Chapter 2 Sets - Relations - Functions

1 Sets Theory

1.1 Sets and elements

A set is a collection of objects that verify certain properties. An object which satisfies the needed rules is called element of the set. If the set is denoted by U and x is an element of U we say x belongs to U and we write $x \in U$.

Example 1.1 *Let us consider the following sets*
$A = \{x \in \mathbb{R} : -7 < x \leq 5\} =]-7\ 5]$
$B = \{x \in \mathbb{N} : -7 < x \leq 5\} = \{0, 1, 2, 3, 4, 5\}$
$C = \{x \in \mathbb{Z} : -7 < x \leq 5\} = \{-6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}$

1.2 cardinality of a finite set

If a set U contains a finite number of elements it is said to be finite, otherwise it is said to be infinite. If U is finite and it contains $n \in \mathbb{N}$ elements, then n is called the cardinality of U we write $card\ U = n$ or $|U| = n$. If $n = 0$ the set U is called an empty set and is denoted by $\emptyset$ and we have $card\emptyset = 0$.
In the previous example A is infinite set, $|B| = 6$.

1.3 Inclusion and Equality

Definition 1.1 *Let A, B be two sets contained in some universal set U*

- *The set A is a proper subset of B provided that $A \subseteq B$ and $A \neq B$. When EA is a proper subset of B we write $A \subset B$. One reason of the definition of proper subset is that each set is a subset of itself. That is $A \subseteq B$.*

- $(A \subset B) \Leftrightarrow (\forall x \in U,\ x \in A \Rightarrow x \in B)$.

- *If $A \subset B$ we say that A is a subset of B or B contains A.*

- $(A \not\subseteq B) \Longleftrightarrow (\exists x \in U) : (x \in A \text{ and } x \notin B)$

Theorem 1.2 *Let A, B, C be subsets of an universal set U.*

- $(A = B) \Longleftrightarrow [(A \subseteq B) \text{ et } (B \subseteq A)]$.
- $[(A \subset B) \wedge (B \subset C)] \Longrightarrow (A \subset C)$.
- *For any subset X of U we have $X \subseteq X$ and $\emptyset \subseteq X$.*

Example 1.2 $A = \left]1 \; \frac{5}{2}\right]$, $B = [-5 \; 3[$.

$A \subset B$. In fact,

$$\begin{aligned} x \in A &\Rightarrow 1 < x \leq \frac{5}{2}, \\ &\Rightarrow -5 \leq 1 < x \leq \frac{5}{2} < 3 \\ &\Rightarrow x \in B. \end{aligned}$$

Example 1.3 $A = \{n \in \mathbb{Z}, \exists k \in \mathbb{Z} : \; n = 5k + 2\}$, $B = \{n \in \mathbb{Z}, \exists k \in \mathbb{Z} : \; n = 5k + 7\}$.
Prove that $A = B$

$$\begin{aligned} n \in A &\Rightarrow \exists k \in \mathbb{Z}, \; n = 5k + 2, \\ &\Rightarrow \exists k \in \mathbb{Z}, \; n = 5k + 2 + 5 - 5, \\ &\Rightarrow \exists k \in \mathbb{Z}, \; n = 5(k-1) + 7, \\ &\Rightarrow \exists k \in \mathbb{Z}, \; n = 5k' + 7 \text{ avec } k' = k - 1 \in \mathbb{Z}, \\ &\Rightarrow A \subset B. \end{aligned}$$

$$\begin{aligned} n \in B &\Rightarrow \exists k \in \mathbb{Z}, \; n = 5k + 7, \\ &\Rightarrow \exists k \in \mathbb{Z}, \; n = 5k + 7 + 2 - 2, \\ &\Rightarrow \exists k \in \mathbb{Z}, \; n = 5(k+1) + 2, \\ &\Rightarrow \exists k \in \mathbb{Z}, \; n = 5k' + 2 \text{ avec } k' = k + 1 \in \mathbb{Z}, \\ &\Rightarrow B \subset A. \end{aligned}$$

$$A \subset B \text{ and } B \subset A \text{ then } A = B.$$

1.4 The Power Set of a Set

Let A be a subset of an universal set U, then the set whose elements are all the subsets of A is called the power set of A and is denoted by $\mathcal{P}(A)$. Symbolically, we write

$$\mathcal{P}(A) = \{X \subseteq U / X \subseteq A\}.$$

That is $X \in \mathcal{P}(A) \Longleftrightarrow X \subseteq A$.

Example 1.4 $A = \Big\{a, b, c\Big\}$,
$\mathcal{P}(A) = \Big\{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\Big\}$.

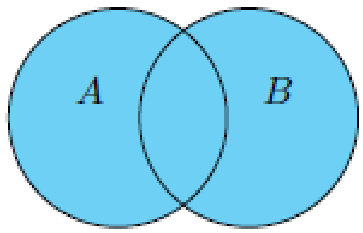
Remark 1.1 $A \in \mathcal{P}(A)$, *and* $\emptyset \in \mathcal{P}(A)$.

Remark 1.2 $\mathcal{P}(\emptyset) = \{\emptyset\}$, *and* $\emptyset \in \{\emptyset\}$.
Moreover, $\textit{card}\, \emptyset = 0$ *and* $\textit{card}\, \{\emptyset\} = 1$. *That is* $\emptyset \neq \{\emptyset\}$.
$\mathcal{P}(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}$,

1.5 Operations on sets

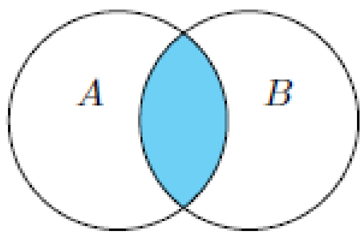
Let A, B be subsets of an universal set U, then

- Unions



$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

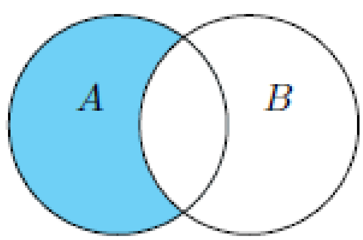
- Intersections



$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

note: when $A \cap B = \emptyset$, then A and B are said to be **disjoint**.

- Complements



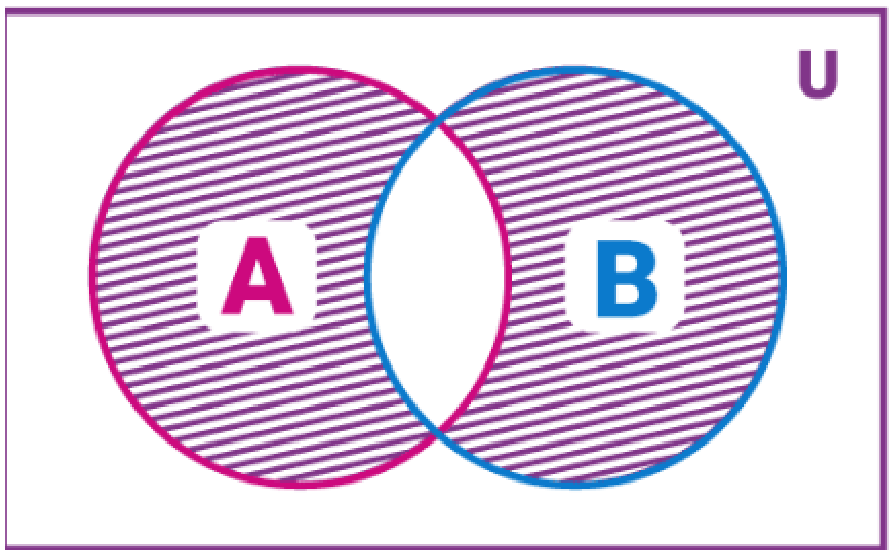
$$A \backslash B = \{x \in A : x \notin B\}$$

- $A \backslash B$ is also called the set difference of A and B, or relative complement of B with respect to A, written $A - B$ and read A minus B or the complement of B with respect to A.

- The complement of the set A in U denoted by A^c, or $\overline{A}$, is the set of all elements of U that are not in A. That is $A^c = \{x \in U\ x \notin A\}$.

- The symmetric difference of two sets A, B denoted by $A\Delta B$, also known as the disjunctive union and set sum, is the set of elements which are in either of the sets, but not in their intersection(the set which contains the elements which are either in set A or in set B but not in both is).

$$\begin{aligned} A\Delta B &= (A \cup B) - (A \cap B), \\ &= (A - B) \cup (B - A). \end{aligned}$$



Example 1.5 $U = \{-1, -5, -3, 0, 2, 7, 11\}$, $A = \{-1, 0, 7\}$, $B = \{-3, 0, -1, 2, 11\}$
$A^c = \{-5, -3, 2, 11\}$.
$A - B = \{7\}$, $B - A = \{-3, 2, 11\}$, $A\Delta B = (A - B) \cup (B - A) = \{7, -3, 2, 11\}$

Example 1.6 *Let us consider the two sets A and B,*
$A = \{0, 1, 3, 5, 8, 10, 17, 20\}$,
$B = \{2, 1, 4, 5, 8, 18, 17, 21\}$,
We have :
$A - B = \{0, 3, 10, 20\}$,
$B - A = \{2, 4, 18, 21\}$

$A \cup B = \{0, 1, 3, 5, 8, 10, 17, 20, 2, 4, 18, 21\}$,
$A \cap B = \{1, 5, 8, 17\}$,
$(A \cup B) - (A \cap B) = \{0, 3, 10, 20, 2, 4, 18, 21\}$,
$(A - B) \cup (B - A) = \{0, 3, 10, 20, 2, 4, 18, 21\}$,
$A \Delta B = \{0, 3, 10, 20, 2, 4, 18, 21\} = (A \cup B) - (A \cap B) = (A - B) \cup (B - A)$.

Theorem 1.3 *Let A be a set, then $A \backslash A = \emptyset$.*

Theorem 1.4 *Let A, B be a subsets of an universal set U.*

$$\textit{If } A \subset B \textit{ then } B^c \subset A^c.$$

1.6 Laws of the algebra of sets

Let A, B be subsets of an universal set U.

Idempotent Laws	(a) A ∪ A = A	(b) A ∩ A = A
Associative Laws	(a) (A ∪ B) ∪ C = A ∪ (B ∪ C)	(b) (A ∩ B) ∩ C = A ∩ (B ∩ C)
Commutative Laws	(a) A ∪ B = B ∪ A	(b) A ∩ B = B ∩ A
Distributive Laws	(a) A ∪ (B ∩ C) = (A ∪ B) ∩ (A ∪ C)	(b) A ∩ (B ∪ C) =(A ∩ B) ∪ (A ∩ C)
De Morgan's Laws	(a) $(A \cup B)^c = A^c \cap B^c$	(b) $(A \cap B)^c = A^c \cup B^c$
Identity Laws	(a) A ∪ ∅ = A (b) A ∪ U = U	(c) A ∩ U =A (d) A ∩ ∅ = ∅
Complement Laws	(a) $A \cup A^c = U$ (b) $A \cap A^c = \emptyset$	(c) $U^c = \emptyset$ (d) $\emptyset^c = U$
Involution Law	(a) $(A^c)^c = A$	

Example 1.7 *Through the following example check Laws of the algebra of sets.*

$A = \{x \in \mathbb{N} \mid x \leq 7\}$
$B = \{x \in \mathbb{N} \mid x \ \ \textit{is a multiple of} \ \ 3\}$
$C = \{x \in \mathbb{N} \mid x \geq 10\}$.

Solution 1.5 $A = \{0, 1, 2, 3, 4, 5, 6, 7\}$,
$B = \{0, 3, 6, 9, 12, \ldots\}$,
$C = \{10, 11, 12, 13, 14, 15, 16, 17, \ldots\}$,

$A \cap B = \{x \in \mathbb{N} \,|\, x \in A \ \textit{et} \ \ x \in B\}$.
$A \cap B = \{x \in \mathbb{N} \,|\, x \leq 7 \ \textit{and} \, x \ \textit{is a multiple of} \ \ 3\}$.
$A \cap B = \{0, 3, 6\}$.

$A \cup B = \{x \in \mathbb{N} \,|\, x \in A \ \textit{ou} \ \ x \in B\}$,
$A \cup B = \{x \in \mathbb{N} \,|\, x \leq 7 \ \textit{or} \, x \ \textit{is a multiple of} \ \ 3\}$,
$A \cup B = \{0, 1, 2, 3, 4, 5, 6, 7, 9, 12, 15, 18, 21, \ldots\}$.

$A^c = \{x \in \mathbb{N} \mid x \notin A\}$,
$A^c = \{x \in \mathbb{N} \mid x > 7\}$,
$A^c = \{8, 9, 10, 11, 12, 13, \ldots\}$,

$B^c = \{x \in \mathbb{N} \mid x \notin B\}$,
$B^c = \{x \in \mathbb{N} \mid x \neq 3k, \ k \in \mathbb{N}\}$,
$B^c = \{x = 3k + 1 \ \textit{or} \, x = 3k + 2 \ k \in \mathbb{N}\}$,
$B^c = \{1, 2, 4, 5, 7, 8, 10, 11, 13, \ldots\}$,

$A^c \cap B^c = \{x \in \mathbb{N} \mid x \in A^c \ \textit{and} \, x \in B^c\}$,
$A^c \cap B^c = \{x \in \mathbb{N} \mid x \notin A \ \textit{and} \, x \notin B\}$,
$A^c \cap B^c = \{x \in \mathbb{N} \mid x > 7 \ \textit{and} \, x \neq 3k, \ k \in \mathbb{N}\}$,
$A^c \cap B^c = \{x \in \mathbb{N} \mid x > 7 \ \textit{and} \, x \neq 3k, \ k \in \mathbb{N}\}$,
$A^c \cap B^c = \{8, 10, 11, 13, 14, 16, \ldots\}$,

$(A \cup B)^c = \left\{x \in \mathbb{N} \,/\, \overline{x \leq 7 \ \textit{or} \, x \ \textit{is a multiple of} \ \ 3}\right\}$,
$(A \cup B)^c = \{x \in \mathbb{N} \,/\, x \notin (A \cup B)\}$,
$(A \cup B)^c = \{x \in \mathbb{N} \,/\, x \notin A \ \textit{and} \, x \notin B\}$
$(A \cup B)^c = \{x \in \mathbb{N} \,/\, x \in A^c \ \textit{and} \, x \in B^c\} = A^c \cap B^c$,

$A^c \cup B^c = \{x \in \mathbb{N} \mid x \in A^c \ \textit{or} \, x \in B^c\}$,
$A^c \cup B^c = \{x \in \mathbb{N} \mid x \notin A \ \textit{or} \, x \notin B\}$,
$A^c \cup B^c = \{x \in \mathbb{N} \mid x \notin (A \cap B)\}$,

$A^c \cup B^c = (A \cap B)^c$.

Exercise 1.6 *Let A and B be two subset of a set U.*
Prove that $A \Delta B = (A \cup B) \cap (A^c \cup B^c)$.

Solution 1.7

$$\begin{aligned} x \in A\Delta B &\Leftrightarrow x \in (A \cup B) - (A \cap B), \\ &\Leftrightarrow x \in (A \cup B) \;\; et\, x \notin (A \cap B), \\ &\Leftrightarrow x \in (A \cup B) \;\; et\, x \in \overline{(A \cap B)}, \\ &\Leftrightarrow x \in (A \cup B) \;\; et\, x \in (A \cap B)^c, \\ &\Leftrightarrow x \in (A \cup B) \;\; et\, x \in (A^c \cup B^c), \\ &\Leftrightarrow x \in (A \cup B) \cap (A^c \cup B^c), \end{aligned}$$

Then, $A \Delta B = (A \cup B) \cap (A^c \cup B^c)$,

1.7 Cartesian Products

Definition 1.8 *The Cartesian product of two sets A and B denoted by* $A \times B$ *is the set of all ordered pairs* (a, b) *where* $a \in A$ *and* $b \in B$, *and we write*

$$A \times B = \{(a, b) \mid a \in A, b \in B\}.$$

Remark 1.3 *If* $a \neq b$ *then* $(a, b) \neq (b, a)$ *and so* $A \times B \neq B \times A$.

Example 1.8 *One can see the following examples of Cartesian products.*

- $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x, y) \mid x \in \mathbb{R},\ y \in \mathbb{R}\}$.
- $[0\ 1[\times \mathbb{R} = \{(x, y) \mid 0 \leq x < 1,\ y \in \mathbb{R}\}$.
- $\{0, 1\} \times [1\ 2] = \{(0, y), (1, y),\ 1 \leq y \leq 2\}$.
- $\{3, 5, 8\} \times \{0, 9\} = \Big\{ (3, 9), (5, 0)\, (3, 0), (5, 9), (8, 0), (8, 9) \Big\}$.

1.7.1 Cartesian Products of Sets Properties

Let A, B and C be three sets. Some of the important properties of Cartesian products of sets are given below.

- Let (a,b) and (c,d) two elements of $A \times B$. We say that $(a,b) = (c,d)$ if only if $a = c$ and $b = d$.
- $card\,(A \times B) = |A \times B| = |A||B| = cardA.cardB$.
- If A and B are non-empty sets and either A or B are infinite set, then $A \times B$ is also an infinite set.
- $A^2 = A \times A = \{(a_1, a_2)\,/a_1, a_2 \in A\}$.
- $$\begin{aligned} A^n &= \underbrace{A \times A \times A \times ... \times A}_{n\ times} \\ &= \Big\{(a_1, a_2, a_3, ..., a_n)/a_1, a_2, a_3, ..., a_n \in A\Big\}. \end{aligned}$$

  $n \in \mathbb{N}$, $\big(a_1, a_2, a_3, ..., a_n\big)$ is called an ordered $n-$uplet
- If the sets are disjoint in pairs, then $A \times (B \times C) \neq (A \times B) \times C$.
- $\emptyset \times A = \emptyset = A \times \emptyset$.
- If A or B are empty sets, then $A \times B = \emptyset$.
- If $A \times B = \emptyset$ this means that either $(A = \emptyset$ and $B \neq \emptyset)$ or $(B = \emptyset$ and $A \neq \emptyset)$ or $(A = \emptyset$ and $B = \emptyset)$.
- $A \times (B \cap C) = (A \times B) \cap (A \times C)$.
  $A \times (B \cup C) = (A \times B) \cup (A \times C)$.
  $(A \cap B) \times C = (A \times C) \cap (B \times C)$.
  $(A \cup B) \times C = (A \times C) \cup (B \times C)$.

2 Relations

Definition 2.1 *Let A and B be sets. A relation $\mathcal{R}$* **from the set** A **to the set** B *is a subset of $A \times B$. That is, $\mathcal{R}$ is a collection of ordered pairs where the first coordinate of each pair is an element of A and the second* *coordinate of each pair is an element of* B. *The set of pairs* $(x, y) \in A \times B$ *which satisfy* $\mathcal{R}(x, y)$ *is called* ***the graph of the relation*** $\mathcal{R}$, *it is denoted by* $G_{\mathcal{R}}$, *such that*

$$G_{\mathcal{R}} = \Big\{ (x, y) \in A \times B \mid x\mathcal{R}y \Big\}.$$

A relation from the set A *to the set* A *is called a relation* **on the set** A. *So a relation on a set* A *is a subset of* $A \times A$

Example 2.1 $A = \{0, 1, 3, 5, 8, 10, 16\}$, $B = \{4, 16, 20, 23\}$.
provied the graph of the relation $\mathcal{R}$ *defined on* $A \times B$ *by:*

$$\forall (x, y) \in A \times B, \ x\mathcal{R}y \Leftrightarrow y = 2x.$$

$$G_{\mathcal{R}} = \Big\{ (8, 16), (10, 20) \Big\}.$$

2.1 Equivalence Relations-Order relations

Let A be a set and $\mathcal{R}$ be a relation on A. The relation $\mathcal{R}$ may have various properties. There are three in particular that we are intrested in

- We say the relation $\mathcal{R}$ is reflexive if for all $a \in A, a\mathcal{R}a$.
- We say the relation $\mathcal{R}$ is symetric if for all $a, b \in A, \ a\mathcal{R}b \Longrightarrow b\mathcal{R}a$.
- We say the relation $\mathcal{R}$ is antisymetric if for all $a, b \in A, \ [(a\mathcal{R}b) \wedge (b\mathcal{R}a)] \Longrightarrow (a = b)$.
- We say the relation $\mathcal{R}$ is transitive if for all $a, b, c \in A, \ [(a\mathcal{R}b) \wedge (b\mathcal{R}c)] \Longrightarrow (a\mathcal{R}c)$.

Definition 2.2 *An equivalence relation is a relation that is reflexive, symmetric and transitive.*

Example 2.2 *Let us consider the relation* $\mathcal{R}$ *defined on* $\mathbb{R}$ *by :*

$$\forall x, y \in \mathbb{R}, \ x\mathcal{R}y \Leftrightarrow xe^{y} = ye^{x}.$$

Prove that $\mathcal{R}$ *is an equivalence relation.*

Solution 2.3 *We show that* $\mathcal{R}$ *is reflexive, symmetric and transitive.*

1. $\forall x \in \mathbb{R}$ *on a* $xe^x = xe^x$. *In other words, we have* $x\mathcal{R}x$ *and then* $\mathcal{R}$ *is reflexive.*

2. $\mathcal{R}$ *is symetric . In fact, let* $x, y \in \mathbb{R}$, *such that* $x\mathcal{R}x$, *hence we have*

$$\begin{aligned} x\mathcal{R}y &\Rightarrow xe^y = ye^x, \\ &\Rightarrow ye^x = xe^y, \\ &\Rightarrow y\mathcal{R}x, \end{aligned}$$

3. $\mathcal{R}$ *is transitive because for all* $x, y, z \in \mathbb{R}$, *such that* $\Big[(x\mathcal{R}y) \wedge (y\mathcal{R}z)\Big]$, *on a*
$x\mathcal{R}y \Rightarrow xe^y = ye^x$*..............(1)*
$y\mathcal{R}z \Rightarrow ye^z = ze^y$*................(2)*
(2) *gives* $y = \dfrac{ze^y}{e^z}$, *moreover, using* (1) *and by substituting* y *we have* $xe^y = \dfrac{ze^y}{e^z}e^x$ *hence* $xe^ye^z = ze^ye^x$. *Since* $e^y \neq 0$ *Thus*

$$xe^z = ze^x,$$

which implies $x\mathcal{R}z$.

4. $\mathcal{R}$ *is reflexive, symetric and transitive then it is an equivalence relation.*

Definition 2.4 *Let* $\mathcal{R}$ *be an equivalence relation on a set* A.

1. *The equivalence class of an element* a *in* A *is the set of all elements* x *in* A *that are in relation with* a. *We denote this set by* $\overline{a}$ *or* $\dot{a}$, *and we write it as follow*
$$\overline{a} = \dot{a} = \Big\{x \in A /\ x\mathcal{R}a\Big\}$$

2. a *is a representative of the equivalence class* $\dot{a}$.

3. *The set of equivalence classes for all elements in* A *is called the* ***"quotient set" of*** A *for the equivalence relation* $\mathcal{R}$. *It is denoted as* $A_{/\mathcal{R}}$, *and written as follows:*

$$A_{/\mathcal{R}} = \Big\{\dot{x},\ x \in A\Big\}$$

Exercise 2.5 *Let use consider the relation* $\mathcal{R}$ *defined on* $\mathbb{R}$ *by*

$$\forall x, y \in \mathbb{R},\ x\mathcal{R}y \Leftrightarrow \left(x^2 - y^2 = x - y\right)$$

Prove that $\mathcal{R}$*is an equivalence relation on* $\mathbb{R}$.
Determine the equivalence classe of an elément a *in* $\mathbb{R}$.

solution

1. $\mathcal{R}$ is reflexive because: $\forall x \in \mathbb{R},\ x^2 - x^2 = 0 = x - x$, that is $x\mathcal{R}x$.

2. $\mathcal{R}$is symetric. In fact , let $x, y \in \mathbb{R}$ such that $x\mathcal{R}y$, hence

$$\begin{aligned} x\mathcal{R}y &\Rightarrow x^2 - y^2 = x - y, \\ &\Rightarrow y^2 - x^2 = y - x, \\ &\Rightarrow y\mathcal{R}x. \end{aligned}$$

3. $\mathcal{R}$is transitive. In fact , let $x, y, z, \in \mathbb{R}$, such that $\Big((x\mathcal{R}y) \wedge (y\mathcal{R}z)\Big)$.
   $x\mathcal{R}y \Rightarrow x^2 - y^2 = x - y$.................(1)
   $y\mathcal{R}z \Rightarrow y^2 - z^2 = y - z$...............(2)
   The sum of (1) and (2) gives $x^2 - z^2 = x - z$, which implies $x\mathcal{R}z$.

4. $\mathcal{R}$ is reflexive, symetric and transitive then it is an equivalence relation.

5. let $a \in \mathbb{R}$.

$$\begin{aligned} \dot{a} &= \Big\{x \in \mathbb{R},\ \mid x\mathcal{R}a\Big\}, \\ &= \Big\{x \in \mathbb{R} \mid x^2 - a^2 = x - a\Big\}, \\ &= \Big\{x \in \mathbb{R} \mid\ (x - a)(x + a) - (x - a) = 0\Big\}, \\ &= \Big\{x \in \mathbb{R} \mid\ (x - a)(x + a - 1) = 0\Big\}, \\ &= \Big\{x \in \mathbb{R} \mid x = a \text{ or } x = 1 - a\Big\}, \\ &= \Big\{a, 1 - a\Big\}. \end{aligned}$$

Remark 2.1 *We distinguish two cases*
If $a = \dfrac{1}{2}$ *we have:* $\dot{a} = \left\{\dfrac{1}{2}\right\}$,
If $a \neq \dfrac{1}{2}$ *on we have:* $\dot{a} = \Big\{a, 1 - a\Big\}$.

Definition 2.6 *An order relation (also called an order or ordering) is a binary relation that is reflexive, antisymmetric and transitive.*

Definition 2.7 *(Totality of an order) Let* $\mathcal{R}$ *be an order relation on* A*. If for all* a, b *in* A *we have* $(a\mathcal{R}b)$ *or* $(a\mathcal{R}b)$ *or both, then the order is called total (total property). We say* A *is totaly ordered by* $\mathcal{R}$*. The total property implies the reflexive property, by setting* $a = b$*. If the order is not total it is partial.*
Let $\mathcal{R}$ *be an order relation on* A*. Two elements* a, b *in* A *are comparable if only if we have* $(\big(a\mathcal{R}b \vee b\mathcal{R}a\big)$*. If* $\mathcal{R}$ *is an order relation on* A*, and if the order is total, then all elements of* A *are comparable.*

Example 2.3 *Let* A *be a non-empty set and* $\mathcal{R}$ *a relation on* A *defined by :*

$$\forall a, b \in A,\ a\mathcal{R}b \Leftrightarrow a = b.$$

$\mathcal{R}$ *is a an order relation on* A*.*
if A *is a singleton, then the order is total. If not, the order is partial.*

3 Functions - Mappings

Now, we are interested in relations that map the elements of two sets: the set of departure and the set of arrival. A function is a rule which operates on one element to give another element. However, not every rule describes a valid function. The following definitions explain how to see whether a given rule describes a valid function, and introduces some of the mathematical terms associated with functions.

3.1 Generalities

Definition 3.1 *A function is a rule that maps an element to another unique element. The input to the function is called the independent variable, and is also called the argument of the function. The output of the function is called the dependent variable.*

Definition 3.2 *We call a function from a set E to a set F any relation from E to F that associates, or maps, every element $x \in E$* ***to at most one*** *in $y \in F$ such that $x\mathcal{R}y$. Generally, functions are denoted by $f, g, h, K, T, \ldots,$ and we write:*

$$f : \left| \begin{array}{lll} E & \longrightarrow & F \\ x & \longmapsto & y = f(x) \end{array} \right.$$

- *y is the image of x by the function f,*
- *x is the pre-image or preimage of y,*
- *E is the set of departure of f, and F is the set of arrival.*

Definition 3.3 *A function from a set E to a set F that associates with each element x of the set E* **exactly one element** *of the set F is called a* **mapping** *from E to F.*

Definition 3.4 *Let $f : E \longrightarrow F$. A domain or set of definition of f is the set of all elements $x \in E$ such that $\exists\, y \in F$ which satisfies $y = f(x)$, is the set denoted D_f and is writen as*

$$D_f = \Big\{ x \in E :\ \exists y \in F \mid y = f(x) \Big\}$$

Remark 3.1 *A mapping f defined from E to F is a function where $D_f = E$.*

Example 3.1 *Determine the domain of definition of the following function:*

$$f : \left| \begin{array}{lll} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & f(x) = \sqrt{x-1} \end{array} \right.$$

$D_f = \Big\{ x \in \mathbb{R}, \mid x - 1 \geq 0 \Big\}$

$D_f = \Big\{ x \in \mathbb{R}, \mid x \geq 1 \Big\}$

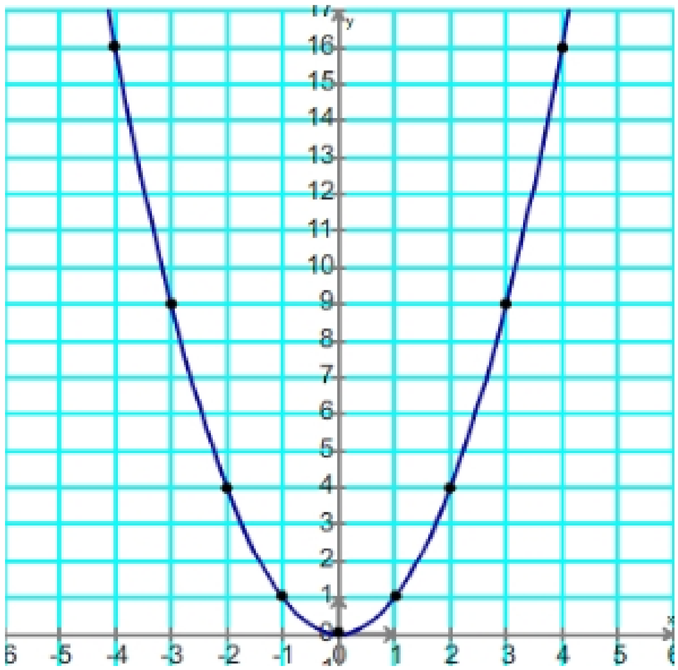
$D_f = [1 \quad +\infty[$

Definition 3.5 1. *The graph of a function $f : E \longrightarrow F$ is the set of all ordered pairs $(x, y) \in E \times F$ where $y = f(x)$ and we write*

$$G_f = \Big\{ (x, y) \in E \times F \ / \ y = f(x) \Big\}$$

2. *The curve representing the function f in a coordinate system $\left(O, \vec{i}, \vec{j}\right)$; which is generally denoted as $(\mathcal{C}_f)$; is the set of all points $M(x, y)$ with ordered pairs $(x, y) \in G_f$.*

Example 3.2 *The curve representing the function $x \longmapsto x^2$.*



Example 3.3 *La fonction*

$$f : \left| \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & f(x) = \dfrac{1}{x} \end{array} \right.$$

This is a function because 0 *has not an image by f, While*

$$f : \left| \begin{array}{ccc} \mathbb{R}^* & \longrightarrow & \mathbb{R} \\ x & \longmapsto & f(x) = \dfrac{1}{x} \end{array} \right.$$

is a mapping and $D_f = \mathbb{R}^*$ the departure set departure of f.

Definition 3.6 *Let E be a set, the mapping*

$$Id_E : \left| \begin{array}{ccc} E & \longrightarrow & E \\ x & \longmapsto & Id_E(x) = x, \end{array} \right.$$

It is called an identity function.

3.2 Composed of two functions

Let $f : E \longrightarrow F$ and $g : F \longrightarrow G$ two mapping. We call the composite of f and g the map denoted gof and defined as follows:

$$gof : \left| \begin{array}{ccc} E & \longrightarrow & F \\ x & \longmapsto & (gof)(x) = g[f(x)] \end{array} \right.$$

Example 3.4 *Does* $(gof) = (fog)$*?*
In general the answer is no. We give the following example:

$$f : \left| \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & f(x) = 5x - 1 \end{array} \right.$$

$$g : \left| \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & g(x) = x^2 \end{array} \right.$$

Let $x \in \mathbb{R}$

$$\begin{array}{rcl} (gof)(x) & = & g[f(x)] \\ & = & (f(x))^2 \\ & = & (5x-1)^2 \end{array}$$

$$\begin{array}{rcl} (fog)(x) & = & f[g(x)] \\ & = & 5g(x) - 1 \\ & = & 5x^2 - 1 \end{array}$$

$\forall x \in \mathbb{R},\ (gof)(x) \neq (fog)(x),$ hence $gof \neq fog$.

3.3 Injection - Surjection - Bijection

Let E, F non-empty sets and $f : E \rightarrow F$ a function

Definition 3.7 *f is* ***injective*** *or an* ***injection*** *if only if for all* $x, x' \in E$ *:*
si $f(x) = f(x')$ *then,* $x = x'$*, and we write*

$$f\,injective \Leftrightarrow \left[\forall x, x' \in E : \left(\left(f(x) = f(x') \right) \Longrightarrow \left(x = x' \right) \right) \right]$$

This definition means that when f is an injective function, it is impossible to find two distinct pre-images that have the same image. In other words, for all $y \in F$, the equation $y = f(x)$ has at most one solution $x \in E$.

Example 3.5 *It is very easy to see that the function*

$$f : \left| \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & f(x) = x^2 \end{array} \right.$$

is not injective. We can take as counterexample the case when $x = -1$ and $x' = 1$, we have $f(-1) = 1 = f(1)$, whereas $1 \neq -1$.

Definition 3.8 *We say that f is **surjective** or a **surjection** if and only if, for every $y \in F$, the equation $y = f(x)$ has at least one solution $x \in E$, and we write:*

$$f \ \textit{ is surjective} \Leftrightarrow \Big[\forall y \in F, \exists\, x \in E \ / \ y = f(x) \Big]$$

This definition shows that when f is a surjective function, we never find an element in F that does not have a pre-image in E.

Example 3.6 *We can immediately see that*

$$f : \left| \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & f(x) = x^2 \end{array} \right.$$

is not surjective because the equation $y = x^2$ has no solution when $y < 0$. In other words, if $y < 0$, then y has no pre-image x.

Definition 3.9 *The function f is **bijective** or a **bijection** if and only if it is both injective and surjective. In other words, f is bijective si est seulemnt if for all $y \in F$, the equation $y = f(x)$ has a unique solution $x \in E$, and we write*

$$f \ \textit{bijective} \ \Leftrightarrow \Big[\forall y \in F, \exists !\, x \in E \ / \ y = f(x) \Big]$$

Proposition 3.10 *Let E, F be two non empty sets. If the function $f : E \to F$ is bijective then f admits a has an **inverse** function denoted by f^{-1} which is also bijective and defined as follow:*

$$f^{-1} : \left| \begin{array}{ccc} F & \longrightarrow & E \\ y & \longmapsto & x = f^{-1}(y). \end{array} \right.$$

Example 3.7 *The function f defined by*

$$f : \left| \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R}^+ \\ x & \longmapsto & f(x) = e^x. \end{array} \right.$$

is bijective, then it admits an inverse function f^{-1} given by

$$f^{-1} : \left| \begin{array}{ccc} \mathbb{R}^+ & \longrightarrow & \mathbb{R} \\ y & \longmapsto & x = f^{-1}(y) = \ln(y), \end{array} \right.$$

and we write

$$f^{-1} : \left| \begin{array}{ccc} \mathbb{R}^+ & \longrightarrow & \mathbb{R} \\ x & \longmapsto & f^{-1}(x) = \ln(x), \end{array} \right.$$

Remark 3.2 *If $f : E \to F$ is bijective, then $fof^{-1} = I_F$ and $f^{-1}of = I_E$.*

Exercise 3.11 *Let us consider the function f defined by*

$$f : \left| \begin{array}{ccc} [1\ +\infty[& \longrightarrow & [0\ +\infty[\\ x & \longmapsto & f(x) = x^2 - 1, \end{array} \right.$$

f is it bijective? If your answer is positive, then give f^{-1}.

Solution 3.12 1. *We check if f is bijective.*

(a) *injectivity:*
Let $x, x' \in [1\ +\infty[$ such that $f(x) = f(x')$ a-t-on $x = x'$?

$$\begin{aligned} f(x) = f(x') &\Rightarrow x^2 - 1 = x'^2 - 1, \\ &\Rightarrow x^2 - x'^2 = 0, \\ &\Rightarrow \big(x - x'\big)\big(x + x'\big) = 0, \\ &\Rightarrow \big(x - x'\big) = 0 \text{ because } x + x' > 0, \\ &\Rightarrow x = x', \quad \text{then } f \text{ is injective.} \end{aligned}$$

(b) *surjectivity:*
Let $y \in [0\ +\infty[$, Does it exist $x \in [1\ +\infty[$ such that $y = f(x)$?

$$\begin{aligned} y = f(x) &\Rightarrow y = x^2 - 1, \\ &\Rightarrow x^2 = y + 1, \text{ avec } y + 1 > 0 \\ &\Rightarrow |x| = \sqrt{y+1}, \quad \big(|x| = x \text{ because } x > 0\big), \\ &\Rightarrow x = \sqrt{y+1} \in [1\ +\infty[, \quad \text{then } f \text{ is surjective.} \end{aligned}$$

   (c) *Since f is injective and surjective, then it is bijective.*

2. *Since f is bijective then it admits an inverse fonction f^{-1}defined as follow*

$$f^{-1}: \left| \begin{array}{rcl} [0\ +\infty[& \longrightarrow & [1\ +\infty[\\ y & \longmapsto & f^{-1}(y) = x = \sqrt{y+1}, \end{array} \right.$$

we write:

$$f^{-1}: \left| \begin{array}{rcl} [0\ +\infty[& \longrightarrow & [1\ +\infty[\\ x & \longmapsto & f^{-1}(x) = \sqrt{x+1}, \end{array} \right.$$

3.4 Direct image - Inverse image of a set

Definition 3.13 *Let E, F, A, B be sets such that $A \subset E$ et $B \subset F$, and f a function from E to F.*

- *The direct image of A by f, is the set denoted by $f(A)$ and defined by*

$$f(A) = \{f(x),\ x \in A\},$$

  or

$$f(A) = \{y \in F \mid y = f(x) \wedge x \in A\}.$$

- *The Inverse image of B, under the function f is the set denoted by $f^{-1}(B)$ and defined as follow*

$$f^{-1}(B) = \{x \in E,\ f(x) \in B\},$$

  or

$$f^{-1}(B) = \{x \in E \mid y = f(x) \wedge y \in B\}.$$

Remark 3.3 *It is important to note here that $f^{-1}(B)$ is just a notation, and f^{-1} does not represent the inverse function of f We do not require f to be bijective to determine $f^{-1}(B)$, It is emphasized that f is merely a function.*

3.5 Properties

Let us consider the function $f: \ E \ \to \ F$.
The following properties are very useful in exercises, especially when trying to find the direct or inverse image of a set expressed as intersections or unions of two or more sets.
Let $A, B \subset E$, and $A', B' \subset F$

- $f(A), f(B) \subset F; \ \ f^{-1}(A'), f^{-1}(B') \subset E.$
- $f(\emptyset) = \emptyset; \ \ f^{-1}(\emptyset) = \emptyset.$
- Si $A \subset B$ alors $f(A) \subset f(B)$.
- Si $A' \subset B'$ alors $f^{-1}(A') \subset f^{-1}(B')$.
- $f\left(A \cup B\right) = f\left(A\right) \cup f\left(B\right); \quad f^{-1}\left(A' \cup B'\right) = f^{-1}\left(A'\right) \cup f^{-1}\left(B'\right).$
- $f\left(A \cap B\right) \subset f\left(A\right) \cap f\left(B\right); \quad f^{-1}\left(A' \cap B'\right) = f^{-1}\left(A'\right) \cap f^{-1}\left(B'\right).$

Example 3.8 *Let us consider the function f*

$$f: \left| \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & f(x) = x^2 + 1, \end{array} \right.$$

$A = \big\{0, \, 3, \, 6, \, 8\big\}, \ \ B = \big\{0\big\}.$
$f(A) = \big\{f(x), \ x \in A\big\} = \big\{f(0), \ f(3), \ f(6), \ f(8)\big\} = \big\{0, \, 10, \, 37, \, 65\big\}.$
$f^{-1}(B) = \big\{x \in \mathbb{R}, \mid y = x^2 + 1 \ \ et \ \ y \in B\big\}.$
Since B contains only one element, which is 0, we solve the equation $y = x^2 + 1$ with $y = 0$. However, the equation $x^2 + 1 = 0$ has no solutions in $\mathbb{R}$, then $f^{-1}(B) = \emptyset$.

3.6 Characteristic function

Definition 3.14 *Let E be a set, and let $A \subset E$. We define the **characteristic** or the **indicator** of A as*

$$\chi_A: \left| \begin{array}{ccc} A & \longrightarrow & \{0, \, 1\} \\ x & \longmapsto & \chi_A(x) = \begin{cases} 1 \ \text{if} \, x \in A, \\ 0 \ \text{if} \, x \notin A. \end{cases} \end{array} \right.$$

Exercise 3.15 *Let us consider the function*

$$f : \left| \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & f(x) = \sqrt{x^2+2}. \end{array} \right.$$

1) *Provide the direct images by* f *of the sets* $A_1 = \{0,\ 2\},\ A_2 = [0\ 2]\,, A_3 = \big[-1\ 1\big]$.
2) *Provide the inverse images of the sets* $B_1 = \{0,\ 2,\ \frac{1}{2},\ \sqrt{2}\},\ B_2 = \big[1\ 3\big],\ B_3 = \{-4\}$.

Solution 3.16 • $f(A_1) = \{f(x),\ x \in A_1\} = \{f(0),\ f(2)\} = \{\sqrt{2},\ \sqrt{6}\}$
• $f(A_2) = \{f(x),\ x \in A_2\}$

As f *is continuous and a no-decreasing function on* $\mathbb{R}^+$, *then for* $x \in A_2$, *i.e.,* $0 \le x \le 2$, *we have* $f(0) \le f(x) \le f(2)$. *Hence* $f(A_2) = \big[\sqrt{2}\ \sqrt{6}\big]$.
• $f(A_3) = \{f(x),\ x \in A_3\} = \{f(x),\ x \in [-1\ \ 1]\}$.
There are two distinct cases: $x \in [-1\ \ 0]$ *and* $x \in [0\ \ 1]$.
Since f *is continuous and decreasing function on* $\mathbb{R}^-$,*for* $-1 \le x \le 0$, *we have* $f(0) \le f(x) \le f(-1)$, *i.e.,* $f([-1\ \ 0]) = \big[\sqrt{2}\ \ \sqrt{3}\big]$.
f *is continuous and a no-decreasing function on* $\mathbb{R}^+$, *then for* $x \in [0\ \ 1]$, *i.e.,* $0 \le x \le 1$, *we have* $f(0) \le f(x) \le f(1)$. *Hence,* $f([0\ 1]) = \big[\sqrt{2}\ \ \sqrt{3}\big]$.
In conclusion, $f(A_3) = \big[\sqrt{2}\ \ \sqrt{3}\big]$.

• $f^{-1}(B_1) = \{x \in \mathbb{R},\ f(x) \in B_1\} = \{x \in \mathbb{R},\ f(x) \in \{0,\ 2,\ \frac{1}{2},\ \sqrt{2}\}\}$.
$f(x) \in \{0,\ 2,\ \frac{1}{2},\ \sqrt{2}\} \Rightarrow \left((f(x) = 0) \vee (f(x) = 2) \vee \left(f(x) = \dfrac{1}{2} \right) \vee \big(f(x) = \sqrt{2}\big) \right)$.

$$\begin{array}{lll} f(x) = 0 & \Rightarrow & \sqrt{x^2+2} = 0, \\ & \Rightarrow & x^2 + 2 = 0, \\ & \Rightarrow & x^2 = -2, \end{array}$$

The equation $f(x) = 0$ *has no solutions* $\mathbb{R}$.

$$\begin{array}{lll} f(x) = 0 & \Rightarrow & \sqrt{x^2+2} = 2, \\ & \Rightarrow & x^2 + 2 = 4, \\ & \Rightarrow & x^2 = 2, \\ & \Rightarrow & \big(x = \sqrt{2}\big) \vee \big(x = -\sqrt{2}\big). \\ f(x) = 0 & \Rightarrow & \sqrt{x^2+2} = \frac{1}{2}, \\ & \Rightarrow & x^2 + 2 = \frac{1}{4}, \\ & \Rightarrow & x^2 = -\frac{7}{4}, \end{array}$$

The equation $f(x) = \dfrac{1}{2}$ *has no solutions in* $\mathbb{R}$.

$$\begin{aligned} f(x)=0 &\Rightarrow \sqrt{x^2+2}=\sqrt{2},\\ &\Rightarrow x^2+2=2,\\ &\Rightarrow x^2=0,\\ &\Rightarrow x=0. \end{aligned}$$

In conclusion, $f^{-1}(B_1)=\{\sqrt{2},\, -\sqrt{2},\, 0\}$

- $f^{-1}(B_2)=\Big\{x\in\mathbb{R},\ f(x)\in B_2\Big\}=\Big\{x\in\mathbb{R},\ f(x)\in\big[1\ 3\big]\Big\}.$

$$\begin{aligned} f(x)\in\big[0\ 3\big] &\Rightarrow \sqrt{x^2+2}\in\big[1\ 3\big],\\ &\Rightarrow 1\le\sqrt{x^2+2}\le 3,\\ &\Rightarrow 1\le x^2+2\le 9,\\ &\Rightarrow -1\le x^2\le 7, \end{aligned}$$

We consider two cases: $-1\le x^2\le 0$ *and* $0\le x^2\le 7$.

In the case: $-1\le x^2\le 0$ *we have not solutions.*

In the case: $0\le x^2\le 7$ *we have*

$$\begin{aligned} 0\le x^2\le 7 &\Rightarrow 0\le |x|\le\sqrt{7},\\ &\Rightarrow \Big(0\le x\le\sqrt{7}\Big)\vee\Big(0\le -x\le\sqrt{7}\Big),\\ &\Rightarrow \Big(0\le x\le\sqrt{7}\Big)\vee\Big(-\sqrt{7}\le x\le 0\Big),\\ &\Rightarrow \Big(x\in\big[0\ \ \sqrt{7}\big]\Big)\vee\Big(x\in\big[-\sqrt{7}\ \ 0\big]\Big),\\ &\Rightarrow x\in\big[0\ \ \sqrt{7}\big]\cup\big[-\sqrt{7}\ \ 0\big], \end{aligned}$$

$f^{-1}(B_2)=\big[0\ \ \sqrt{7}\big]\cup\big[-\sqrt{7}\ \ 0\big].$

- $f^{-1}(B_3)=\Big\{x\in\mathbb{R},\ f(x)\in B_3\Big\}=\Big\{x\in\mathbb{R},\ f(x)\in\big\{-4\big\}\Big\}.$

$$\begin{aligned} f(x)\in\big\{-4\big\} &\Rightarrow f(x)=-4,\\ &\Rightarrow \sqrt{x^2+2}=-4, \end{aligned}$$

l'équation $f(x)=0$ *has no solutions in* $\mathbb{R}$, *then* $f^{-1}(B_3)=\emptyset$.
