Chapter 3 Real Functions of a Real Variable

1 Introduction

This chapter serves to reinforce the notions already acquired by the student during their secondary education. We will review known concepts to refresh the student's memory and provide the necessary tools that will enable them to approach the new concepts they will need in their university-level analysis studies. In the previous chapter, we discussed the subject of functions and mappings in a general context where f is a function from a given set E to another set F. In this chapter, the set E will be $\mathbb{R}$ or a subset of $\mathbb{R}$, which is why it's called functions of a real variable. The set F will also be $\mathbb{R}$ or a subset of $\mathbb{R}$, which explains the designation of real-valued functions. In other words, this chapter is dedicated to functions where the variable is real, and the values of these functions are also real.

2 Generalities

Definition 2.1 *We call a real function of a real variable any function*

$$f : \left| \begin{array}{rcl} \mathcal{D} \subseteq \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & f(x) \end{array} \right.$$

In all what follows, we consider the function f as in definition5.1

Definition 2.2 • *The domain of definition of the function f is*

$$D_f = \{x \in \mathbb{R} \mid \exists y \in \mathbb{R},\ y = f(x)\}.$$

- *The graph of f is a subset of $\mathbb{R}^2$ denoted by G_f and is defined by* $G_f = \{ (x, y) \mid x \in D_f \;\; et \; (y = f(x)) \}$.

- *The representative curve of f often denoted by (C_f) where (Γ) is the set of points $M(x,y)$ with $(x,y) \in G_f$.*

Definition 2.3 • *A function f is called an even function if*

$$\forall x \in \mathcal{D},\ (-x \in \mathcal{D}) \wedge (f(-x) = f(x)).$$

- *The curve of an even function is symmetric with respect to the y-axis.*

- *A function f is called an odd if*

$$\forall x \in \mathcal{D},\ (-x \in \mathcal{D}) \wedge (f(-x) = -f(x)).$$

- *Studying the parity of a function involves determining whether it is even, odd, or neither.*

- *The curve of an odd function is symmetric with respect to the origin of the coordinate system*

- *A function f is said periodic and p is its period if p is the smallest positive real number that satisfies*

$$\forall x \in \mathcal{D},\ \big((x+p) \in \mathcal{D}\big) \wedge \big(f(x+p) = f(x)\big).$$

Example 2.1 • *The function $x \longmapsto x^4$ is an even function because* $\forall x \in \mathbb{R}, -x \in \mathbb{R} \wedge f(-x) = (-x)^4 = f(x)$.

- *The function $x \longmapsto \dfrac{1}{x}$ is an odd function because* $\forall x \in \mathbb{R}^*, -x \in \mathbb{R}^* \wedge f(-x) = \dfrac{1}{-x} = -\dfrac{1}{x} = -f(x)$.

- *The function $x \longmapsto \cos(x)$ is periodic of period 2π because* $\forall x \in \mathbb{R}, \big((x+2\pi) \in \mathbb{R}\big) \wedge f(x+2\pi) = \cos(x+2\pi) = \cos(x) = f(x)$.

Remark 2.1 *Before studying the parity of a function, it is essential to ensure that its domain of definition is symmetric with respect to 0.*

Example 2.2 *Let us consider the function*

$$f: \left| \begin{array}{lll} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & f(x) = \dfrac{\ln(1+x)}{|x|} \end{array} \right.$$

$D_f =]-1\ \ 0[\cup]0\ \ +\infty[$

We cannot discuss the parity of this function because its domain of definition is not symmetric with respect to zero.

Definition 2.4

- *It is said that f is bounded above on $\mathcal{D}$ if and only if*

$$\exists\, M \in \mathbb{R} \text{ tel-que } \forall x \in \mathcal{D}, \quad f(x) \leq M$$

- *It is said that f is bounded below on $\mathcal{D}$ if and only if*

$$\exists\, m \in \mathbb{R} \text{ tel-que } \forall x \in \mathcal{D}, \quad f(x) \geq m$$

- *It is said that f is bounded on $\mathcal{D}$ if and only if it is both bounded above and bounded below.*

$$\exists\, M \in \mathbb{R},\ \exists\, m \in \mathbb{R} \text{ such that } \forall x \in \mathcal{D}, \quad m \leq f(x) \leq M$$

  or

$$\exists M \in \mathbb{R}_+^* \text{ such that } \forall x \in \mathcal{D}, \quad \big|f(x)\big| \leq M$$

- *It is said that f is non- decreasing or that it preserves order on $\mathcal{D}$ if and only if*

$$\forall x, x' \in \mathcal{D}; \quad \big(x < x' \Longrightarrow f(x) \leq f(x')\big)$$

- *It is said that f is decreasing or that it does not preserve order on $\mathcal{D}$ if and only if*

$$\forall x, x' \in \mathcal{D}; \quad \big(x < x' \Longrightarrow f(x) \geq f(x')\big)$$

- *The non- decreasing function preserves order, while the decreasing function reverses the order.*

3 Limits

Definition 3.1 *Let $x_0 \in \mathcal{D}$,*

- *We say that* $f(x)$ *tends towards* l *as* x *tends to* x_0 *and we write* $\lim\limits_{x \to x_0} f(x) = l$ *iff* $\forall \varepsilon > 0, \exists \eta > 0;\ \forall x \in \mathcal{D}, \Big(|x - x_0| < \eta \Longrightarrow |f(x) - l| < \varepsilon \Big)$

- *It is said that* $f(x)$ *tends to* l *as* x *approaches* x_0 *from above iff* $\forall \varepsilon > 0, \exists \eta > 0;\ \forall x \in \mathcal{D}, \Big(0 < x - x_0 < \eta \Longrightarrow |f(x) - l| < \varepsilon \Big)$ *and we write* $\lim\limits_{\substack{x \to x_0 \\ >}} f(x) = \lim\limits_{x \to x_0^+} f(x) = l$. *This limit is also called a right-hand limit of* x_0.

- *It is said that* $f(x)$ *tends to* l *when* x *approaches* x_0 *from below iff* $\forall \varepsilon > 0, \exists \eta > 0;\ \forall x \in \mathcal{D}, \Big(-\eta < x - x_0 < 0 \Longrightarrow |f(x) - l| < \varepsilon \Big)$ *and we write* $\lim\limits_{\substack{x \to x_0 \\ <}} f(x) = \lim\limits_{x \to x_0^-} f(x) = l$. *This limit is also called a left-hand limit of* x_0.

Proposition 3.2 $\big(\lim\limits_{x \to x_0} f(x) = l \big) \Longleftrightarrow \Big(\lim\limits_{\substack{x \to x_0 \\ <}} f(x) = \lim\limits_{\substack{x \to x_0 \\ >}} f(x) = l \Big)$

Example 3.1 $f(x) = \dfrac{x}{|x|} + 1.$

We cannot calculate the limit of this function at 0 *because it is not defined at* 0. *However, we can calculate the right-hand and left-hand limits of* 0.

$$\lim_{\substack{x \to 0 \\ >}} f(x) = \lim_{\substack{x \to 0 \\ >}} \frac{x}{x} + 1 = 2$$

and

$$\lim_{\substack{x \to 0 \\ <}} f(x) = \lim_{\substack{x \to 0 \\ <}} \frac{x}{-x} + 1 = 0$$

then $\lim\limits_{x \to 0} f(x)$ *doesn't exist because* $\lim\limits_{\substack{x \to 0 \\ >}} f(x) \neq \lim\limits_{\substack{x \to 0 \\ <}} f(x)$

Proposition 3.3 *The limit at a point, when it exists, is unique.*

3.1 Elementary Properties

Proposition 3.4 *Let f and g be two functions defined in the neighborhood of x_0 such that*

$$\begin{cases} \lim\limits_{x\to x_0} f(x) = 0, \\ et \\ \ g \ \text{ is bounded,} \end{cases}$$

$$\text{then } \lim_{x\to x_0} f(x)g(x) = 0.$$

Example 3.2 *According to the proposition3.4*

$$\lim_{\substack{x\to 0 \\ >}} \sqrt{x}\cos\frac{\pi}{x} = 0$$

because $\lim\limits_{\substack{x\to 0 \\ >}} \sqrt{x} = 0$ *and* $\forall x \in \mathbb{R}^* : \left|\cos\left(\frac{\pi}{x}\right)\right| \leq 1$ *(boundedness).*

Proposition 3.5 *(Squeeze Theorem)*

The squeeze theorem (also known as sandwich theorem) states that if a function $f(x)$ lies between two functions $g(x)$ and $h(x)$ and the limits of each of $g(x)$ and $h(x)$ at a particular point are equal (to L), then the limit of $f(x)$ at that point is also equal toL.
Mathematically the squeeze theorem is defined as follows:
Let f, g, h be real functions definied in the neighborhood of x_0 such that

$$\begin{cases} g(x) \leq f(x) \leq h(x), \\ \ and \\ \lim\limits_{x\to x_0} g(x) = \lim\limits_{x\to x_0} h(x) = l, \end{cases}$$

$$\text{then } \lim_{x\to x_0} f(x) = l.$$

Remark 3.1 *The Squeeze Theorem remains valid as x approaches infinity.*

Example 3.3 *soit* $f(x) = \dfrac{1}{1+\cos(x)}$. *On a:* $\lim\limits_{x\to+\infty} \dfrac{1}{1+\cos(x)} = 0 \qquad (*).$
En effet,

$$\forall x \in D_f, \ -1 \leq \cos(x) \leq 1$$

et en ajoutant x *à tous les membres de l'inégalité, on obtient*

$$x - 1 \leq x + \cos(x) \leq x + 1,$$

quand on passe à l'inverse on trouve

$$\frac{1}{x+1} \leq \frac{1}{x+\cos(x)} \leq \frac{1}{x-1}.$$

Il est facile de voir que

$$\lim_{x\to+\infty} \frac{1}{x+1} = \lim_{x\to+\infty} \frac{1}{x-1} = 0.$$

The result $(*)$ *is obtained directly using the Squeeze Theorem.*

3.2 Indeterminate forms

When the rules for direct limit calculation don't work, we end up with so-called indeterminate forms. Some common indeterminate forms in mathematics include: $\infty - \infty, 0.\infty, \frac{\infty}{\infty}, \frac{0}{0}, 1^{\infty}, 0^0, \infty^0$. To resolve the indeterminacy, you can refer to classical methods typically learned in secondary school when dealing with polynomials or simple fractions. In many cases, you can use a rule called L'Hôpital's Rule to resolve indeterminate forms like $\frac{\infty}{\infty}$ and $\frac{0}{0}$. However, for forms like 1^{∞}, 0^0, and ∞^{0}", it's common to use logarithms to simplify them into more manageable forms.

Example 3.4 $\lim_{x\to 0} x^x = 0^0$ *FI*
To resolve this indeterminacy, you can use the following technique:

$$\lim_{x\to 0} x^x = \lim_{x\to 0} e^{x\ln(x)} = e^o = 1.$$

It is worth noting that

$$\lim_{x\to 0} x\ln(x) = 0, \quad a^x = e^{x\ln(a)}.$$

The following example serves as a basic tool for resolving the indeterminate form 1^{∞}, where we often reduce it to one of the following representations:

$$(1+x)^{\frac{1}{x}}, (1-x)^{\frac{1}{x}}, \left(1+\frac{1}{x}\right)^x, \left(1-\frac{1}{x}\right)^x.$$

Example 3.5 $\lim_{x\to 0}(1+x)^{\frac{1}{x}} = e.$

Solution 3.6 $\lim_{x\to 0}(1+x)^{\frac{1}{x}} = \lim_{x\to 0} e^{\frac{1}{x}\ln(1+x)}$

We know that: $\lim_{x\to 0}\dfrac{\ln(x+1)}{x} = 1 = \lim_{x\to 0}\dfrac{1}{x}\ln(x+1).$

Consequently: $\lim_{x\to 0}(1+x)^{\frac{1}{x}} = \lim_{x\to 0} e^{\frac{1}{x}\ln(1+x)} = e^1 = e.$

Through a simple change of variable, it can be shown that

$\lim_{x\to 0}(1-x)^{\frac{1}{x}} = e^{-1}$

$\lim_{x\to +\infty}\left(1+\frac{1}{x}\right)^x = e$

$\lim_{x\to +\infty}\left(1-\frac{1}{x}\right)^x = e^{-1}$

In conclusion, $\lim_{A\to 0}(1+A)^{\frac{1}{A}} = e$. This important result is generally used to calculate the limit in the case of the indeterminate form 1^∞.

Example 3.6 *Calculate* $\lim_{x\to +\infty}\left(1+\dfrac{3}{x}\right)^x$

Solution 3.7 *We make the change of variable* $y = \dfrac{3}{x}$ *alors* $x = \dfrac{3}{y}$, *So when* $x \to +\infty$ *we have* $y \to 0$, *and it yields*

$\lim_{x\to +\infty}\left(1+\dfrac{3}{x}\right)^x = \lim_{y\to 0}(1+y)^{\frac{3}{y}} = \lim_{y\to 0}\left[\,(1+y)^{\frac{1}{y}}\,\right]^3 = e^3.$

4 continuity

Let $x_0 \in \mathcal{D}$.

$$f : \left| \begin{array}{ccc} \mathcal{D} \subseteq \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & y = f(x) \end{array} \right.$$

Definition 4.1 *The function f is* ***continuous*** *in x_0 if and only if*

$$\lim_{x\to x_0} f(x) = f(x_0).$$

Remark 4.1 *We can only study the continuity of f at x_0 if f is defined at that point, meaning $f(x_0)$ exists.*

Definition 4.2 1. *We say that f is* ***right-continuous at the point*** x_0. *if and only if*

$$\lim_{\substack{x\to x_0 \\ >}} f(x) = f(x_0)$$

2. *We say that f is* ***left-continuous at the point*** x_0. *if and only if*

$$\lim_{\substack{x \to x_0 \\ <}} f(x) = f(x_0)$$

3. $\big($*f is* ***continuous*** *en* $x_0\big) \Longleftrightarrow \big(\lim_{\substack{x \to x_0 \\ >}} f(x) = \lim_{\substack{x \to x_0 \\ <}} f(x) = f(x_0)\big)$

4. *If f and g are two functions that are continuous at x_0, then $f+g$ and $f \cdot g$ are continuous at x_0, and if, furthermore, $g(x_0) \neq 0$, then $\dfrac{f}{g}$ is continuous at x_0.*

5. *If $f: \mathcal{D} \longrightarrow \mathbb{R}$ is continuous at x_0 and if $g: f(\mathcal{D}) \longrightarrow \mathbb{R}$ is continuous at $f(x_0)$ then (gof) is continuous at x_0.*

Example 4.1 *We consider the function f defined on $\mathbb{R}$ by*

$$f(x) = \begin{cases} 0 \quad si \quad x \leq 0, \\ x \quad si \quad 0 < x \leq 1, \\ -x^2 + 4x - 2 \quad si \quad 1 < x \leq 3, \\ 4 - x \quad si \quad x > 3 \end{cases}$$

It is obvious that we must study the continuity of f at points $0, 1$ and 3.

- *We study the continuity of f at the point 0*

$$\lim_{\substack{x \to 0 \\ >}} f(x) = \lim_{\substack{x \to 0 \\ >}} x = 0 = \lim_{\substack{x \to 0 \\ <}} f(x) = f(0).$$

- *We study the continuity of f at the point 1*

$$\lim_{\substack{x \to 1 \\ >}} f(x) = \lim_{\substack{x \to 1 \\ >}} -x^2 + 4x - 2 = 1 = \lim_{\substack{x \to 1 \\ <}} x = f(1).$$

- *We study the continuity of f at the point 3*

$$\lim_{\substack{x \to 3 \\ >}} f(x) = \lim_{\substack{x \to 3 \\ >}} 4 - x = 1 = \lim_{\substack{x \to 3 \\ <}} -x^2 + 4x - 2 = f(3).$$

Theorem 4.3 *(Intermediate Value Theorem)*

- *If f is continuous on $[a\,b]$ and if λ is a real number between $f(a)$ et $f(b)$, i.e., $(f(a) \leq \lambda \leq f(b))$ ou $(f(b) \leq \lambda \leq f(a))$ then there exists at least one real number $c \in [a\,b]$ such that $f(c) = \lambda$.*

- *If f is continuous on $[a\,b]$ and if $f(a).f(b) < 0$ then there exists at least one real $c \in [a\,b]$ such that $f(c) = 0$.*

Remark 4.2 *If f is a function defined from a domain I to a domain J and we want to prove that for all $y \in J$, there exists an $x \in I$ such that $y = f(x)$, we invoke the Intermediate Value Theorem. It should be noted that this theorem confirms only the existence of x.*

Example 4.2 $f : \left| \begin{array}{lll} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & f(x) = x^3 + x^2 - x - 2, \end{array} \right.$
On peut utiliser le théorème des valeurs intermédiaires pour montrer que f est surjective. En effet,
f est continue sur $\mathbb{R}$, de plus $\lim\limits_{x \to -\infty} f(x) = -\infty$ et $\lim\limits_{x \to +\infty} f(x) = +\infty$ alors d'aprš le théorème des valeurs intermédiaires, pour tout $y \in \left]-\infty\ +\infty\right[$, il existe au moins $x \in \mathbb{R}$ tel-que $y = f(x)$.

We can use the Intermediate Value Theorem to demonstrate that f is surjective. Indeed,
f is continuous on $\mathbb{R}$, and furthermore, $\lim_{x \to -\infty} f(x) = -\infty$ and $\lim_{x \to +\infty} f(x) = +\infty$. According to the Intermediate Value Theorem, for any $y \in \left]-\infty, +\infty\right[$, there is at least one $x \in \mathbb{R}$ such that $y = f(x)$.

Theorem 4.4 *(The Bijection Theorem)*

- *If f is continuous and strictly monotonous on $[a\,b]$ then f is a bijection from $[a\,b]$ to $f([a\,b])$.*
- *f^{-1} is also a bijection from $f([a\,b])$ to $[a\,b]$, and it has the same direction of variation as f.*
- *The two representative curves of f et f^{-1} are symmetric with respect to the line with equation $y = x$.*

Example 4.3 $f : \left| \begin{array}{lll} \mathbb{R} & \longrightarrow & \mathbb{R}^+ \\ x & \longmapsto & f(x) = e^x \end{array} \right.$
f is continuous on $\mathbb{R}$ and $\forall x \in \mathbb{R}, f'(x) = e^x > 0$, so f is continuous and strictly increasing, then it has an inverse function.
$f^{-1} : \left| \begin{array}{lll} \mathbb{R}^+ & \longrightarrow & \mathbb{R} \\ x & \longmapsto & f(x) = \ln(x) \end{array} \right.$
We have $x \to \ln(x)$ is continuous and strictly increasing, $(\forall x \in \mathbb{R}^+ f'(x) = \dfrac{1}{x} > 0)$, Furthermore $\ln(e^x) = x$, and $e^{\ln(x)} = x$.

4.1 Elementary Inverse Functions

1. The function

$$f : \left| \begin{array}{ccc} \left[-\frac{\pi}{2}\ \frac{\pi}{2}\right] & \longrightarrow & [-1\ 1] \\ x & \longmapsto & f(x) = \sin(x) \end{array} \right.$$

is continuous and strictly increasing, so it is bijective, and consequently, it has an inverse function denoted as arcsin such that

$$f^{-1} : \left| \begin{array}{ccc} [-1\ 1] & \longrightarrow & \left[-\frac{\pi}{2}\ \frac{\pi}{2}\right] \\ x & \longmapsto & f^{-1}(x) = \arcsin(x) \end{array} \right.$$

and w have

$$y = \sin(x) \Longleftrightarrow x = \arcsin(y)$$

$$\sin(\arcsin(x)) = x, \text{ et } \arcsin(\sin(x)) = x.$$

2. The function

$$f : \left| \begin{array}{ccc} [0\ \pi] & \longrightarrow & [-1\ 1] \\ x & \longmapsto & f(x) = \cos(x) \end{array} \right.$$

is continuous and strictly decreasing, so it is bijective, and consequently, it has an inverse function denoted as arccos such that

$$f^{-1} : \left| \begin{array}{ccc} [-1\ 1] & \longrightarrow & [0\ \pi] \\ x & \longmapsto & f^{-1}(x) = \arccos(x) \end{array} \right.$$

and w have

$$y = \cos(x) \Longleftrightarrow x = \arccos(y)$$

$$\cos(\arccos(x)) = x, \text{ et } \arccos(\cos(x)) = x.$$

3. The function

$$f : \left| \begin{array}{ccc} \left]-\frac{\pi}{2}\ \frac{\pi}{2}\right[& \longrightarrow & \mathbb{R} \\ x & \longmapsto & f(x) = \tan(x) \end{array} \right.$$

is continuous and strictly increasing, so it is bijective, and consequently, it has an inverse function denoted as arctan such that

$$f^{-1} : \left| \begin{array}{ccc} \mathbb{R} & \longrightarrow & \left]-\frac{\pi}{2}\ \frac{\pi}{2}\right[\\ x & \longmapsto & f^{-1}(x) = \arctan(x) \end{array} \right.$$

and w have

$$y = \tan(x) \Longleftrightarrow x = \arctan(y)$$

$$\tan(\arctan(x)) = x, \text{ et } \arctan(\tan(x)) = x.$$

4. The function

$$f : \left| \begin{array}{ccc}]0\ \pi[& \longrightarrow & \mathbb{R} \\ x & \longmapsto & \text{f(x) = cotan(x)} \end{array} \right.$$

is continuous and strictly decreasing, so it is bijective, and consequently, it has an inverse function denoted as *arccotan* such that

$$f^{-1} : \left| \begin{array}{ccc} \mathbb{R} & \longrightarrow &]0\ \pi[\\ x & \longmapsto & f^{-1}(x) = arcotan(x) \end{array} \right.$$

and w have

$$\text{y = cotan(x)} \iff \text{x = arccotan(y)}$$

$$\text{cotan(arccotan(x)) = x,\ \ et arccotan(cotan(x)) = x.}$$

4.2 Continuity Extension

Let $f:\ \mathcal{D} \to \mathbb{R}$ A function which is not defined at x_0, but $\lim\limits_{x \to x_0} f(x) = a \in \mathbb{R}$.

Then we can define a new function that is continuous at x_0 denoted by $\tilde{f}$ as follow:

$$\tilde{f}(x) = \begin{cases} f(x) \ \text{ if } \ x \neq x_0, \\ a \ \text{ if } \ x = x_0, \end{cases}$$

$\tilde{f}$ is called the **continuity extension** of f at x_0.

Example 4.4 *The function* $x \to f(x) = \dfrac{\ln(x+1)}{x}$ *is not definet at* $x_0 = 0$, *but its limit as* $x \to 0$ *exists and we have* $\lim\limits_{x \to 0} \dfrac{\ln(x+1)}{x} = 1$. *Then* f *admits a* ***continuity extension*** *at* $x_0 = 0$

$$\tilde{f}(x) = \begin{cases} \dfrac{\ln(x+1)}{x} & \text{if } \ x \neq 0, \\ 1 & \text{if } \ x = 0, \end{cases}$$

5 Derivative

5.1 Definitions

In all that follows, we consider a function $f:\ \mathcal{D} \subseteq \mathbb{R} \longrightarrow \mathbb{R}$, and $x_0 \in \mathbb{R}$.

Definition 5.1 *1. f is differentiable at $x_0 \in \mathcal{D}$ if and only if the following limit exists and is finite*

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = a \in \mathbb{R} \tag{5.1}$$

2. *The number a, when it exists, is called the derivative of f at point x_0, and we write $f'(x_0) = a$.*

3. *The geometric interpretation of a: it represents the slope of the tangent to the curve (C_f) at the point $A(x_0, f(x_0))$. Furthermore, the equation of this tangent is*

$$(\triangle) : \ y = a(x - x_0) + f(x_0)$$

4. *If the right-hand and left-hand derivatives at x_0 exist and are equal, then f is differentiable at x_0, and conversely. In other words,*

$$\Big(f \ \text{ is differentiable at } \ x_0\Big) \Longleftrightarrow \left(\lim_{\substack{x \to x_0 \\ >}} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{\substack{x \to x_0 \\ <}} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0)\right)$$

5. *Si $\lim\limits_{\substack{x \to x_0 \\ >}} \frac{f(x) - f(x_0)}{x - x_0} = a_1$, and $\lim\limits_{\substack{x \to x_0 \\ <}} \frac{f(x) - f(x_0)}{x - x_0} = a_2$, with $a_1 \neq a_2$, then f is not differentiable at x_0*

6. *If in the equation 5.1 we remplace $x - x_0$ by h we will have*

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = a$$

7. *If f is differentiable at x_0 then f is continuous at x_0. Warning, the converse is not true, meaning that if f is continuous at x_0, we cannot make any claims about its differentiability at x_0; it needs to be studied. Furthermore, if f is not differentiable at x_0, we cannot make any conclusions regarding continuity at x_0; it also needs to be studied.*

Remark 5.1 *To answer the question of studying the continuity and differentiability of f, you can follow one of the two following paths.*

1. *We start by studying continuity, and if f is not continuous at x_0, we stop and conclude that f is not differentiable at x_0. However, if f is continuous at x_0, we must then proceed to study differentiability at x_0.*

2. *We start by studying differentiability at x_0, and if f is differentiable at x_0, we directly conclude that f is continuous at x_0. However, if f is not differentiable at x_0, we must then proceed to study continuity at x_0.*

3. *A function f may not be differentiable at x_0 and not continuous at x_0.*

Example 5.1 *We consider the function f defined on $\mathbb{R}$ as follow $f(x) = |x-1|$*

It is clear that we should study the continuity and differentiability at $x_0 = 1$. We choose to begin with the study of differentiability.

$$\begin{aligned} \lim_{\substack{x \to 1 \\ >}} \frac{f(x) - f(1)}{x - 1} &= \lim_{\substack{x \to 1 \\ >}} \frac{(|x-1|) - 0}{x - 1} \\ &= \lim_{\substack{x \to 1 \\ >}} \frac{x-1}{x-1} \\ &= 1 \end{aligned}$$

$$\begin{aligned} \lim_{\substack{x \to x_0 \\ <}} \frac{f(x) - f(1)}{x - 1} &= \lim_{\substack{x \to 1 \\ >}} \frac{(|x-1|) - 0}{x - 1} \\ &= \lim_{\substack{x \to 1 \\ <}} \frac{-(x-1)}{x-1} \\ &= -1 \end{aligned}$$

Since $1 \neq -1$, then f is not differentiable at $x_0 = 1$ and in this case, we cannot make any conclusions regarding continuity; we need to study it. To do this, we calculate the following two limits:

$$\lim_{\substack{x \to 1 \\ <}} |x-1| = \lim_{\substack{x \to 1 \\ <}} -x + 1 = 0 = f(1)$$

$$\lim_{\substack{x \to 1 \\ >}} |x-1| = \lim_{\substack{x \to 1 \\ >}} x - 1 = 0 = f(1)$$

hence f is continuous at $x_0 = 1$ because $\lim_{\substack{x \to 1 \\ <}} |x-1| = \lim_{\substack{x \to 1 \\ >}} |x-1| = f(1)$.

5.2 Rules of Differentiation

We begin by recalling the rules for differentiating common functions.

$f(x)$	a	x^n	$\frac{1}{x}$	$\sqrt{x}$	e^x	$\ln(x)$	$\cos(x)$	$\sin(x)$
$f'(x)$	0	nx^{n-1}	$-\frac{1}{x^2}$	$\frac{1}{2\sqrt{x}}$	e^x	$\frac{1}{x}$	$-\sin(x)$	$\cos(x)$
D_f	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}^*$	$\mathbb{R}^+$	$\mathbb{R}$	$\mathbb{R}_+^*$	$\mathbb{R}$	$\mathbb{R}$

If f and g are two differentiable functions on $\mathcal{D} \subseteq \mathbb{R}$, then the functions constructed from these two functions are differentiable, and we have

1. $(f+g)^{'} = f^{'} + g^{'}$

2. $(fg)^{'} = f^{'}g + g^{'}f$

3. If $g \neq 0$ on $\mathcal{D}$, $\left(\frac{f}{g}\right)^{'} = \frac{f^{'}g - g^{'}f}{g^2}$

4. $(f^n)^{'} = nf^{'}f^{n-1}$, $n \in \mathbb{N}$

5. We use the following notations $f = f^{(0)}, f^{'} = f^{(1)}, f^{''} = f^{(2)}, \ldots, f^{(n)} = \left(f^{(n-1)}\right)^{'}$,

6. If a function f is differentiable several times on $\mathcal{D} \subset \mathbb{R}$, we can define the following sets

   Definition 5.2 *Let $\mathcal{D} \subset \mathbb{R}$ be an open interval. Let $n \in \mathbb{N}^*$. We say that $f : \mathcal{D} \longrightarrow \mathbb{R}$ is of class $\mathcal{C}^n$ if it is differentiable n times, and if the nth derivative is continuous. If f is of class $\mathcal{C}^n$ for all $n \in \mathbb{N}$, it is called of class $\mathcal{C}^\infty$ or infinitely differentiable. f is said to be of class $\mathcal{C}^0$ if it is continuous on $\mathcal{D}$.*

7. $(fg)^{(n)} = \sum_{i=0}^{n} \frac{n!}{i!(n-i)!} f^{(i)}g^{(n-i)}$.

8. Now, suppose that f is differentiable on $\mathcal{D}$ and g is differentiable on $f(\mathcal{D})$ alors gof est dérivable sur $\mathcal{D}$ et on a

$$\forall x \in \mathcal{D}, \ \ (gof)^{'}(x) = f^{'}(x).g'\left[f(x)\right].$$

9. Let f a bijection from $\mathcal{D}$ to $f(\mathcal{D})$ and let f^{-1} its inverse function. If f is differentiable on $\mathcal{D}$ and if f' does not equal zero on $\mathcal{D}$ then f^{-1} is differentiable $f(\mathcal{D})$ et on a

$$\forall y_0 \in f(\mathcal{D}), \quad \left(f^{-1}\right)'(y_0) = \frac{1}{f'\left[f^{-1}(y_0)\right]}$$

Remark 5.2 *To calculate $f^{-1}(y_0)$, you need to find $x_0 \in \mathcal{D}$ that satisfies the equation $y_0 = f(x_0)$, and then you will have $f^{-1}(y_0) = x_0$. Consequently,*

$$\left(f^{-1}\right)'(y_0) = \frac{1}{f'(x_0)}.$$

Example 5.2 *Let $f : x \to f(x) = x + e^x$*
Prove that f is bijective on $\mathbb{R}$
Calculate $\left(f^{-1}\right)'(1)$.

Solution 5.3 *f is continuous on $\mathbb{R}$, and $\forall x \in \mathbb{R}$, $f'(x) = 1 + e^x > 0$, hence f is continuous and strictly monotonous and therefore f^{-1} exists. To calculate $\left(f^{-1}\right)'(1)$, we first solve the equation $f(x) = 1$. The solution to this equation is as follows:*

$$\begin{aligned} f(x) = 1 &\implies x + e^x = 1, \\ &\implies x = 0, \end{aligned}$$

It yields

$$f(0) = 1 \implies f^{-1}(1) = 0$$

$$\begin{aligned} \left(f^{-1}\right)'(1) &= \frac{1}{f'\left[f^{-1}(1)\right]}, \\ &= \frac{1}{f'(0)}, \\ &= \frac{1}{1 + e^0}, \\ &= \frac{1}{2}. \end{aligned}$$

5.3 Derivatives of Inverse Circular Functions

Proposition 5.4 *The Circular Functions* arcsin, arccos, arctan, *arccotan are differentiable over their domains of definition.*

$$\forall x \in]-1\ \ 1[,\ \ (\arcsin)'(x) = \frac{1}{\sqrt{1-x^2}}$$

$$\forall x \in]-1\ \ 1[,\ \ (\arccos)'(x) = \frac{-1}{\sqrt{1-x^2}}$$

$$\forall x \in \mathbb{R},\ \ (\arctan)'(x) = \frac{1}{1+x^2}$$

$$\forall x \in \mathbb{R},\ \ (\mathit{arccotan})'(x) = \frac{-1}{1+x^2}$$

Example 5.3 *Calculate the derivative of the function defined by*

$$f(x) = \arctan(x^2+1)$$

We apply the rule of differentiation for a composite function and arctan *.*
The technique is very simple: we assume that

$$f(x) = \arcsin(g(x)) \text{ with } g(x) = 1 + x^2$$

and

$$\forall x \in \mathbb{R},\ \ f'(x) = g'(x) \times (\arctan)'(g(x)),$$

$g'(x) = 2x,$

$(\arctan)'(x) = \dfrac{1}{1+x^2},$

$(\arctan)'(g(x)) = \dfrac{1}{1+(g(x))^2},$

Which gives

$$\forall x \in \mathbb{R},\ \ f'(x) = \frac{2x}{1+(1+x^2)^2}$$

5.4 Important Theorems

Theorem 5.5 ***Rolle's Theorem***
Let f *a continuous function on* $[a\ b]$, *and differentiable on* $]a\ b[$ *such that* $f(a) = f(b)$ *then,*

$$\exists c \in]a\ b[,\ f'(c) = 0.$$

Theorem 5.6 ***Finite-Increment Theorem***
Let f a continuous function on $[a\ b]$ and differentiable on $]a\ b[$ then

$$\exists c \in]a\ b[\,,\ f'(c) = \frac{f(b) - f(a)}{b - a}$$

Theorem 5.7 ***L'Hôpital's Rule***
Let f, g two continuous functions over an interval $\mathcal{D} \subseteq \mathbb{R}$, except perhaps at $x_0 \in \mathcal{D}$, if $f(x_0) = g(x_0) = 0$ and if g' does not vanish on $\mathcal{D} - \{x_0\}$, and if $\lim_{x \to x_0} \frac{f'(x)}{g'(x)} = a$ then

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = a$$

Remark 5.3

- *L'Hôpital's Rule is a widely used tool in the evaluation of limits to resolve indeterminate forms $\frac{0}{0}$ where $\frac{\infty}{\infty}$.*
- *We can apply L'Hôpital's Rule as many times as necessary until we obtain the desired limit*

Example 5.4 $\lim_{x \to 0} \frac{\sin(x)}{x} = \frac{0}{0}$, *we apply L'Hôpital's Rule*

$$\lim_{x \to 0} \frac{\sin(x)}{x} = \lim_{x \to 0} \frac{\cos(x)}{1} = \frac{1}{1} = 1.$$

Example 5.5 $\lim_{x \to +\infty} \frac{\ln(x)}{x^2} = \frac{+\infty}{\infty}$, *We apply L'Hôpital's Rule twice*

$$\lim_{x \to +\infty} \frac{\ln(x)}{x^2} = \lim_{x \to +\infty} \frac{\frac{1}{x}}{2x} = \lim_{x \to +\infty} \frac{1}{2x^2} = 0.$$