in\mathbb{R},\\
&= -\frac{1}{a}\frac{1}{x+\frac{b}{2a}}+k,\quad k\in\mathbb{R}.
\end{aligned}$$

Example 1.10

$$\begin{aligned}
\int \frac{1}{3x^2+6x+3}\,dx &= \int \frac{1}{3\,(x-1)^2}\,dx,\\
&= \frac{1}{3}\int \frac{1}{u^2}\,du,\\
&= \frac{1}{3}\left(\frac{u^{-1}}{-1}\right)+k,\quad k\in\mathbb{R},\\
&= \frac{-1}{3\,(x-1)}+k,\quad k\in\mathbb{R}.
\end{aligned}$$

3. If $\Delta > 0$, on a, $ax^2+bx+c = a\,(x-x_1)\,(x-x_2)\,.$

   We decompose the fraction $\frac{1}{ax^2+bx+c}$ in the form

$$\frac{1}{ax^2+bx+c} = \frac{1}{a}\left[\frac{A}{x-x_1}+\frac{B}{x-x_2}\right],$$

   where A, B are real numbers that we will determine after reducing to the same denominator and identifying

$$\begin{aligned}
\int \frac{1}{ax^2+bx+c}\,dx &= \int \frac{1}{a}\left[\frac{A}{x-x_1}+\frac{B}{x-x_2}\right]dx\\
&= \frac{1}{a}\left[\int \frac{A}{x-x_1}\,dx+\int \frac{B}{x-x_2}\,dx\right],\\
&= \frac{A}{a}\ln|x-x_1|+\frac{B}{a}\ln|x-x_2|+k,\quad k\in\mathbb{R}.
\end{aligned}$$

Example 1.11 *Calculate* $\int \frac{1}{x^2-5x+6}\,dx$
$x^2-5x+6 = (x-2)\,(x-3)\,.$ *On en déduit que*

$$\int \frac{1}{x^2-5x+6}\,dx = A\ln|x-2|+B\ln|x-3|+k,$$

where A and B satisfy

$$\begin{aligned}\frac{1}{(x-2)(x-3)} &= \frac{A}{(x-2)}+\frac{B}{(x-3)},\\ &= \frac{Ax-3A+Bx-2B}{x^2-5x+6},\\ &= \frac{(A+B)x-3A-2B}{x^2-5x+6},\\ &\Longrightarrow [(A+B)=0]\wedge[-3A-2B=1],\\ &\Longrightarrow [A=-B]\wedge[-3A-2B=1],\\ &\Longrightarrow [(A=-1)\wedge(B=1)],\end{aligned}$$

$$\int\frac{1}{x^2-5x+6}\,dx=-\ln|x-2|+\ln|x-3|+k=\ln\left|\frac{x-3}{x-2}\right|+k,\quad k\in\mathbb{R}.$$

1.1.6 Integration of the function $x\longmapsto\dfrac{Ax+B}{ax^2+bx+c},\ a\neq 0, A\neq 0$

We start by simplifying the fraction $\dfrac{Ax+B}{ax^2+bx+c}$ by making the derivative of the denominator appear. We have

$$\begin{aligned}Ax+B &= Ax+B+\frac{Ab}{2a}-\frac{Ab}{2a}\\ &= \left(Ax+\frac{Ab}{2a}\right)+\left(B-\frac{Ab}{2a}\right)\\ &= \frac{A}{2a}(2ax+b)+\left(B-\frac{Ab}{2a}\right)\end{aligned}$$

So,

$$\begin{aligned}\frac{Ax+B}{ax^2+bx+c} &= \frac{\frac{A}{2a}(2ax+b)+\left(B-\frac{Ab}{2a}\right)}{ax^2+bx+c}\\ &= \frac{A}{2a}\frac{(2ax+b)}{ax^2+bx+c}+\left(B-\frac{Ab}{2a}\right)\frac{1}{ax^2+bx+c}\end{aligned}$$

$$\int\frac{Ax+B}{ax^2+bx+c}\,dx=\frac{A}{2a}\int\frac{(2ax+b)}{ax^2+bx+c}\,dx+\left(B-\frac{Ab}{2a}\right)\int\frac{1}{ax^2+bx+c}\,dx$$

By using integration by parts and setting,

$$I_1=\int\frac{1}{ax^2+bx+c}\,dx,$$

we get

$$\int\frac{Ax+B}{ax^2+bx+c}\,dx=\frac{A}{2a}\ln\left|ax^2+bx+c\right|+\left(B-\frac{Ab}{2a}\right)I_1+k,\quad k\in\mathbb{R}.$$

Remark 1.2 *We recall that I_1 is calculated using one of the methods from the previous subsection.*

Example 1.12

$$\int \frac{3x+4}{2x^2-5x+6} = \frac{3}{4}\ln\left|2x^2-5x+6\right| + \frac{31}{4}\int \frac{1}{2x^2-5x+6}\,dx + k, \quad k \in \mathbb{R}.$$

1.1.7 Integration of the function $x \longmapsto \dfrac{1}{\sqrt{ax^2+bx+c}}$, $a \neq 0$

1. If $a > 0$ et $\Delta = 0$, then $\sqrt{ax^2+bx+c} = \sqrt{a}\left|x + \dfrac{b}{2a}\right|$,
   hence we have :
   $$\int \frac{1}{\sqrt{ax^2+bx+c}}\,dx = \frac{1}{\sqrt{a}}\int \frac{1}{\left|x+\frac{b}{2a}\right|}\,dx$$

2. If $a > 0$ and $\Delta < 0$, then $\sqrt{ax^2+bx+c} = \sqrt{a}\sqrt{\left(x+\dfrac{b}{2a}\right)^2 + m^2}$, where $m^2 = \dfrac{-\Delta}{4a^2}$

$$\begin{aligned}
\int \frac{1}{\sqrt{ax^2+bx+c}}\,dx &= \int \frac{1}{\sqrt{a}\sqrt{\left(x+\frac{b}{2a}\right)^2+m^2}} \\
&= \frac{1}{\sqrt{a}}\int \frac{1}{\sqrt{u^2+m^2}}\,du \\
&= \frac{1}{\sqrt{a}}\ln\left|u+\sqrt{u^2+m^2}\right| + k, \quad k \in \mathbb{R} \\
&= \frac{1}{\sqrt{a}}\ln\left|\left(x+\frac{b}{2a}\right) + \sqrt{\left(x+\frac{b}{2a}\right)^2 - \frac{\Delta}{4a^2}}\right| + k, \quad k \in \mathbb{R}.
\end{aligned}$$

   A simple calculation shows that the derivative of the function $u \longmapsto \ln\left|u+\sqrt{u^2+m^2}\right|$ is the function $u \longmapsto \dfrac{1}{\sqrt{u^2+m^2}}$

   Example 1.13 $\displaystyle\int \frac{1}{\sqrt{5x^2+1}}\,dx = \frac{1}{\sqrt{5}}\ln\left|x+\sqrt{x^2+\frac{1}{5}}\right| + k, \qquad k \in \mathbb{R}.$

3. Si $a > 0$ et $\Delta > 0$, alors $ax^2+bx+c > 0$ sur $]-\infty\, x_1[\bigcup]x_2\, +\infty[$, and we have

$$\sqrt{ax^2+bx+c} = \sqrt{a}\sqrt{\left(x+\frac{b}{2a}\right)^2 - m^2}, \quad \text{où} \quad m^2 = \frac{\Delta}{4a^2}$$

$$\begin{aligned}
\int \frac{1}{\sqrt{ax^2+bx+c}}\,dx &= \int \frac{1}{\sqrt{a}\sqrt{\left(x+\frac{b}{2a}\right)^2 - m^2}} \\
&= \frac{1}{\sqrt{a}}\int \frac{1}{\sqrt{u^2-m^2}}\,du \\
&= \frac{1}{\sqrt{a}}\ln\left|u+\sqrt{u^2-m^2}\right| + k, \quad k\in\mathbb{R} \\
&= \frac{1}{\sqrt{a}}\ln\left|\left(x+\frac{b}{2a}\right)+\sqrt{\left(x+\frac{b}{2a}\right)^2 - \frac{\Delta}{4a^2}}\right| + k, \quad k\in\mathbb{R}.
\end{aligned}$$

A simple calculation shows that the derivative of the function $u \longmapsto \ln\left|u+\sqrt{u^2-m^2}\right|$ is the function $u \longmapsto \dfrac{1}{\sqrt{u^2-m^2}}$

Example 1.14 $\displaystyle\int \frac{1}{\sqrt{4x^2-1}}\,dx = \frac{1}{2}\ln\left|x+\sqrt{x^2-\frac{1}{4}}\right| + k, \qquad k\in\mathbb{R}.$

4. If $a<0$ and $\Delta<0$ then $\sqrt{ax^2+bx+c}$ is not defined.
5. If $a<0$ and $\Delta=0$ then $\sqrt{ax^2+bx+c}$ is defined on $\mathbb{R}-\left\{-\dfrac{b}{2a}\right\}$.
6. If $a<0$ and $\Delta>0$ then $\sqrt{ax^2+bx+c}$ is defined on $]x_1\ x_2[$.
   On the other hand,

$$ax^2+bx+c = a\left[\left(x+\frac{b}{2a}\right)^2 - \frac{\Delta}{4a^2}\right].$$

In this case, we set $a=-a^{'},\ \ a^{'}>0$ and we obtain

$$\begin{aligned}
ax^2+bx+c &= -a^{'}\left[\left(x+\frac{b}{2a}\right)^2 - \frac{\Delta}{4a^2}\right] \\
&= a^{'}\left[-\left(x+\frac{b}{2a}\right)^2 + \frac{\Delta}{4a^2}\right] \\
&= \frac{\Delta}{4a^{'}}\left[1-\left(\frac{2a^{'}}{\sqrt{\Delta}}x - \frac{b}{\sqrt{\Delta}}\right)^2\right].
\end{aligned}$$

By setting $u=\left(\frac{2a^{'}}{\sqrt{\Delta}}x-\frac{b}{\sqrt{\Delta}}\right)$, on a : $dx=\frac{\sqrt{\Delta}}{2a^{'}}\,du$, and consequently,

$$\begin{aligned}\int\frac{1}{\sqrt{ax^2+bx+c}}\,dx &= \frac{1}{\sqrt{a'}}\int\frac{1}{\sqrt{1-u^2}}\,du\\ &= \frac{1}{\sqrt{a'}}\arcsin(u)+k,\quad k\in\mathbb{R}\\ &= \frac{1}{\sqrt{a'}}\arcsin\left(\frac{2a^{'}}{\sqrt{\Delta}}x-\frac{b}{\sqrt{\Delta}}\right)+k,\quad k\in\mathbb{R}.\end{aligned}$$

Example 1.15 $\int\frac{1}{\sqrt{1-4x^2}}\,dx=\frac{1}{2}\arcsin(2x)+k,\quad k\in\mathbb{R}$

1.1.8 Techniques for decomposing a fraction into partial fractions

In the following, we seek to integrate a fraction of the form $\frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials of degrees n and m $(n,m\in\mathbb{N})$, with $Q(x)$ a factorizable polynomial. To do this, we start by simplifying the fraction $\frac{P(x)}{Q(x)}$, depending on the values of m and n. We distinguish three cases $n<m, n=m$ and $n>m$.

1. Si $n<m$
   — $Q(x)=(a_1x+b_1)(a_2x+b_2)\cdots(a_mx+b_m)$. Alors

$$\frac{P(x)}{Q(x)}=\frac{A_1}{(a_1x+b_1)}+\frac{A_2}{(a_2x+b_2)}+\cdots\frac{A_m}{(a_mx+b_m)}.$$

$A_1,A_2,\cdots,A_m$ sont des réels qu'on détermine par identification.

Example 1.16 *Simplify the fraction* $\frac{x}{x^2-1}$.
We have $degr(x)=1<(x^2-1)=2,$ *et* $x^2-1=(x-1)(x+1)$ *Donc*

$$\begin{aligned}\frac{x}{x^2-1} &= \frac{A_1}{x-1}+\frac{A_2}{x+1}\\ &= \frac{A_1(x+1)}{x^2-1}+\frac{A_2(x-1)}{x^2-1}\\ &= \frac{(A_1+A_2)x+(A_1-A_2)}{x^2-1},\end{aligned}$$

by identification, we have

$$x=(A_1+A_2)x+(A_1-A_2)$$

which gives $\begin{cases}A_1+A_2=1\\ A_1-A_2=0\end{cases}$ *therefore,* $A_1=A_2=\frac{1}{2}$ *and*

$$\frac{x}{x^2-1}=\frac{\frac{1}{2}}{x-1}+\frac{\frac{1}{2}}{x+1}$$

Application

$$\begin{aligned}\int \frac{x}{x^2-1}\,dx &= \int \frac{\frac{1}{2}}{x-1}\,dx + \int \frac{\frac{1}{2}}{x+1}\,dx \\ &= \frac{1}{2}\int \frac{1}{x-1}\,dx + \frac{1}{2}\int \frac{1}{x+1}\,dx \\ &= \frac{1}{2}\ln|x-1| + \frac{1}{2}\ln|x+1| + k, \quad k\in\mathbb{R}.\end{aligned}$$

— If $Q(x)$ contains at least one term with a power
Example 1.17

$$\frac{x+1}{x(x-2)^2(x-1)^3} = \frac{A_1}{x} + \frac{A_2}{(x-2)} + \frac{A_3}{(x-2)^2} + \frac{A_4}{(x-1)} + \frac{A_5}{(x-1)^2} + \frac{A_6}{(x-1)^3}$$

— If $Q(x)$ contains a term of the form ax^2+bx+c, with $\Delta < 0$.
Example 1.18

$$\frac{x^3+x^2+1}{x(x-2)^2(x^2+1)^3(x^2+x+2)} = \frac{A_1}{x} + \frac{A_2}{(x-2)} + \frac{A_3}{(x-2)^2} + \frac{A_4x+A_5}{(x^2+1)} + \frac{A_6x+A_7}{(x^2+1)^2} + \frac{A_8x+A_9}{(x^2+1)^2} + \frac{B_1x+B_2}{(x^2+x+2)}$$

2. Si $n=m$. In this case,
$$\frac{P(x)}{Q(x)} = A + \frac{R(x)}{Q(x)},$$
où $R(x)$ is a polynomial such that $degr(R(x)) < degr(Q(x))$ and $A\in\mathbb{R}$ are determined by Euclidean division or by identification. We decompose $\frac{R(x)}{Q(x)}$ as in the case 1.

Example 1.19 $\dfrac{x^2+4}{3x^2+x+1} = \dfrac{1}{3} + \dfrac{\frac{1}{3}(-x+11)}{3x^2+x+1}$

3. If $n>m$. In this case
$$\frac{P(x)}{Q(x)} = K(x) + \frac{R(x)}{Q(x)},$$
where $K(x), R(x)$ are respectively the quotient and the remainder of the Euclidean division of $P(x)$ by $Q(x)$. It is clear that $degr(R(x)) < degr(Q(x))$ and therefore, to simplify the fraction $\frac{R(x)}{Q(x)}$ we return to the case 1

1.1.9 Rules of Bioche

In this section, we are interested in calculating the integrals of rational functions in *sinus* and in *cosinus*. The Bioche rules allow simplifying fractions of the form $\frac{P(\sin(x),\cos(x))}{Q(\sin(x),\cos(x))}$, by reducing them to integrals of simple rational fractions. We aim to calculate the following integral.

$$I = \int \frac{P(\sin(x),\cos(x))}{Q(\sin(x),\cos(x))}\,dx. \tag{1.1}$$

we set

$$f(x) = \frac{P(\sin(x), \cos(x))}{Q(\sin(x), \cos(x))}\,dx.$$

The following Bioche rules allow us to determine the type of integral (1.1).

1. If $f(-x) = f(x)$ we set $t = \cos(x)$
2. If $f(\pi - x) = f(x)$ we set $t = \sin(x)$
3. If $f(\pi + x) = f(x)$ we set $t = \tan(x)$
4. Ifnone of the three previous conditions is satisfied, then we set $t = tan(\frac{x}{2})$ and in this case, we have :
   $\sin(x) = \dfrac{2t}{1+t^2}$, $\cos(x) = \dfrac{1-t^2}{1+t^2}$, $\tan(x) = \dfrac{2t}{1-t^2}$, et $dx = \dfrac{2}{1+t^2}\,dt$.

Example 1.20 *Find* $I = \int \frac{\sin^3(x)}{1+\cos^2(x)}\,dx$

Solution 1.1 *We set* $f(x) = -\frac{\sin^3(x)}{1+\cos^2(x)}\,dx$, *we have*

$$f(-x) = \frac{-\sin^3(x)}{1+\cos^2(x)}\,d(-x) = f(x)$$

So, we use the change of variable $\begin{cases} t = \cos(x), \\ \dfrac{dt}{dx} = -\sin(x), \\ dt = -\sin(x)dx, \end{cases}$

to obtain the integral I.

$$\begin{aligned} I = \int \frac{\sin^3(x)}{1+\cos^2(x)}\,dx &= \int \frac{\sin(x)\sin^2(x)}{1+\cos^2(x)}\,dx \\ &= -\int \frac{1-\cos^2(x)}{1+\cos^2(x)}\,dt \\ &= \int \frac{t^2-1}{1+t^2}\,dt \\ &= \int dt - 2\int \frac{1}{1+t^2}\,dt \\ &= t - 2\arctan(t) + k, \quad k \in \mathbb{R} \\ &= \cos(x) - 2\arctan(\cos(x)) + k, \quad k \in \mathbb{R} \end{aligned}$$

1.2 Definite integrals

Definition 1.3 *Let* f *a continuous function on* $[a\ \ b]$, *and* F *a primitive of* f *on this interval. The definite integral of* f *between* a *et* b *is the real number defined by*

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$$

Remarks 1.2 *1. One must take the difference between* $\int_a^b f(x)\,dx$ *which gives a number, and* $\int f(x)\,dx$ *which gives all functions whose derivative is* $f(x)$.

2. $\int_a^x f(t)\,dt$: *it is a function in* x *and which is zero for* $x = a$.

Examples 1.2 *1.* $\int \frac{1}{x}\,dx = \ln(x) + k, \quad k \in \mathbb{R}.$

2. $\int_1^2 \frac{1}{x}\,dx = [\ln(x)]_1^2 = \ln(2) - \ln(1) = \ln(2).$

3. $\int_1^x \frac{1}{t}\,dt = [\ln(t)]_1^x = \ln(x) - \ln(1) = \ln(x).$

1.2.1 Propriétés

Let f and g two functions defined and continuous on $[a\,b]$ and $\lambda \in \mathbb{R}$. All the properties of indefinite integrals remain valid for definite integrals ; moreover, we have :

1. $\int_a^a f(x)\,dx = 0$
2. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
3. $\int_a^\lambda f(x)\,dx + \int_\lambda^b f(x)\,dx = \int_a^b f(x)\,dx.$
4. $\int_a^b dx = (b - a)\,.$
5. If $f(x) = 0$ on $[a\,b]$, then $\int_a^b f(x)\,dx = 0$ (Note that the converse is not always true).
6. If $f(x)$ is positive on $[a\,b]$, then $\int_a^b f(x)\,dx \geq 0.$
7. If $f(x) - g(x) \geq 0$ on $[a\,b]$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx.$
8. If there exist two real numbers α, β such that for every $x \in [a\,b]$, $\alpha \leq f(x) \leq \beta$, then $\alpha\,(b-a) \leq \int_a^b f(x)\,dx \leq \beta\,(b-a)\,.$
9. If f is a continuous and even function on $[-a\;a]$, then $\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx.$
10. If f is a continuous and odd function on $[-a\;a]$, then $\int_{-a}^a f(x)\,dx = 0.$
11. If f is a periodic function with period T then $\int_a^{a+T} f(x)\,dx = \int_a^T f(x)\,dx.$

1.3 Corrected exercises

We remind that to calculate a definite integral of a continuous function, one must first find the antiderivative of that function and then evaluate this antiderivative between the bounds of the integral.

Exercise 1.2 *(Direct calculation)*
Calculate the following integrals

$$\int_0^{\frac{\pi}{2}} \cos(x)\, dx, \quad \int_0^1 \frac{\sqrt[3]{x^2} + 2\sqrt{x} + 1}{x^4}\, dx, \quad \int_{-1}^1 \Big(3x^2 - x + 1\Big)\, dx.$$

Solution :
1) We calculate all the primitives of the function $x \longmapsto \cos(x)$, which are given by the following indefinite integral,

$$\int \cos(x)\, dx = \sin(x) + k, \quad k \in \mathbb{R},$$

We evaluate these antiderivatives between the bounds of the integral, and we obtain

$$\int_0^{\frac{\pi}{2}} \cos(x)\, dx = [\sin(x)]_0^{\frac{\pi}{2}} = \sin(\frac{\pi}{2}) - \sin(0) = 1.$$

2) We follow the same procedure as before.

$$\begin{aligned}
\int \Big(3x^2 - x + 1\Big)\, dx &= \int 3x^2 - \int x\, dx + \int dx \\
\int_{-1}^1 \Big(3x^2 - x + 1\Big)\, dx &= \int_{-1}^1 3x^2 - \int_{-1}^1 x\, dx + \int_{-1}^1 dx \\
&= 3\left[\frac{x^3}{3}\right]_{-1}^1 - \left[\frac{x^2}{2}\right]_{-1}^1 + [x]_{-1}^1 \\
&= \left[3\frac{x^3}{3} - \frac{x^2}{2} + x\right]_{-1}^1 \\
&= \left[(1)^3 - \frac{(1)^2}{2} + (1)\right] - \left[(-1)^3 - \frac{(-1)^2}{2} + (-1)\right] \\
&= 4.
\end{aligned}$$

$$\begin{aligned}
3)\ \int \frac{\sqrt[3]{x^2} + 2\sqrt{x} + 1}{x^4}\, dx &= \int \frac{\sqrt[3]{x^2}}{x^4}\, dx + 2\int \frac{\sqrt{x}}{x^4}\, dx + \int \frac{1}{x^4}\, dx \\
&= \int x^{\frac{-10}{3}}\, dx + 2\int x^{\frac{-11}{3}}\, dx + \int x^{-4}\, dx \\
&= \frac{-3}{7} x^{\frac{-7}{3}} - \frac{6}{8} x^{\frac{-8}{3}} - \frac{1}{3} x^{-3} + k, \quad k \in \mathbb{R} \\
\int_0^1 \frac{\sqrt[3]{x^2} + 2\sqrt{x} + 1}{x^4}\, dx &= \left[\frac{-3}{7} x^{\frac{-7}{3}} - \frac{6}{8} x^{\frac{-8}{3}} - \frac{1}{3} x^{-3}\right]_0^1 \\
&= \frac{-3}{7}(1)^{\frac{-7}{3}} - \frac{6}{8}(1)^{\frac{-8}{3}} - \frac{1}{3}(1)^{-3} - 0 \\
&= -\frac{127}{84}.
\end{aligned}$$

Exercise 1.3 *(Integration by change of variable)*
Calculer $\displaystyle\int \frac{x}{\sqrt{1 - x^2}}\, dx.$

Solution :
Set $u = 1 - x^2$, then $\dfrac{du}{dx} = -2x$, therefore, $x\,dx = -\dfrac{1}{2}du$.

$$\begin{aligned}
\int \frac{x}{\sqrt{1-x^2}}\,dx &= -\frac{1}{2}\int \frac{du}{\sqrt{u}} \\
&= -\frac{1}{2}\int u^{\frac{-1}{2}}\,du \\
&= -\frac{1}{2}\left[\frac{u^{\frac{-1}{2}+1}}{\frac{-1}{2}+1}\right] \\
&= -\sqrt{u} + k, \quad k \in \mathbb{R} \\
&= -\sqrt{1-x^2} + k, \quad k \in \mathbb{R}.
\end{aligned}$$

Exercise 1.4 *(Integration by change of variable)*
Find $\int \arccos(x)\,dx$.

Solution :
First, we start by expressing the function to integrate as a product of two functions. Thus, we calculate $\int 1 \times \arccos(x)\,dx$, we have

$$\begin{cases} u^{'} = 1 \Longrightarrow u = x \\ v = \arccos(x) \Longrightarrow v^{'} = \dfrac{-1}{\sqrt{1-x^2}}. \end{cases}$$

The principle of integration by parts gives

$$\begin{aligned}
\int 1 \times \arccos(x)\,dx &= [x\arccos(x)] - \int \frac{-x}{\sqrt{1-x^2}}\,dx \\
&= [x\arccos(x)] - \sqrt{1-x^2} + k, \quad k \in \mathbb{R}.
\end{aligned}$$

Exercise 1.5 *(Using the decomposition)*
Calculate the following integrals

$$I_1 = \int \frac{-x+1}{x^2+2x+5}\,dx, \quad I_2 = \int \frac{\ln\left(x^2+2x+5\right)}{\left(x-1\right)^2}\,dx.$$

Solution :
1) We can easily see that

$$\int \frac{-x+1}{x^2+2x+5}\,dx = \int \frac{-x+1}{\left(x+1\right)^2+4}\,dx,$$

we use the change of variable $u = x+1$, it follows that $\dfrac{du}{dt} = 1$ and $x = u-1$. The integral I_1 is written as

$$I_1 = \int \frac{-u+2}{u^2+4}\,du = \int \frac{2}{u^2+4}\,du - \int \frac{u}{u^2+4}\,dx$$

$$\int \frac{2}{u^2+4}\,du = \int \frac{\frac{1}{2}}{\left(\frac{u}{2}\right)^2+1}\,du = \arctan\left(\frac{u}{2}\right) + k_1,$$

$$\int \frac{u}{u^2+4}\,dx = \frac{1}{2}\int \frac{2u}{u^2+4}\,dx = \ln\left|u^2+4\right| + k_2,$$

Donc

$$I_1 = \arctan\left(\frac{u}{2}\right) - \frac{1}{2}\ln\left|u^2+4\right| + k = \arctan\left(\frac{x+1}{2}\right) - \frac{1}{2}\ln\left|(x+1)^2+4\right| + k,\quad k\in\mathbb{R}$$

$$\int \frac{-x+1}{x^2+2x+5}\,dx = \arctan\left(\frac{x+1}{2}\right) - \frac{1}{2}\ln\left|\left(x^2+2x+5\right)\right| + k,\quad k\in\mathbb{R}.$$

2) Pour calculer I_2, on pense à une intégration par partie.

$$I_2 = \int \frac{\ln\left(x^2+2x+5\right)}{(x-1)^2}\,dx = \int \frac{1}{(x-1)^2}\ln\left(x^2+2x+5\right)\,dx.$$

$$\begin{cases} u' = \dfrac{1}{(x-1)^2} \Longrightarrow u = \displaystyle\int \frac{1}{(x-1)^2}\,dx = \frac{-1}{x-1} \\ v = \ln\left(x^2+2x+5\right) \Longrightarrow v' = \dfrac{2x+2}{x^2+2x+5} \end{cases}$$

The principle of integration by parts gives us

$$\int \frac{\ln\left(x^2+2x+5\right)}{(x-1)^2}\,dx = \left[\frac{-1}{x-1}\ln\left(x^2+2x+5\right)\right] - \int \frac{-1}{x-1}\frac{2x+2}{x^2+2x+5}\,dx$$

$$I_2 = \frac{-\ln\left(x^2+2x+5\right)}{x-1} + \int \frac{2x+2}{(x-1)\left(x^2+2x+5\right)}\,dx.$$

We decompose the fraction $\dfrac{2x+2}{(x-1)\left(x^2+2x+5\right)}$.

$$\begin{aligned} \frac{2x+2}{(x-1)\left(x^2+2x+5\right)} &= \frac{A_1}{x-1} + \frac{A_2x+A_3}{x^2+2x+5}, \\ &= \frac{A_1\left(x^2+2x+5\right) + (A_2x+A_3)(x-1)}{(x-1)\left(x^2+2x+5\right)}. \end{aligned}$$

Après simplification et identification on trouve $A_1 = \dfrac{1}{2},\ A_2 = -\dfrac{1}{2},\ A_3 = \dfrac{1}{2}$.

En décomposant la fraction $\dfrac{-x+1}{x^2+2x+5}$, on trouve

$$\begin{aligned} \int \frac{2x+2}{(x-1)\left(x^2+2x+5\right)}\,dx &= \frac{1}{2}\int \frac{1}{x-1}\,dx + \frac{1}{2}\int \frac{-x+1}{x^2+2x+5}\,dx, \\ &= \frac{1}{2}\ln\left|x-1\right| + \frac{1}{2}\arctan\left(\frac{x+1}{2}\right) - \frac{1}{4}\ln\left|\left(x^2+2x+5\right)\right| + k,\quad k\in\mathbb{R}. \end{aligned}$$

$$I_2 = \frac{-\ln\left(x^2+2x+5\right)}{x-1} + \frac{1}{2}\ln\left|x-1\right| + \frac{1}{2}\arctan\left(\frac{x+1}{2}\right) - \frac{1}{4}\ln\left|\left(x^2+2x+5\right)\right| + k,\quad k\in\mathbb{R}.$$

Exercise 1.6 *(Bioche rules)*
Find $\displaystyle\int \frac{1}{\sin(2x)}\,dx$, *et en déduire* $\displaystyle\int \frac{1}{\sin(x)}\,dx$.

Solution :
On pose
$$f(x) = \frac{1}{\sin(2x)}\,dx,$$
on a
$$f\left(x+\pi\right) = \frac{1}{\sin 2\left(x+\pi\right)}\,d\left(x+\pi\right) = \frac{1}{\sin(2x)}\,dx = f(x).$$
According to the Bioche rules, we use the change of variable $u = \tan(x)$. We will then have $\dfrac{du}{dx} = \dfrac{1}{\cos^2(x)}$ and $u = \dfrac{1}{\cos^2(x)}\,dx$. D'autre part,
$$\begin{aligned}\frac{1}{\sin(2x)} = \frac{1}{2\sin(x)\cos(x)} &= \frac{1}{2}\frac{\cos(x)}{\sin(x)\cos^2(x)}\\ &= \frac{1}{2}\frac{1}{\tan(x)}\frac{1}{\cos^2(x)},\end{aligned}$$
Therefore :
$$\begin{aligned}\int \frac{1}{\sin(2x)}\,dx &= \frac{1}{2}\int \frac{1}{\tan(x)}\frac{1}{\cos^2(x)}\,dx\\ &= \frac{1}{2}\int \frac{1}{u}\,du\\ &= \frac{1}{2}\ln|u| + k, \quad k\in\mathbb{R}\\ &= \frac{1}{2}\ln\Big|\tan(x)\Big| + k \quad k\in\mathbb{R}.\end{aligned}$$
To calculate $\displaystyle\int \frac{1}{\sin(x)}\,dx$, we use the change of variable $x = 2u$, which gives $u = \dfrac{x}{2}$ and $dx = 2\,du$.
$$\begin{aligned}\int \frac{1}{\sin(x)}\,dx &= 2\int \frac{1}{\sin(2u)}\,du\\ &= \ln\Big|\tan(u)\Big| + k, \quad k\in\mathbb{R}\\ &= \ln\left|\tan\left(\frac{x}{2}\right)\right| + k, \quad k\in\mathbb{R}.\end{aligned}$$

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