Chapter1:Logic and Methods of proofs

1 Introduction

Calculating and proving are the two main activities of mathematics. When we are interested in the activity of demonstrating, logic explains how a fact or an affirmation can flow from other facts already admitted. In mathematics, logic is the practice of rigor and exactness in thought. Therefore one can never do sound mathematical reasoning if one does not master the fundamental notions of mathematical logic. In this chapter, we present the elements of logic essential for any mathematical reasoning by direct or indirect methods.

2 Element of logic

Logic is interested in the rules of construction of mathematical sentences and on the other hand in their truth.Statements are atoms in mathematical logic.

2.1 Statements

Definition 2.1 *A statement is a sentence which is either true or false, but not both simultaneously*

- *when the statement is true we assign the value* 1 *or* T.
- *when the statement is false, it is assigned the value* 0 *or* F.

Truth and falisity of a statement is called its truth value. These truth values are organized in a table called a "truth table." This table shows all the possible combinations of truth values for different statements that make up a logical expression and indicates the resulting truth value of the expression. In mathematical logic, propositions are often denoted by letters such as $P, Q, \ldots,$

P
1
0

Example 2.1 *1. The sentence*

Ain Defla is in Algeria ; is true. So it is a statement.

2. *The sentence*

   "Every rectangle is a square" is false. So it is a statement.

3. *The sentence "open the door" can not be assigned true or false (Infact, it is a command). So it can not be called a statement.*

4. *The sentence "How old are you?" can not be assigned true or false (In fact, it is a question). So it is not a statement.*

5. *The truth or falsity of the sentence "x is a natural number" depends on the value of x. So it is not considered as a statement. However, in some books it is called an open statement.*

Remark 2.1 *No sentence can be called a statement if*

- *It is an exclamation*
- *It is an order or request*
- *It is a question*
- *It involves variable time such as 'today', 'tomorrow', 'yesterday' etc.*
- *It involves variable places such as 'here', 'there', 'everywhere' etc.*
- *It involves pronouns such as 'she', 'he', 'they' etc.*

Definition 2.2 *Simple statements*
A statement is called simple if it can not be broken down into two or more statements

Definition 2.3 *Compound statements*
A compound statement is the one which is made up of two or more simple statements.

2.2 Basic logical connectives

With the help of two (or several) statements one can define new propositions using logical connectives. We give the rules for these five connectors *'not,' 'and', 'or', 'if...then', 'if and only if'.*

Definition 2.4 ***(Negation: 'No','Note')***
An assertion that a statement fails or denial of a statement is called the negation of the statement. The negation of a statement is generally formed by introducing the word "not" at some proper place in the statement or by prefixing the statement with "It is not the case that" or It is false that" Let P be a statement. The negation of P is the statement denoted as $\overline{P}$, which is true if P is false, and vice versa.

Example 2.2 *The negation of P : "Khemis Miliana is a a city " is*
$\overline{P}$: *" Miliana is not a city " or* $\overline{P}$: *" It is false that Khemis Miliana is a a city. "*

Definition 2.5 ***(Conjunction: 'And')***
If two simple statements P and Q are connected by the word 'and' , then the resulting compound statement $(P \text{ and} Q)$ is called a conjunction of P and Q and is written in symbolic form as $(P \wedge Q)$. This new statement can be true only if both P and Q are true at the same time.

Example 2.3 *" 3 is a prime number and odd." And this statement is true. The number 3 is a prime number because it has no divisors other than 1 and itself. Additionally, it is odd because it is not divisible by 2.*

Definition 2.6 ***(Disjunction: 'Or')***
Let P and Q be two statements. The disjunction of P and Q is the proposition $(P \vee Q)$, which can only be false if both P and Q are false at the same time.

Definition 2.7 ***(The conditional statement)***
Recall that if P and Q are any two statements, then the compound statement "if p then q" formed by joining P and Q by a connective 'if then' is called a conditional statement or an implication and is written in symbolic form as $P \to Q$ or $P \Rightarrow Q$. Here, P is called hypothesis (or antecedent) and Q is called conclusion (or consequent) of the conditional statement $(P \Rightarrow Q)$.

Remark 2.2 *The conditional statement $(P \Rightarrow Q)$ can be expressed in several different ways. Some of the common expressions are :*

- *if* P*, then* Q
- Q *if* P
- P *only if* Q
- P *is sufficient for* Q
- Q *is necessary for* P*. Observe that the conditional statement* $(P \Rightarrow Q)$ *reflects the idea that whenever it is known that* P *is true, it will have to follow that* Q *is also true.*

Definition 2.8 ***(The biconditional statement)***
If two statements P *and* Q *are connected by the connective* ***'if and only if'*** *then the resulting compound statement " P if and only if Q" is called a biconditional of p and q and is written in symbolic form as* $(P \Longleftrightarrow Q)$*. We say that* P *and* Q *are equivalent when* P *and* Q *have the same truth values.*

Remark 2.3 *By using truth tables, we can show that:*

- $\overline{\overline{P}} \Leftrightarrow P$
- $(P \Leftrightarrow Q) \Leftrightarrow [(P \Rightarrow Q) \wedge (Q \Rightarrow P)]$.
- $(P \Rightarrow Q) \Leftrightarrow (\overline{P} \vee Q)$
- $(P \Rightarrow Q) \Leftrightarrow (\overline{Q} \Rightarrow \overline{P})$

Definition 2.9 ***(Contrapositive of a conditional statement)***
The statement $(\overline{Q} \Rightarrow \overline{P})$ *is called the contrapositive of the statement* $(P \Rightarrow Q)$.

Definition 2.10 ***(Converse of a conditional statement)***
The conditional statement $Q \Rightarrow P$ *is called the converse of the conditional statement* $(P \Rightarrow Q)$

Remark 2.4 *Attention! It is necessary to distinguish between the negation of an implication and the inverse of an implication. The converse of* $(P \Rightarrow Q)$ *is* $(Q \Rightarrow P)$*, but the negation of* $(P \Rightarrow Q)$ *is* $\overline{(P \Rightarrow Q)}$.

Exercise 2.11 *Find the statement which is equivalent to* $\overline{(P \Rightarrow Q)}$.

2.3 Morgan's Rules

Proposition 2.12 ***(Morgan's Rules)***
The Rules of Morgan are fundamental laws of logic that allow manipulation of logical operators for negation, conjunction, and disjunction. They are named after the British mathematician and logician Augustus De Morgan, who formulated them in the 19th century. Here are the two Rules of Morgan: Let P and Q be two logical statements, then:

1. *Negation of conjunction: The negation of a conjunction is equivalent to the disjunction of the negations of the individual propositions.*
2. *Negation of disjunction: The negation of a disjunction is equivalent to the conjunction of the negations of the individual propositions.*

In other words:

1. $\left(\overline{P \wedge Q}\right) \Leftrightarrow \left(\overline{P} \vee \overline{Q}\right)$.
2. $\left(\overline{P \vee Q}\right) \Leftrightarrow \left(\overline{P} \wedge \overline{Q}\right)$.

Proof. By using truth tables we get the required results.

P	Q	$\overline{P}$	$\overline{Q}$	$P \wedge Q$	$\overline{P} \vee \overline{Q}$	$\overline{P \wedge Q}$
1	1	0	0	1	0	0
0	0	1	1	0	1	1
0	1	1	0	0	1	1
1	0	0	1	0	1	1

In the same way one can do $\overline{P \vee Q}$ and $\overline{P} \wedge \overline{Q}$. ■

Theorem 2.13 *Let P, Q, R be three statements*

1. $(P \wedge Q) \Leftrightarrow (Q \wedge P)$ *et* $(P \vee Q) \Leftrightarrow (Q \vee P)$.
2. $[(P \wedge Q) \wedge R] \Leftrightarrow [P \wedge (Q \wedge R)]$ *et* $[(P \vee Q) \vee R] \Leftrightarrow [P \vee (Q \vee R)]$.
3. $[(P \wedge Q) \vee R] \Leftrightarrow [(P \vee R) \wedge (Q \vee R)]$ *et* $[(P \vee Q) \wedge R] \Leftrightarrow [(P \wedge R) \vee (Q \wedge R)]$.

The $\vee$ and the $\wedge$ are said to be commutative, associative and distributive with respect to each other. Commutativity means that the order of propositions does not matter when performing a disjunction or conjunction between them. We can swap the propositions without changing the truth value of the expression. Associativity means that we can group the propositions in any order, using parentheses, without changing the truth value of the expression.Distributivity allows us to distribute the conjunction or disjunction over the propositions that are combined inside parentheses.

Example 2.4 *Soit* $n \geq 2$, *We consider the implication* (I)*:*

$$[(n \text{ prime and } \; n \neq 2) \Rightarrow (n \text{ is odd})]\,.$$

The contrapositive of the proposition (I) *is:*

$$[(n \;\text{ is even}) \Rightarrow (n = 2 \text{ or } \; n \text{ is not prime})]\,.$$

The negation of (I) *is:*

$$[(n \text{ is prime and } \; n \neq 2) \text{ and } \; (n \text{ is even})]\,.$$

The converse of (I) *is:*

$$[(n \text{ is odd}) \Rightarrow (n \text{ is prime and } \; n \neq 2)]\,.$$

2.4 Quantifiers

2.4.1 Predicate

Let E be a set (for example: $\mathbb{R}, \mathbb{Z}, \mathbb{N}$) A predicate on E is any statement that contains one or more variables x, and if we replace x with a fixed element from E, we obtain a true or false proposition. A predicate is denoted by: $p(x), q(x), \ldots$

Example 2.5 *Here are two predicate examples*
$p(x) : x \;$ *is a multiple of* $7, \; x \in \mathbb{Z}$.
$q(x) : n \;$ *is a prime number* , $n \in \mathbb{N}$.

2.4.2 Quantificateur universel

The universal quantifier whose symbol is" $<< \forall >>$ means ¡¡ For all ¿¿, ¡¡ For any .... ¿¿. Let E be a given set, and $p(x)$ be a predicate on E. The following three statements: (a) for any x in $E : p(x)$,
(b) for every element x in $E : p(x)$,
(c) for any x in $E : p(x)$,
can be replaced by the notation "$\forall x \in E :; p(x)$."

Example 2.6 *The sentence "every real number is greater than or equal to 5." is written* " $\forall x \in \mathbb{R} : x \geq 5$"

Example 2.7 *The first proposition is true while the second is false*
- $\forall x \in \mathbb{N} \quad x + 1 > 0.$
- $\forall x \in \mathbb{Z} \quad x^2 + 3x - 2 < 0.$

2.4.3 Existential quantifier

In mathematical logic, the symbol $<< \exists >>$ (read as "there exists") represents the existential quantifier. It is used to state that there exists at least one element that satisfies a given predicate.

So, the statement: $\exists x \in E : P(x)$, can be read as "There exists an x in E such that $P(x)$. This means that there is at least one element x in the set E for which the predicate $P(x)$ is true.

Example 2.8 *The following statement*
"there exists at least one real number such that $x^2 - 5 > 0$."

In mathematical notation, it is written as:

$\exists x \in \mathbb{R} : x^2 - 5 > 0.$

The symbol "$\exists$" represents the existential quantifier, and it signifies the existence of at least one element that satisfies the predicate $x^2 - 5 > 0$ in the set of real numbers $\mathbb{R}$.

It is clear that a quantifier associated with a predicate gives a proposition which can be true or false.

Exercise 2.14 *Write the following sentences using quantifiers.*

1. *The square of any real number is positive.*
2. *For all real numbers the square of the sum of two numbers is equal to the sum of their squares.*
3. *Every integer has an opposite.*
4. *There is at least one integer which is opposite to all integers.*

Solution 2.15 *We obtain the following propositions:*

1. $\forall x \in \mathbb{R} \quad x^2 \geq 0.$ (T)
2. $\forall x \in \mathbb{R}, \forall y \in \mathbb{R} \quad (x+y)^2 = x^2 + y^2.$ (F)
3. $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, \quad x + y = 0.$ (T)
4. $\exists y \in \mathbb{Z}, \forall x \in \mathbb{Z}, \quad x + y = 0.$ (F)

2.4.4 Negation of a quantifier

Let E be a non-empty set and $p(x)$ a predicate on E.

The negation of a quantifier involves changing the meaning of the quantified statement from "for all" to "there exists" or vice versa.
In the case of the universal quantifier "$\forall$," the negation is the existential quantifier "$\exists$." Similarly, for the existential quantifier "$\exists$," the negation is the universal quantifier "$\forall$." the negation of a quantified statement is given as follows:
Negation of the universal quantifier $\forall x$:

$$\overline{(\forall x \in E : P(x))} \Longleftrightarrow \left(\exists x \in E : \overline{P(x)}\right)$$

$\left(\exists x \in E : \overline{P(x)}\right)$: means "There exists an $x \in E$ for which the predicate $P(x)$ is not true."
Negation of the existential quantifier $\exists x$:

$$\overline{(\exists x \in E : P(x))} \Longleftrightarrow \left(\forall x \in E : \overline{P(x)}\right)$$

$\left(\forall x \in E : \overline{P(x)}\right)$: means "For all $x \in E$, the predicate $P(x)$ is not true."
In simpler terms, negating a universal quantifier changes the statement from "All elements have property P" to "There exists at least one element that does not have property P." Negating an existential quantifier changes the statement from "At least one element has property P" to "All elements do not have property P."

Example 2.9 *The negation of the following two propositions is given as follows:*

- $\overline{\forall x \in \mathbb{R} \;\; x^2 \geq 0}$ *is* $\exists x \in \mathbb{R},\ x^2 < 0$.
- $\overline{\forall x \in \mathbb{Z},\ \exists y \in \mathbb{Z},\ \ x + y = 0}$ *is* $\exists x \in \mathbb{Z},\ \forall y \in \mathbb{Z},\ \ x + y \neq 0$.

3 Methods of Proofs

3.1 Direct Methods

Let P, Q be two given propositions.

- To show that the implication $(p \Rightarrow Q)$ is true, it is sufficient to assume that P is true and demonstrate that Q is also true.

- To prove that the equivalence $(p \Leftrightarrow Q)$ is true, it is sufficient to demonstrate that both implications $(P \Rightarrow Q)$ and $(Q \Rightarrow P)$ are true at the same time.

Example 3.1 *Prove that* $\forall x, y \in \mathbb{R}, \ \left(x^2 = y^2\right) \Rightarrow (|x| = |y|)$.

Solution 3.1 *Let x, y be two real numbers, we assume that $x^2 = y^2$ and we prove that $|x| = |y|$.*
Since x^2, y^2 are positive real numbers then by considering their square roots we have

$$\begin{aligned} x^2 = y^2 &\Rightarrow \sqrt{x^2} = \sqrt{y^2} \\ &\Rightarrow |x| = |y|. \end{aligned}$$

When direct methods of reasoning are not effective, we use other types of reasoning called indirect methods.

3.2 Indirect Methods

3.2.1 Proof by contradiction

In order to prove that a proposition P is true, we assume that $\overline{P}$ is true and from this assumption, we are able to arrive at or deduce a statement that contradicts some assumption we made in the proof or some known fact. We deduce that to prove that $(P \Rightarrow Q)$ is true, we assume that P is true and Q is false. Such assumptions will lead to a contradiction.
In fact, $\overline{(P \Rightarrow Q)} \Longleftrightarrow \left(P \wedge \overline{Q}\right)$. Assuming that $(P \Rightarrow Q)$ is false, this means that $\overline{(P \Rightarrow Q)}$ is true. Then obviosly $\left(P \wedge \overline{Q}\right)$ is true. From the truth table of a conjunction we must have P and $\overline{Q}$ true at the same time. In other words P true and Q false at the same time.

Example 3.2 *Using proof by contradiction, show that for every integer n, we have: $\left(n^2 \text{ is even } \Rightarrow n \text{ is even}\right)$.*

Solution 3.2 *we assume that n^2 is even and n is odd.*
n odd $\Rightarrow \exists k \in \mathbb{N}, n = 2k + 1$, hence :
$n^2 = (2k+1)^2 = 4k^2+4k+1 = 2(2k^2+2k)+1$. *Since $(2k^2+2k)$ is an integer we set: $k' = (2k^2 + 2k) \in \mathbb{N}$. Therefore, $n^2 = 2k' + 1$, which contradict the hypothesis n^2 is even.*

Example 3.3 *Prove that $\sqrt{2}$ is not a rational number.*
we recall that a rational number is an element of $\mathbb{Q} = \left\{ \frac{a}{b}, \;\; a \in \mathbb{Z}, \; b \in \mathbb{Z}^* \right\}.$

Proof. Proof by contradiction consists in assuming that $\sqrt{2} \in \mathbb{Q}$. and we must find a contradiction related to this assumption. $\sqrt{2} \in \mathbb{Q} \Longrightarrow \exists a \in \mathbb{Z}, and b \in \mathbb{Z}^*$ such that $\sqrt{2} = \frac{a}{b}$ with $GCD\,(a, b) = 1$. Then $\frac{a^2}{b^2} = 2$, which implies that $a^2 = 2b^2$ is even. By example 1 we deduce that a will be even. So it exists a integer k such that $a = 2k$ and $a^2 = 4k^2$. On the other hand $4k^2 = 2b^2$ implies that b^2 is even. Consequently, b is even. Since 2 divides a and 2 divides b, then it divides $GCD\,(a, b)$. In other words, $GCD\,(a, b) = 2n$, $n \in \mathbb{N}^*$. This is contradiction with hypothesis $GCD\,(a, b) = 1$. The contradiction comes from the assuption $\sqrt{2} \in \mathbb{Q}$. Then $\sqrt{2} \notin \mathbb{Q}$. ■

3.2.2 Proof by cases

Example 3.4 *Prove that the product of two consicutive integers is even*

Proof. Let n be an integer. Then n is either even or odd. The product of two consicutive integers is of the form $n\,(n + 1)$.

- Case 1:If n is even
  It exists $k \in \mathbb{N}$ such that $n = 2k$. Therefore, $n\,(n + 1) = 2k\,(2k + 1) = 2k'$ with $k' = k\,(2k + 1) \in \mathbb{N}$.

- Case 2: If n is odd
  It exists $k \in \mathbb{N}$ such that $n = 2k+1$. Therefore, $n\,(n + 1) = (2k + 1)\,(2k + 2) = 2k'$ with $k' = (2k + 1)\,(k + 1) \in \mathbb{N}$.

- Conclusion: The proposition is true for all $n \in \mathbb{N}$.

■

3.2.3 Proof by Contrapositive

It is based on the following property

$$(P \Longrightarrow Q) \Longleftrightarrow \left(\overline{Q} \Longrightarrow \overline{P}\right)$$

If the direct method does not help to prove that $(P \Longrightarrow Q)$, then we prove that its contrapositive hold. In other we prove that $\left(\overline{Q} \Longrightarrow \overline{P}\right)$ is true.

Example 3.5 *Let n be an integer. Prove that if $n^2 + 2n < 0$ then $n < 0$.*

Proof. $P: n^2+2n<0, \quad Q: n<0, \quad \overline{P}: n^2+2n \geq 0, \quad \overline{Q}: n \geq 0$. We use the contrapositive method. We will prove that if $n \geq 0$ then $n^2+2n \geq 0$. Assume that $n \geq 0$ then $2n \geq 0$. Since $n^2 \geq$, the sum of two positive numbers is positive. That is $n^2+2n \geq 0$. ■

3.2.4 Proof by counterexample

To prove that the proposition $(\forall x \in D, p(x))$ is not true, it is enought to find an element in D such that $p(x)$ is false for this element.

Example 3.6 *The proposition*
For all positive integer n, the number n^2-n+41 is prime. is false
Counterexample
For $n=41$, we have $(41)^2-41+41=(41)^2$ which is not prime.

3.2.5 if only if

We recall that $[(p \Longleftrightarrow q) \Longleftrightarrow [(p \Longrightarrow q) \wedge (q \Longrightarrow p))]$. In order to prove that $(p \Longleftrightarrow q)$ is true, we prove that $(p \Longrightarrow q)$ is true and $(q \Longrightarrow p)$ is also true.

Example 3.7 *Let n be an integer. To prove that $\left(n \text{ is odd if only if } n^2-1 \text{ is even}\right)$.*
we have to prove that $\left(n \text{ is odd } \Longrightarrow n^2-1 \text{ is even}\right)$ is true, and $\left(n^2-1 \text{ is even } \Longrightarrow n \text{ is odd}\right)$ is true.

3.2.6 Proof by induction

Let $n \in \mathbb{N}_0 \subseteq \mathbb{N}$ be a positive integer and $p(n)$ a predicate. To prove that the proposition $(\forall n \in \mathbb{N}_0, p(n))$ is true, we use a proof by induction.
Inductive process(Steps for proof by induction)

1. The basic step: we have to validate our statement by proving it is true when $n=n_0$ where n_0 is the first integer in $\mathbb{N}_0$,

2. The hypothesis: this step consists in assuming that the statement is true for some $n \geq n_0$ and we show that is also true for $n+1$,

3. The inductive step: then we conclude that $p(n)$ is true for all $n \in \mathbb{N}_0$.

Example 3.8 *Prove that* $\forall n \geq 1, 5+10+15+20+\ldots+5n=\dfrac{5n\,(n+1)}{2}$

Proof. *Let* $p(n): 5+10+15+20+\ldots+5n=\dfrac{5n\,(n+1)}{2}$ *and* $T_1 = 5+10+15+20+\ldots+5n$ *and* $T_2=\dfrac{5n\,(n+1)}{2}$.

1. *Step1: for* $n = 1, T_1 = 5, T_2 = \dfrac{5 \times 2}{2} = 5$. *That is* $p(1)$ *is satisfied.*

2. *Stem2: We assume that* $p(n)$ *is true for some* $n \geq 1$ *and we prove that* $p(n+1)$ *is true. That is we assume that*

$$5 + 10 + 15 + 20 + ... + 5n = \frac{5n\,(n+1)}{2}$$

and prove that

$$5 + 10 + 15 + 20 + ... + 5(n+1) = \frac{5(n+1)\,(n+2)}{2}$$

In fact,

$$\begin{aligned} 5 + 10 + 15 + 20 + ... + 5(n+1) &= 5 + 10 + 15 + 20 + ... + 5n + (n+1), \\ &= \frac{5n\,(n+1)}{2} + 5(n+1), \\ &= \frac{5n\,(n+1) + 10n + 10}{2}, \\ &= \frac{5\,(n+1)\,(n+2)}{2}, \end{aligned}$$

we deduce that $p(n+1)$ *is true,*

3. *Conclusion: from step1 and step2 we deduce that* $p(n)$ *is true for all* $n \in \mathbb{N}^*$.

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Example 3.9 *Prove that* $\forall n \in \mathbb{N},\ 2^n > n$.
Proof. $n \in \mathbb{N}$. *Let* $P(n)$ *be the predicate* $2^n > n$,

1. *For* $n = 0$ *we have* $2^0 = 1 > 0$. *Then* $P(0)$ *is true .*

2. *let* $n \geq 0$. *We assume that* $P(n)$ *is true and prove that* $P(n+1)$ *is true.*
In other words, we assume that $2^n > n$ *and prove that* $2^{n+1} > n + 1$.

$$\begin{aligned} 2^{n+1} &= 2.2^n, \\ &= 2^n + 2^n, \\ &> n + 1 \text{ since } 2^n > n \text{ and } 2^n \geq 1. \end{aligned}$$

Hence, $P(n+1)$ *is true.*

3. *Conclusion: from step1 and step2 we deduce that* $p(n)$ *is true for all* $n \in \mathbb{N}$.

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