Chapter 1

Simple and Multiple Integrals

The goal of this chapter is to review fundamental notions about integrals and primitives(antiderivatives), which are essential tools for differential equations and applications in physics

1.1 Review of the Riemann Integral and the Calculation of Primitives

The concept of the integral arises naturally in physics and mathematics whenever we want to compute a continuous quantity as the limit of discrete approximations. For instance, one can obtain the total mass from a density distribution, the total charge from a charge density, or the work from a force applied along a trajectory. The Riemann integral provides the rigorous framework for this transition from discrete sums to continuous accumulation.

Example: The work W done by a force $F(x)$ along a straight line:

$$W = \int_{x_0}^{x_1} F(x)\, dx$$

We can approximate this work by dividing the interval $[x_0, x_1]$ into small subintervals of width Δx. For each subinterval, we approximate the work by $F(x_i)\,\Delta x$. Taking the limit as $\Delta x \to 0$ gives the integral:

$$W = \lim_{\Delta x \to 0} \sum_i F(x_i)\,\Delta x$$

This provides an intuitive definition of the integral as the limit of the sum of small local contributions.

1.1.1 Function Riemann integrable

Let us consider some continuous positive function f defined on a bounded interval $[a\ b]$ where a and b are real numbers. A **partition** of $[a\ b]$ is a finite sequence

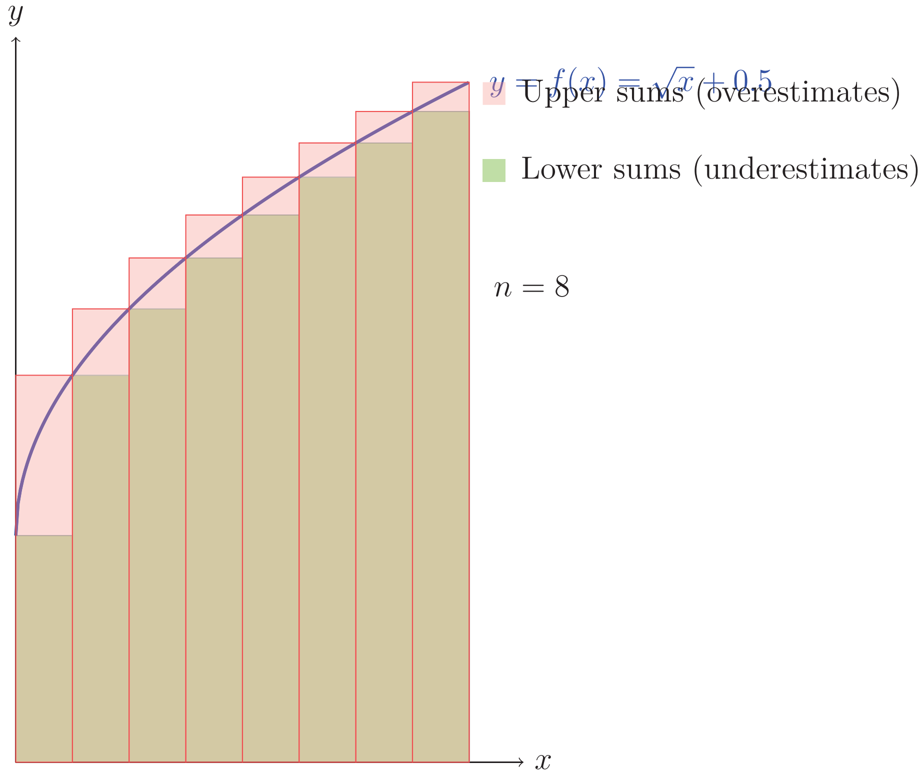
$$P = \{a = x_0 < x_1 < x_2 < \cdots < x_n = b\}.$$

For each subinterval $[x_{i-1}\ x_i]$, we define

$$m_i = \inf_{x \in [x_{i-1}\ x_i]} |f(x)|, \qquad M_i = \sup_{x \in [x_{i-1}\ x_i]} |f(x)|.$$

We then define the lower sum and upper sum

$$L(f,P)=\sum_{i=1}^{n}(x_i-x_{i-1})\,m_i,\quad U(f,P)=\sum_{i=1}^{n}(x_i-x_{i-1})\,M_i,$$



Definition 1.1 *The positive function f is Riemann integrable if*

$$\sup_P L(f,P)=\inf_P U(f,P),$$

and we call this common value the Riemann integral of f over $[a\ b]$ and ti is denoted by $\int_a^b f(x)dx$. Geometrically, the Riemann integral represents the area limited by the abscise axis and the curve with equation $y=f(x)$ and the straight lines with equations $x=a$ and $x=b$.

Definition 1.2 *A function $f:[a,b]\to\mathbb{R}$ is called* ***Riemann-integrable*** *if it is bounded and if the following limit exists:*

$$\lim_{\|P\|\to 0}\sum_{i=1}^{n}f(\xi_i)\,\Delta x_i,$$

where:

- $P=\{x_0,x_1,\ldots,x_n\}$ *is a partition of* $[a,b]$,
- $\Delta x_i=x_i-x_{i-1}$,
- $\xi_i\in[x_{i-1},x_i]$ *is any sample point in the subinterval, and*

- $\|P\| = \max_i \Delta x_i$ *is the norm of the partition (the length of the largest subinterval).*

If this limit exists and is independent of the choice of points ξ_i, it is called the ***Riemann integral*** *of f over $[a, b]$, denoted by*

$$\int_a^b f(x)\,dx.$$

Remark 1.1 *Functions Do Not Need to Be Positive to Be Riemann-Integrable.In fact A function $f : [a, b] \to \mathbb{R}$ does* ***not need to be positive*** *to have a Riemann integral. What matters is that f is* ***bounded*** *and that the limit of its Riemann sums exists as the partition is refined. In other words, Riemann-integrability depends on* ***boundedness*** *and the existence of the limit of Riemann sums,* ***not on the sign*** *of the function.*

Examples:

- *$f(x) = x$ on $[-1, 1]$ changes sign (negative for $x < 0$, positive for $x > 0$), but it is Riemann-integrable:*

$$\int_{-1}^{1} x\,dx = 0$$

- *$f(x) = \sin(x)$ on $[0, 2\pi]$ also changes sign, yet it is Riemann-integrable:*

$$\int_0^{2\pi} \sin(x)\,dx = 0$$

- *The Riemann integral always produces a number, which represents a cumulative quantity of the function over the interval.*

- *If the function $f(x)$ is* ***non-negative*** *on $[a, b]$, the integral corresponds to the* ***geometric area*** *under the curve.*

- *If the function $f(x)$* ***changes sign****, the integral represents the* ***net area*** *(algebraic sum of positive and negative contributions), and not the total geometric area.*

- *Therefore, the Riemann integral does* ***not always measure the literal surface****, but it always gives a meaningful accumulated value of the function over the interval.*

Examples of Geometric Area, Net Area, and Total Geometric Area

Example 1: Geometric Area (Positive Function)

Consider the function

$$f(x) = x^2 \quad \text{on } [0, 1].$$

Since $f(x) \geq 0$ for all $x \in [0, 1]$, the Riemann integral represents the actual geometric area under the curve:

$$\int_0^1 x^2\,dx = \frac{1}{3}.$$

Example 2: Net Area (Function Changing Sign)

Consider the function

$$f(x) = \begin{cases} 1, & 0 \leq x \leq 1, \\ -1, & 1 < x \leq 2 \end{cases}$$

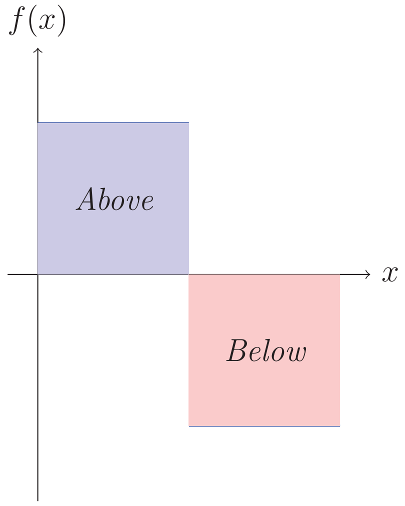
The integral gives the ***net area****, taking positive contributions above the x-axis and negative contributions below the x-axis: Although the geometric area of each rectangle is 1, the integral sums them algebraically, giving a net area of 0. Indeed:*

The geometric areas are:

- *Area above x-axis:* $1 \times 1 = 1$
- *Area below x-axis:* $1 \times 1 = 1$

The ***net area*** *(Riemann integral) is:*

$$\int_0^2 f(x)\,dx = 1 + (-1) = 0$$



Example 3: Total Geometric Area (Ignoring Sign)

The ***total geometric area*** *is the actual area under the curve,* ***ignoring the sign*** *of $f(x)$. We sum all parts, whether the function is above or below the x-axis.*

Consider the same function as in Example 2:

$$f(x) = \begin{cases} 1, & 0 \leq x \leq 1, \\ -1, & 1 < x \leq 2 \end{cases}$$

The total geometric area is

$$\int_0^1 |f(x)|\,dx + \int_1^2 |f(x)|\,dx = 1 + 1 = 2.$$

Here, we take the absolute value of the function to compute the ***real area under the curve****.*

Proposition 1.1 *Every continuous function on a closed interval $[a, b]$ is Riemann integrable.*

Remark 1.2 *This is a very useful property because it guarantees that most functions encountered in physics and engineering are automatically Riemann integrable.*

Properties of the Riemann Integral

Let f and g be Riemann-integrable functions on $[a, b]$, and let $\alpha, \beta \in \mathbb{R}$. The main properties are:

1. **Linearity:**
$$\int_a^b \Big(\alpha f(x) + \beta g(x)\Big)\,dx = \alpha \int_a^b f(x)\,dx + \beta \int_a^b g(x)\,dx$$
Interpretation: Constants can be factored out, and the integral of a sum is the sum of the integrals.

2. **Positivity:** If $f(x) \geq 0$ for all $x \in [a, b]$, then
$$\int_a^b f(x)\,dx \geq 0$$
Interpretation: The area under the curve is non-negative when the function is above the x-axis.

3. **Order (Monotonicity):** If $f(x) \leq g(x)$ for all $x \in [a, b]$, then
$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$$
Interpretation: A function that is larger everywhere has a larger integral.

4. **Additivity over intervals:** For $a < c < b$,
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$
Interpretation: The integral over an interval can be split into the sum of integrals over subintervals.

5. **Inequality with absolute value:**
$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx$$
Interpretation: The integral of the absolute value is always greater than or equal to the absolute value of the integral.

6.

Mean Value of a Riemann Integrable Function

Let f be a Riemann integrable function on the interval $[a, b]$. The **mean value or average value** of f over $[a, b]$ is defined by:

$$f_{\text{avg}} = m = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Explanation: This quantity represents the constant value that the function would need to take on the entire interval $[a, b]$ to produce the same total integral as $f(x)$ does. It is widely used in physics to compute mean quantities, such as mean velocity, mean force, or mean energy.

Remark 1.3 *If $f(x)$ is non-negative, f_{avg} can also be interpreted as the height of a rectangle over $[a, b]$ whose area equals the area under the curve of f.*

Root Mean Square (RMS) of a Riemann Integrable Function

Let f be a Riemann integrable function on the interval $[a, b]$. The **root mean square (RMS) value** of f over $[a, b]$ is defined by:

$$f_{\text{RMS}} = \sqrt{\frac{1}{b-a}\int_a^b [f(x)]^2\, dx}.$$

Explanation: The RMS value is a measure of the *effective magnitude* of a function. It is widely used in physics and engineering, for example to compute the effective voltage, current, or other quantities that vary in time.

Remark 1.4 *For a constant function $f(x) = C$, we have $f_{RMS} = |C|$. This generalizes the idea of magnitude for varying functions.*

1.1.2 Indefinite Integrals

Primitives

Example 1.1 (Introductory Example) *Consider the functions f and F defined on the interval $]0, +\infty[$ by*

$$F(x) = \ln(x) + \frac{1}{2}x^2 \quad \text{and} \quad f(x) = \frac{1}{x} + x.$$

We can easily see that both functions are continuous. Moreover, F is differentiable on $]0, +\infty[$ and

$$\forall x \in \,]0, +\infty[\,, \quad F'(x) = f(x).$$

*Such a function F is called an **antiderivative** (or **primitive**) of f on I.*

Definition 1.3 *Let $f(x)$ be a continuous function defined on an interval I. A function $F(x)$ is called an **indefinite integral** or **primitive** of f if*

$$F'(x) = f(x) \quad \text{for all } x \in I.$$

Example 1.2 *The function $F : x \longmapsto F(x) = \frac{1}{2}e^{2x+1}$ is a primitive of the function $f : x \longmapsto f(x) = e^{2x+1}$ because $F'(x) = e^{2x+1}$.*

Notation: The indefinite integral of f is denoted by

$$\int f(x)\, dx = F(x) + k,$$

where k is an arbitrary constant of integration.

Remark 1.5 *The indefinite integral represents the* family of all antiderivatives *of the function f. That is if F is a primitive of the continuous function f then $\int f(x)\, dx = F(x) + k$.*

Example 1.3 *Let* $F(x) = x\ln(x) - x$, *defined for* $x > 0$.
1. Prove that F *is a primitive of the function* $f(x) = \ln(x)$.
2. Find the set of all primitives of $f(x)$.

Solution 1.1 1. *By the product rule,*

$$F'(x) = \frac{d}{dx}[x\ln(x) - x] = 1 \cdot \ln(x) + x \cdot \frac{1}{x} - 1 = \ln(x) = f(x).$$

Thus, F *is a primitive of* $f(x) = \ln(x)$ *on* $(0, +\infty)$.

2. *Using the indefinite integral notation, we write*

$$\int \ln(x)\,dx = x\ln(x) - x + C, \quad C \in \mathbb{R}.$$

This expression represents the set of all primitives of $\ln(x)$ *on* $(0, +\infty)$.

Proposition 1.2 *Let* f *be a continuous function on* $I \subset \mathbb{R}$.

1. *If* F *is an antiderivative of* f, *then* $F + k$ *is also an antiderivative of* f, *where* k *is a constant function on* I.

2. *If* F *and* G *are two antiderivatives of* f, *then* $F - G$ *is constant. In other words, if* F *is an antiderivative of* f, *any other antiderivative* G *of* f *is of the form* $G = F + k$, *where* k *is constant.*

3. *If* F *is an antiderivative of* f *and* $x_0 \in I$, *then there exists a unique antiderivative* G *of* f *such that* $G(x_0) = k$.

Example 1.4 *Let* $F(x) = (x-1)e^x$ *defined* $\mathbb{R}$ *be a primitive of* $f(x) = xe^x$.
Find the primitive of f *which is equal to* 3 *at* 0

Solution 1.2 *Since* $F(x) = (x-1)e^x$ *is a primitive of* $f(x) = xe^x$, *then any other primitive* G *of* f *is of the form* $G(x) = F(x) + C$ *for all* $x \in \mathbb{R}$ *and where* C *is an arbitrary real number. Hence,* $G(0) = F(0) + C = -1 + C$.
Therefore,

$$G(0) = 3 \Longleftrightarrow -1 + C = 3,$$

that is $C = 4$. *and the required primitive is* $G(x) = (x-1)e^x + 4$.

Properties

1. Linearity: $\int \Big(af(x) + bg(x)\Big)\,dx = a\int f(x)\,dx + b\int g(x)\,dx$, with $a, b \in \mathbb{R}$.

2. Integration of a derivative: $\int F'(x)\,dx = F(x) + k$.

Example 1.5 $\int(-5x^2)\,dx = -5\int x^2\,dx$

Example 1.6 $\int(-5x^2 + \frac{1}{x})\,dx = \int -5x^2\,dx + \int \frac{1}{x}\,dx$

Some Immediate Indefinite Integrals

$\int 0\, dx = k, \quad k \in \mathbb{R}$	$\int \frac{1}{\cos^2(x)}\, dx = \tan(x) + k$
$\int a\, dx = ax + k, \quad k \in \mathbb{R}$	$\int \frac{1}{\sin^2(x)}\, dx = -\cot(x) + k$
$\int x^m\, dx = \frac{x^{m+1}}{m+1} + k, \quad k \in \mathbb{R}, m \neq -1$	$\int \frac{1}{\sqrt{1-x^2}}\, dx = \arcsin(x) + k$
$\int \frac{1}{x}\, dx = \ln(\lvert x\rvert) + k, \quad k \in \mathbb{R}$	$\int \frac{-1}{\sqrt{1-x^2}}\, dx = \arccos(x) + k$
$\int e^x\, dx = e^x + k, \quad k \in \mathbb{R}$	$\int \frac{1}{1+x^2}\, dx = \arctan(x) + k$
$\int \sin(x)\, dx = -\cos(x) + k, \quad k \in \mathbb{R}$	$\int \frac{-1}{1+x^2}\, dx = \mathrm{arccotan}(x) + k$
$\int \cos(x)\, dx = \sin(x) + k, \quad k \in \mathbb{R}$	$k \in \mathbb{R}$

Remark 1.6 *In the general case,*

$$\int (f(x) \times g(x))\, dx \neq \int f(x)\, dx \times \int g(x)\, dx.$$

The question we ask is: how can we compute an antiderivative of a product of two functions?

Integration by Parts

Integration by parts is a method used to find the integral of a product of two functions.

Theorem 1.1 (Integration by Parts) *Let $u(x)$ and $v(x)$ be differentiable functions. Then,*

$$\int u(x)\, v'(x)\, dx = u(x)v(x) - \int u'(x)\, v(x)\, dx.$$

Remark 1.7 *The formula is derived from the product rule for differentiation:*

$$(u(x)v(x))' = u'(x)v(x) + u(x)v'(x).$$

Example 1.7 *Compute $\int xe^x\, dx$.*

Solution: *Take $u = x$ and $dv = e^x dx$. Then $du = dx$ and $v = e^x$. Applying the formula:*

$$\int xe^x\, dx = xe^x - \int 1 \cdot e^x\, dx = xe^x - e^x + k.$$

Although integration by parts is an important tool for computing the antiderivatives of products of functions, it does not allow us to calculate all types of function products. Another widely used integration technique is integration by substitution (change of the integration variable).

Integration by Substitution (Change of Variable)

The following integration formulas follow directly from the rules of differentiation by using a simple change of variable.
We set $y = f(x)$, which implies

$$\frac{dy}{dx} = f'(x),$$

hence

$$f'(x)\,dx = dy.$$

The following table is obtained by using the technic of variable change

$$\int (f(x))^{n+1} f'(x)\,dx = \frac{(f(x))^{n+1}}{n+1} + k, \quad k \in \mathbb{R}.$$
$$\int f'(x)e^{f(x)}dx = e^{f(x)} + k, \quad k \in \mathbb{R}.$$
$$\int \frac{f'(x)}{f(x)} = \ln|f(x)| + k, \quad k \in \mathbb{R}.$$
$$\int f'(x)\sin(f(x))\,dx = -\cos(f(x)) + k, \quad k \in \mathbb{R}.$$
$$\int f'(x)\cos(f(x))\,dx = -\sin(f(x)) + k, \quad k \in \mathbb{R}.$$
$$\int \frac{f'(x)}{1+f^2(x)} = \arctan(f(x)) + k, \quad k \in \mathbb{R}.$$
$$\int \frac{f'(x)}{\sqrt{1-f^2(x)}} = \arcsin(f(x)) + k, \quad k \in \mathbb{R}.$$
$$\int \frac{-f'(x)}{\sqrt{1-f^2(x)}} = \arccos(f(x)) + k, \quad k \in \mathbb{R}.$$

Exercise 1.1 *Prove the results of the table above.*

Examples 1.1 *1)* $\int (2x+5)^3\,dx = \frac{1}{2}\int 2(2x+5)^3\,dx = \frac{1}{8}(2x+5)^4 + k, \quad k \in \mathbb{R}.$
2) $\int e^{3x+1}\,dx = \frac{1}{3}\int 3e^{3x+1}\,dx = \frac{1}{3}e^{3x+1} + k, \quad k \in \mathbb{R}.$
3) $\int \frac{2x}{1+(x)^2} = \arctan(x^2) + k, \quad k \in \mathbb{R}.$
4) $\int \frac{2x}{\sqrt{1-(x)^2}} = \arcsin(x^2) + k, \quad k \in \mathbb{R}.$

We can generalize this result as follows:

Proposition 1.3 *Let g be a continuous function on $I \subset \mathbb{R}$ and G its antiderivative on I, and let f be a differentiable function on I with derivative f'. Then*

$$\int f'(x)g(f(x))\,dx = G(f(x)) + k, \quad k \in \mathbb{R}.$$

Proof. Set $u = f(x)$, hence $\frac{du}{dx} = f'(x)$, which gives $du = f'(x)\,dx$.
It follows that

$$\int f'(x)g(f(x))\,dx = \int g(u)\,du = G(u) + k = G(f(x)) + k, \quad k \in \mathbb{R}.$$

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Example 1.8 *Compute*

$$\int x(3x+4)^5\,dx.$$

Set $u = 3x+4$, *so that* $\dfrac{du}{dx} = 3$, *hence* $dx = \dfrac{1}{3}du$.

Moreover, $x = \dfrac{u-4}{3} = \dfrac{1}{3}u - \dfrac{4}{3}$.
Then we obtain

$$\begin{aligned}\int x(3x+4)^5\,dx &= \int \left(\frac{1}{3}u - \frac{4}{3}\right) u^5 \cdot \frac{1}{3}\,du \\ &= \frac{1}{9}u^6 - \frac{4}{9}u^5 + k, \quad k \in \mathbb{R} \\ &= \frac{1}{9}(3x+4)^6 - \frac{4}{9}(3x+4)^5 + k, \quad k \in \mathbb{R}.\end{aligned}$$

Unfortunately, the previous rules are not always directly applicable. In some cases, we need to simplify the function before calculating its antiderivative. This is, for example, the case for rational functions.

Integration of the function $x \longmapsto \dfrac{1}{ax^2+bx+c}$

We aim to calculate the antiderivatives of the function $x \mapsto \dfrac{1}{ax^2+bx+c}$, $a, b, c \in \mathbb{R}$.
First case: If $a = 0, b \neq 0, c \in \mathbb{R}$.

$$\int \frac{1}{bx+c}\,dx = \frac{1}{b}\int \frac{b}{bx+c}\,dx = \frac{1}{b}\ln|bx+c| + k, \quad k \in \mathbb{R}.$$

Example 1.9 $\displaystyle\int \frac{1}{2x+1}\,dx = \frac{1}{2}\ln|2x+1| + k, \quad k \in \mathbb{R}.$

Second case: If $a \neq 0, b, c \in \mathbb{R}$
We have

$$ax^2+bx+c = a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{\Delta}{4a^2}\right].$$

The integral to calculate depends on Δ, and we distinguish three situations.

1. If $\Delta < 0$ we set $\Delta = -\rho$, $\rho > 0$. Then we have:

$$ax^2+bx+c = \frac{\rho}{4a}\left[\left(\frac{2a}{\sqrt{\rho}}x + \frac{b}{\sqrt{\rho}}\right)^2 + 1\right],$$

hence

$$\frac{1}{ax^2+bx+c} = \frac{4a}{\rho}\frac{1}{\left(\frac{2a}{\sqrt{\rho}}x + \frac{b}{\sqrt{\rho}}\right)^2 + 1}.$$

Using the change of variable $u = \dfrac{2a}{\sqrt{\rho}}x + \dfrac{b}{\sqrt{\rho}}$,

we have $dx = \dfrac{\sqrt{\rho}}{2a}\, du$. Therefore,

$$\begin{aligned}
\int \frac{1}{ax^2+bx+c}\, dx &= \frac{2}{\sqrt{\rho}} \int \frac{1}{u^2+1}\, du \\
&= \frac{2}{\sqrt{\rho}} \arctan(u) + k, \quad k \in \mathbb{R}, \\
&= \frac{2}{\sqrt{\rho}} \arctan\left(\frac{2a}{\sqrt{\rho}}x + \frac{b}{\sqrt{\rho}}\right) + k, \quad k \in \mathbb{R}.
\end{aligned}$$

Example 1.10 *Calculate* $\displaystyle\int \frac{1}{2x^2+x+5}\, dx$.

$\Delta = -39, \rho = 39, a = 2, u = \dfrac{4}{\sqrt{39}}x + \dfrac{1}{\sqrt{39}}$

$$\begin{aligned}
\int \frac{1}{2x^2+x+5}\, dx &= \frac{2}{\sqrt{39}} \int \frac{1}{u^2+1}\, du, \\
&= \frac{2}{\sqrt{39}} \arctan(u) + k, \quad k \in \mathbb{R}, \\
&= \frac{2}{\sqrt{39}} \arctan\left(\frac{4}{\sqrt{39}}x + \frac{1}{\sqrt{39}}\right) + k, \quad k \in \mathbb{R}.
\end{aligned}$$

2. If $\Delta = 0$, then $ax^2 + bx + c = a\left(x + \dfrac{b}{2a}\right)^2$.

   Using the change of variable $u = x + \dfrac{b}{2a}$, $du = dx$, we have

$$\begin{aligned}
\int \frac{1}{ax^2+bx+c}\, dx &= \int \frac{1}{a\left(x+\frac{b}{2a}\right)^2}\, dx, \\
&= \frac{1}{a} \int \frac{1}{u^2}\, du, \\
&= \frac{1}{a} \int u^{-2}\, du, \\
&= \frac{1}{a} \frac{u^{-1}}{-1} + k, \quad k \in \mathbb{R}, \\
&= -\frac{1}{a}\frac{1}{u} + k, \quad k \in \mathbb{R}, \\
&= -\frac{1}{a} \frac{1}{x + \frac{b}{2a}} + k, \quad k \in \mathbb{R}.
\end{aligned}$$

Example 1.11

$$\begin{aligned}\int \frac{1}{3x^2+6x+3}\,dx &= \int \frac{1}{3\left(x-1\right)^2}\,dx,\\ &= \frac{1}{3}\int \frac{1}{u^2}\,du,\\ &= \frac{1}{3}\left(\frac{u^{-1}}{-1}\right)+k, \quad k\in\mathbb{R},\\ &= \frac{-1}{3\left(x-1\right)}+k, \quad k\in\mathbb{R}.\end{aligned}$$

3. If $\Delta > 0$, then $ax^2+bx+c = a\left(x-x_1\right)\left(x-x_2\right)$.

   We decompose the fraction $\dfrac{1}{ax^2+bx+c}$ as

$$\frac{1}{ax^2+bx+c} = \frac{1}{a}\left[\frac{A}{x-x_1}+\frac{B}{x-x_2}\right],$$

   where A, B are real numbers determined after reducing to a common denominator and identification.

$$\begin{aligned}\int \frac{1}{ax^2+bx+c}\,dx &= \int \frac{1}{a}\left[\frac{A}{x-x_1}+\frac{B}{x-x_2}\right]dx,\\ &= \frac{1}{a}\left[\int \frac{A}{x-x_1}\,dx+\int \frac{B}{x-x_2}\,dx\right],\\ &= \frac{A}{a}\ln\left|x-x_1\right|+\frac{B}{a}\ln\left|x-x_2\right|+k, \quad k\in\mathbb{R}.\end{aligned}$$

Example 1.12 *Calculate*

$$\int \frac{1}{x^2-5x+6}\,dx$$

We have

$$x^2-5x+6=(x-2)(x-3).$$

It follows that

$$\int \frac{1}{x^2-5x+6}\,dx = A\ln|x-2|+B\ln|x-3|+k,$$

where A and B satisfy

$$\begin{aligned}\frac{1}{(x-2)(x-3)} &= \frac{A}{x-2}+\frac{B}{x-3},\\ &= \frac{Ax-3A+Bx-2B}{x^2-5x+6},\\ &= \frac{(A+B)x-3A-2B}{x^2-5x+6},\\ &\Longrightarrow \left[(A+B)=0\right]\wedge\left[-3A-2B=1\right],\\ &\Longrightarrow \left[A=-B\right]\wedge\left[-3A-2B=1\right],\\ &\Longrightarrow \left[A=-1\wedge B=1\right].\end{aligned}$$

Hence,

$$\int \frac{1}{x^2-5x+6}\,dx = -\ln|x-2|+\ln|x-3|+k = \ln\left|\frac{x-3}{x-2}\right|+k, \quad k\in\mathbb{R}.$$

Integration of the function $x \longmapsto \dfrac{Ax+B}{ax^2+bx+c}$, $a \neq 0, A \neq 0$

We start by simplifying the fraction $\dfrac{Ax+B}{ax^2+bx+c}$ by making the derivative of the denominator appear. We have:

$$\begin{aligned} Ax+B &= Ax+B+\frac{Ab}{2a}-\frac{Ab}{2a} \\ &= \left(Ax+\frac{Ab}{2a}\right)+\left(B-\frac{Ab}{2a}\right) \\ &= \frac{A}{2a}(2ax+b)+\left(B-\frac{Ab}{2a}\right), \end{aligned}$$

hence

$$\begin{aligned} \frac{Ax+B}{ax^2+bx+c} &= \frac{\frac{A}{2a}(2ax+b)+\left(B-\frac{Ab}{2a}\right)}{ax^2+bx+c} \\ &= \frac{A}{2a}\frac{2ax+b}{ax^2+bx+c}+\left(B-\frac{Ab}{2a}\right)\frac{1}{ax^2+bx+c}. \end{aligned}$$

Therefore,

$$\int \frac{Ax+B}{ax^2+bx+c}\,dx = \frac{A}{2a}\int \frac{2ax+b}{ax^2+bx+c}\,dx + \left(B-\frac{Ab}{2a}\right)\int \frac{1}{ax^2+bx+c}\,dx.$$

By using direct integration, and letting

$$I_1 = \int \frac{1}{ax^2+bx+c}\,dx,$$

we get

$$\int \frac{Ax+B}{ax^2+bx+c}\,dx = \frac{A}{2a}\ln\left|ax^2+bx+c\right| + \left(B-\frac{Ab}{2a}\right)I_1 + k, \quad k \in \mathbb{R}.$$

Remark 1.8 *We recall that I_1 can be computed using one of the methods described in the previous subsection.*

Example 1.13

$$\int \frac{3x+4}{2x^2-5x+6} = \frac{3}{4}\ln\left|2x^2-5x+6\right| + \frac{31}{4}\int \frac{1}{2x^2-5x+6}\,dx + k, \quad k \in \mathbb{R}.$$

Integration of the function $x \longmapsto \dfrac{1}{\sqrt{ax^2+bx+c}}$, $a \neq 0$

1. If $a > 0$ and $\Delta = 0$, then

$$\sqrt{ax^2+bx+c} = \sqrt{a}\left|x+\frac{b}{2a}\right|,$$

and we have:

$$\int \frac{1}{\sqrt{ax^2+bx+c}}\,dx = \frac{1}{\sqrt{a}}\int \frac{1}{\left|x+\frac{b}{2a}\right|}\,dx.$$

2. If $a > 0$ and $\Delta < 0$, then

$$\sqrt{ax^2+bx+c} = \sqrt{a}\sqrt{\left(x+\frac{b}{2a}\right)^2 + m^2}, \quad \text{where } m^2 = \frac{-\Delta}{4a^2}.$$

$$\begin{aligned}
\int \frac{1}{\sqrt{ax^2+bx+c}}\,dx &= \int \frac{1}{\sqrt{a}\sqrt{\left(x+\frac{b}{2a}\right)^2 + m^2}}\,dx \\
&= \frac{1}{\sqrt{a}}\int \frac{1}{\sqrt{u^2+m^2}}\,du \\
&= \frac{1}{\sqrt{a}} \ln\left|u+\sqrt{u^2+m^2}\right| + k, \quad k \in \mathbb{R} \\
&= \frac{1}{\sqrt{a}} \ln\left|\left(x+\frac{b}{2a}\right) + \sqrt{\left(x+\frac{b}{2a}\right)^2 - \frac{\Delta}{4a^2}}\right| + k, \quad k \in \mathbb{R}.
\end{aligned}$$

A simple calculation shows that the derivative of the function $u \longmapsto \ln\left|u+\sqrt{u^2+m^2}\right|$ is $u \longmapsto \dfrac{1}{\sqrt{u^2+m^2}}$.

Example 1.14

$$\int \frac{1}{\sqrt{5x^2+1}}\,dx = \frac{1}{\sqrt{5}} \ln\left|x+\sqrt{x^2+\frac{1}{5}}\right| + k, \quad k \in \mathbb{R}.$$

3. If $a > 0$ and $\Delta > 0$, then $ax^2+bx+c > 0$ on $]-\infty, x_1[\,\cup\,]x_2, +\infty[$, and we have

$$\sqrt{ax^2+bx+c} = \sqrt{a}\sqrt{\left(x+\frac{b}{2a}\right)^2 - m^2}, \quad \text{where } m^2 = \frac{\Delta}{4a^2}.$$

$$\begin{aligned}
\int \frac{1}{\sqrt{ax^2+bx+c}}\,dx &= \int \frac{1}{\sqrt{a}\sqrt{\left(x+\frac{b}{2a}\right)^2 - m^2}}\,dx \\
&= \frac{1}{\sqrt{a}}\int \frac{1}{\sqrt{u^2-m^2}}\,du \\
&= \frac{1}{\sqrt{a}} \ln\left|u+\sqrt{u^2-m^2}\right| + k, \quad k \in \mathbb{R} \\
&= \frac{1}{\sqrt{a}} \ln\left|\left(x+\frac{b}{2a}\right) + \sqrt{\left(x+\frac{b}{2a}\right)^2 - \frac{\Delta}{4a^2}}\right| + k, \quad k \in \mathbb{R}.
\end{aligned}$$

A simple calculation shows that the derivative of the function

$$u \longmapsto \ln\left|u+\sqrt{u^2-m^2}\right|$$

is the function

$$u \longmapsto \frac{1}{\sqrt{u^2-m^2}}.$$

Example 1.15

$$\int \frac{1}{\sqrt{4x^2-1}}\,dx = \frac{1}{2}\ln\left|x+\sqrt{x^2-\frac{1}{4}}\right| + k, \quad k \in \mathbb{R}.$$

4. If $a < 0$ and $\Delta < 0$ then $\sqrt{ax^2+bx+c}$ is not defined.

5. If $a < 0$ and $\Delta = 0$ then $\sqrt{ax^2+bx+c}$ is not defined on $\mathbb{R} \setminus \left\{-\frac{b}{2a}\right\}$.

6. If $a < 0$ and $\Delta > 0$ then $\sqrt{ax^2+bx+c}$ is defined on the interval $]x_1, x_2[$. Moreover,

$$ax^2+bx+c = a\left[\left(x+\frac{b}{2a}\right)^2 - \frac{\Delta}{4a^2}\right].$$

In this case, let $a = -a'$, with $a' > 0$, and we get

$$\begin{aligned} ax^2+bx+c &= -a'\left[\left(x+\frac{b}{2a}\right)^2 - \frac{\Delta}{4a^2}\right] \\ &= a'\left[-\left(x+\frac{b}{2a}\right)^2 + \frac{\Delta}{4a^2}\right] \\ &= \frac{\Delta}{4a'}\left[1-\left(\frac{2a'}{\sqrt{\Delta}}x - \frac{b}{\sqrt{\Delta}}\right)^2\right]. \end{aligned}$$

Setting

$$u = \frac{2a'}{\sqrt{\Delta}}x - \frac{b}{\sqrt{\Delta}}, \quad dx = \frac{\sqrt{\Delta}}{2a'}\,du,$$

we then have

$$\begin{aligned} \int \frac{1}{\sqrt{ax^2+bx+c}}\,dx &= \frac{1}{\sqrt{a'}}\int \frac{1}{\sqrt{1-u^2}}\,du \\ &= \frac{1}{\sqrt{a'}}\arcsin(u) + k, \quad k \in \mathbb{R} \\ &= \frac{1}{\sqrt{a'}}\arcsin\left(\frac{2a'}{\sqrt{\Delta}}x - \frac{b}{\sqrt{\Delta}}\right) + k, \quad k \in \mathbb{R}. \end{aligned}$$

Example 1.16

$$\int \frac{1}{\sqrt{1-4x^2}}\,dx = \frac{1}{2}\arcsin(2x) + k, \quad k \in \mathbb{R}.$$

Techniques for Decomposing a Fraction into Partial Fractions

In the following, we aim to integrate a fraction of the form

$$\frac{P(x)}{Q(x)},$$

where $P(x)$ and $Q(x)$ are polynomials of respectively degrees n and m ($n, m \in \mathbb{N}$), with $Q(x)$ factorable.

We start by simplifying the fraction $\frac{P(x)}{Q(x)}$ depending on the values of n and m. We distinguish three cases: $n < m$, $n = m$, and $n > m$.

1. If $n < m$:

   - If
   $$Q(x) = (a_1x + b_1)(a_2x + b_2)\cdots(a_mx + b_m),$$
   then
   $$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \cdots + \frac{A_m}{a_mx + b_m},$$
   where $A_1, A_2, \ldots, A_m$ are real numbers determined by identification.

   Example 1.17 *Simplify the fraction* $\frac{x}{x^2-1}$.
   We have $\deg(x) = 1 < \deg(x^2 - 1) = 2$*, and* $x^2 - 1 = (x - 1)(x + 1)$*. Therefore,*
   $$\begin{aligned}\frac{x}{x^2 - 1} &= \frac{A_1}{x - 1} + \frac{A_2}{x + 1} \\ &= \frac{A_1(x + 1)}{x^2 - 1} + \frac{A_2(x - 1)}{x^2 - 1} \\ &= \frac{(A_1 + A_2)x + (A_1 - A_2)}{x^2 - 1}.\end{aligned}$$
   By identification:
   $$\begin{cases} A_1 + A_2 = 1 \\ A_1 - A_2 = 0 \end{cases} \Rightarrow A_1 = A_2 = \frac{1}{2}.$$
   Hence,
   $$\frac{x}{x^2 - 1} = \frac{\frac{1}{2}}{x - 1} + \frac{\frac{1}{2}}{x + 1}.$$
   Application:
   $$\begin{aligned}\int \frac{x}{x^2 - 1}\,dx &= \int \frac{\frac{1}{2}}{x - 1}\,dx + \int \frac{\frac{1}{2}}{x + 1}\,dx \\ &= \frac{1}{2}\int \frac{1}{x - 1}\,dx + \frac{1}{2}\int \frac{1}{x + 1}\,dx \\ &= \frac{1}{2}\ln|x - 1| + \frac{1}{2}\ln|x + 1| + k, \quad k \in \mathbb{R}.\end{aligned}$$

   - If $Q(x)$ contains at least one factor with a power:

   Example 1.18
   $$\frac{x + 1}{x(x - 2)^2(x - 1)^3} = \frac{A_1}{x} + \frac{A_2}{x - 2} + \frac{A_3}{(x - 2)^2} + \frac{A_4}{x - 1} + \frac{A_5}{(x - 1)^2} + \frac{A_6}{(x - 1)^3}$$

   - If $Q(x)$ contains a quadratic factor $ax^2 + bx + c$ with $\Delta < 0$:

Example 1.19

$$\frac{x^3+x^2+1}{x(x-2)^2(x^2+1)^3(x^2+x+2)} = \frac{A_1}{x} + \frac{A_2}{x-2} + \frac{A_3}{(x-2)^2} + \frac{A_4x+A_5}{x^2+1} + \frac{A_6x+A_7}{(x^2+1)^2} + \frac{A_8x+A_9}{(x^2+1)^3} + \frac{B_1x+B_2}{x^2+x+2}$$

2. If $n = m$, then
$$\frac{P(x)}{Q(x)} = A + \frac{R(x)}{Q(x)},$$
where $R(x)$ is a polynomial of degree less than $\deg(Q(x))$, and $A \in \mathbb{R}$ is determined by Euclidean division or identification. The fraction $\frac{R(x)}{Q(x)}$ is then decomposed as in case 1.

Example 1.20
$$\frac{x^2+4}{3x^2+x+1} = \frac{1}{3} + \frac{\frac{1}{3}(-x+11)}{3x^2+x+1}.$$

3. If $n > m$, then
$$\frac{P(x)}{Q(x)} = K(x) + \frac{R(x)}{Q(x)},$$
where $K(x)$ and $R(x)$ are the quotient and remainder of the Euclidean division of $P(x)$ by $Q(x)$. Clearly, $\deg(R(x)) < \deg(Q(x))$, so we reduce $\frac{R(x)}{Q(x)}$ to case 1.

Bioche's Rules

In this section, we focus on the computation of integrals of rational functions in sine and cosine. Bioche's rules allow us to simplify fractions of the form

$$\frac{P(\sin(x), \cos(x))}{Q(\sin(x), \cos(x))}$$

by reducing them to integrals of simpler rational fractions. We aim to calculate the following integral:
$$I = \int \frac{P(\sin(x), \cos(x))}{Q(\sin(x), \cos(x))}\, dx. \tag{1.1}$$

We set
$$f(x) = \frac{P(\sin(x), \cos(x))}{Q(\sin(x), \cos(x))}\, dx.$$

The following Bioche's rules allow us to determine the appropriate substitution to compute the integral of type (1.1):

1. If $f(-x) = f(x)$, we set $t = \cos(x)$.

2. If $f(\pi - x) = f(x)$, we set $t = \sin(x)$.

3. If $f(\pi + x) = f(x)$, we set $t = \tan(x)$.

4. If none of the previous three conditions is satisfied, then we set $t = \tan\left(\frac{x}{2}\right)$. In this case, we have:

$$\sin(x) = \frac{2t}{1+t^2}, \quad \cos(x) = \frac{1-t^2}{1+t^2}, \quad \tan(x) = \frac{2t}{1-t^2}, \quad dx = \frac{2}{1+t^2}\,dt.$$

Example 1.21 *Calculate*

$$I = \int \frac{\sin^3(x)}{1+\cos^2(x)}\,dx$$

Solution 1.3 *Let*

$$f(x) = \frac{\sin^3(x)}{1+\cos^2(x)}\,dx.$$

We have

$$f(-x) = \frac{-\sin^3(x)}{1+\cos^2(x)}\,d(-x) = f(x),$$

so we use the substitution

$$\begin{cases} t = \cos(x), \\ \dfrac{dt}{dx} = -\sin(x), \\ dt = -\sin(x)\,dx, \end{cases}$$

to compute the integral I:

$$\begin{aligned} I &= \int \frac{\sin^3(x)}{1+\cos^2(x)}\,dx \\ &= \int \frac{\sin(x)\sin^2(x)}{1+\cos^2(x)}\,dx \\ &= -\int \frac{1-\cos^2(x)}{1+\cos^2(x)}\,dt \\ &= \int \frac{t^2-1}{1+t^2}\,dt \\ &= \int dt - 2\int \frac{1}{1+t^2}\,dt \\ &= t - 2\arctan(t) + k, \quad k \in \mathbb{R} \\ &= \cos(x) - 2\arctan(\cos(x)) + k, \quad k \in \mathbb{R}. \end{aligned}$$

1.1.3 Definite Integrals

Definition 1.4 *Let f be a continuous function on $[a, b]$, and let F be an antiderivative of f on this interval. The definite integral of f from a to b is the real number defined by*

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a).$$

Remarks 1.1 *1. One must distinguish between $\int_a^b f(x)\,dx$, which gives a number, and $\int f(x)\,dx$, which gives all functions whose derivative is $f(x)$.*

2. $\int_a^x f(t)\,dt$*: this is a function of x that equals zero when $x = a$.*

Examples 1.2 *1.* $\int \frac{1}{x}\,dx = \ln|x| + k, \quad k \in \mathbb{R}.$

2. $\int_1^2 \frac{1}{x}\,dx = [\ln|x|]_1^2 = \ln 2 - \ln 1 = \ln 2.$

3. $\int_1^x \frac{1}{t}\,dt = [\ln|t|]_1^x = \ln x - \ln 1 = \ln x.$

Properties

Let f and g be two functions defined and continuous on $[a, b]$ and $\lambda \in \mathbb{R}$. All the properties of indefinite integrals remain valid for definite integrals. In addition, we have:

1. $\int_a^a f(x)\,dx = 0$
2. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
3. $\int_a^\lambda f(x)\,dx + \int_\lambda^b f(x)\,dx = \int_a^b f(x)\,dx$
4. $\int_a^b dx = b - a$
5. If $f(x) = 0$ on $[a, b]$, then $\int_a^b f(x)\,dx = 0$ (Note: the converse is not always true).
6. If $f(x)$ is positive on $[a, b]$, then $\int_a^b f(x)\,dx \geq 0$.
7. If $f(x) - g(x) \geq 0$ on $[a, b]$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.
8. If there exist two real numbers α and β such that for all $x \in [a, b]$, $\alpha \leq f(x) \leq \beta$, then
$$\alpha(b - a) \leq \int_a^b f(x)\,dx \leq \beta(b - a)$$
9. If f is a continuous and even function on $[-a, a]$, then
$$\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$$
10. If f is a continuous and odd function on $[-a, a]$, then
$$\int_{-a}^a f(x)\,dx = 0$$
11. If f is a periodic function with period T, then
$$\int_a^{a+T} f(x)\,dx = \int_0^T f(x)\,dx$$

1.2 Generalization: Multiple Integrals (Several Dimensions)

1.2.1 Double Integrals

1. Double integral over a rectangle

Theorem 1.2 (Fubini) *Let $f : [a,b]\times[c,d] \to \mathbb{R}$ be a continuous function. Then the double integral of f over the rectangle $[a,b]\times[c,d]$ exists, and we have*

$$\iint_{[a,b]\times[c,d]} f(x,y)\,dx\,dy = \int_a^b \left(\int_c^d f(x,y)\,dy\right) dx = \int_c^d \left(\int_a^b f(x,y)\,dx\right) dy.$$

Moreover, if $f(x,y) = g(x)h(y)$, then

$$\iint_{[a,b]\times[c,d]} f(x,y)\,dx\,dy = \left(\int_a^b g(x)\,dx\right)\left(\int_c^d h(y)\,dy\right).$$

Example 1.22 *Compute the double integral*

$$I = \int_0^1 \int_{-2}^{-1} xy\,dy\,dx.$$

Step 1: Integration with respect to y.

$$\int_{-2}^{-1} xy\,dy = x\int_{-2}^{-1} y\,dy = x\left[\frac{y^2}{2}\right]_{-2}^{-1} = x\left(\frac{1}{2}-2\right) = -\tfrac{3}{2}x.$$

Step 2: Integration with respect to x.

$$I = \int_0^1 \left(-\tfrac{3}{2}x\right) dx = -\tfrac{3}{2}\int_0^1 x\,dx = -\tfrac{3}{2}\left[\tfrac{x^2}{2}\right]_0^1 = -\tfrac{3}{4}.$$

Final answer $\boxed{I = -\tfrac{3}{4}}$.

2. Double integral of a function with separated variables

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be continuous on $[a,b]\times[c,d]$, such that $f(x,y) = h(x)g(y)$. Then

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \left(\int_a^b h(x)\,dx\right)\left(\int_c^d g(y)\,dy\right).$$

Example 1.23

$$\int_a^b \int_c^d xe^y\,dx\,dy = \left(\int_a^b x\,dx\right)\left(\int_c^d e^y\,dy\right) = \left[\frac{x^2}{2}\right]_a^b . \left[e^y\right]_c^d = \left(\frac{b^2}{2}-\frac{a^2}{2}\right)\left(e^d - e^c\right).$$

3. Double integral over a bounded domain

Let $D_1 = \{(x,y) \in \mathbb{R}^2 \ / \ a \leq x \leq b, \ \ \psi_1(x) \leq y \leq \psi_2(x) \ \}$ and $D_2 = \{(x,y) \in \mathbb{R}^2 \ / \ a \leq y \leq b, \ \ \psi_1(y) \leq$ the integrals of $f(x,y)$ on D_1 and respectively on D_1, are given by:

$$\iint_{D_1} f(x,y)\, dx\, dy = \int_a^b \left[\int_{\psi_1(x)}^{\psi_2(x)} f(x,y)\, dy \right] dx.$$

$$\iint_{D_2} f(x,y)\, dx\, dy = \int_a^b \left[\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\, dx \right] dy.$$

Example 1.24 *Calculate*

$$\int\int_D dx\, dy \ \ \textit{tel-que} \ \ D = \left\{ (x,y) \in \mathbb{R}^2 \ / \ 0 \leq x \leq 1, \ \ x^2 \leq y \leq \sqrt{x} \ \right\}.$$

$$\int\int_D dx\, dy = \int_0^1 \left(\int_{x^2}^{\sqrt{x}} dy \right) dx = \int_0^1 \left(\sqrt{x} - x^2 \right) dx = \left[\frac{2}{3} x^{\frac{3}{2}} - \frac{1}{3} x^3 \right]_0^1 = \frac{1}{3}.$$

4. Compute a double integral by change of variables

$$\int\int_D f\,(x,y)\ dx\, dy = \int\int_S g\,(u,v)\ detJ\, du\, dv,$$

where

$$\det J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

Application: Compute a double integral by using Polar coordinates

$$x = r\cos(\theta), \ y = r\sin(\theta), \ 0 \leq \theta \leq 2\pi, \ \ r > 0.$$

$$J = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix}$$

$$\frac{\partial x}{\partial r} = \cos(\theta), \frac{\partial y}{\partial r} = \sin(\theta), \frac{\partial x}{\partial \theta} = -r\sin(\theta), \frac{\partial y}{\partial \theta} = r\cos(\theta),$$

$$\det J = \begin{vmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{vmatrix} = r\cos^2(\theta) + r\sin^2(\theta) = r\left(\cos^2(\theta) + \sin^2(\theta) \right) = r,$$

that is,

$$\int\int_D f\,(x,y)\ dx\, dy = \int\int_S g\,(r,\theta)\ r\, dr\, d\theta.$$

Example 1.25

$$\int\int_D \frac{1}{1+x^2+y^2} dx\, dy \ \ \textit{tel-que} \ \ D = \left\{ (x,y) \in \mathbb{R}^2 \ / x^2 + y^2 \leq 1 \right\}.$$

Step 1: Identify the region.
The condition $x^2 + y^2 \leq 1$ describes the unit disk centered at the origin. In polar coordinates we have

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad r \geq 0,$$

and

$$x^2 + y^2 = r^2 \leq 1 \quad \Rightarrow \quad 0 \leq r \leq 1.$$

Step 2: Range of θ.
To cover the whole disk, the angle θ must make a full revolution around the origin:

$$0 \leq \theta < 2\pi.$$

Then

$$\int\int_D \frac{1}{1+x^2+y^2}\,dx\,dy = \int_0^{2\pi}\int_0^1 \frac{1}{1+r^2}\,r\,dr\,d\theta = \frac{1}{2}\int_0^{2\pi}\int_0^1 \frac{2r}{1+r^2}\,dr\,d\theta,$$

$$\int\int_D \frac{1}{1+x^2+y^2}\,dx\,dy = \frac{1}{2}\int_0^{2\pi}\left(\int_0^1 \frac{2r}{1+r^2}\,dr\right)\,d\theta = \frac{1}{2}\int_0^{2\pi}\left[\ln\left(r^2+1\right)\right]_0^1\,d\theta,$$

$$\int\int_D \frac{1}{1+x^2+y^2}\,dx\,dy = \frac{1}{2}\int_0^{2\pi}\ln(2)d\theta = \frac{\ln(2)}{2}\,[\theta]_0^{2\pi} = \frac{\ln(2)}{2}\,(2\pi) = \pi\ln(2).$$

1.2.2 Triple integrals

1. Integration by iterated integrals

According to **Fubini's theorem,** a triple integral can be computed as an iterated integral, that is, by integrating successively with respect to each variable.

For example, if f is continuous on the rectangular box $[a,b]\times[c,d]\times[e,f]$, then

$$\iiint_{[a,b]\times[c,d]\times[e,f]} f(x,y,z)\,dx\,dy\,dz = \int_a^b\int_c^d\int_e^f f(x,y,z)\,dz\,dy\,dx.$$

The order of integration may be changed:

$$\iiint_{[a,b]\times[c,d]\times[e,f]} f(x,y,z)\,dx\,dy\,dz = \int_c^d\int_e^f\int_a^b f(x,y,z)\,dx\,dz\,dy,$$

and similarly for the other possible orders of integration.

Example 1.26

$$\iiint_{[0,1]^3}(x+y+z)\,dz\,dy\,dx = \int_0^1\int_0^1\int_0^1(x+y+z)\,dz\,dy\,dx.$$

a. inner integration (integration with respect to z) :

$$\int_0^1(x+y+z)\,dz = \left[(x+y)z+\frac{z^2}{2}\right]_{z=0}^{z=1} = (x+y)\cdot 1+\frac{1^2}{2}-0 = x+y+\frac{1}{2}.$$

b. intermediate integration (with respect to one variable in the middle y) :

$$\int_0^1 \left(x + y + \frac{1}{2}\right) dy = \left[xy + \frac{y^2}{2} + \frac{y}{2}\right]_{y=0}^{y=1} = x \cdot 1 + \frac{1}{2} + \frac{1}{2} = x + 1.$$

c. outer integration (with respect to the last variable x) :

$$\int_0^1 (x+1)\, dx = \left[\frac{x^2}{2} + x\right]_0^1 = \frac{1}{2} + 1 = \frac{3}{2}.$$

$$\boxed{\iiint_{[0,1]^3} (x+y+z)\, dz\, dy\, dx = \frac{3}{2}.}$$

2. Triple Integrals in Cylindrical Coordinates

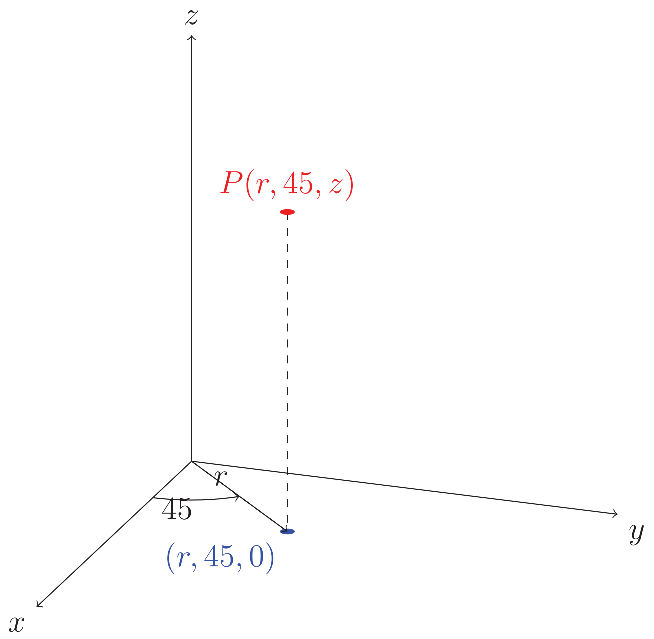
a. Cylindrical Coordinates

In cylindrical coordinates, a point (x, y, z) in $\mathbb{R}^3$ is represented as:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$$

where:

- $r \geq 0$ is the radial distance from the z-axis,
- $0 \leq \theta < 2\pi$ is the azimuthal angle,
- z is the height as in Cartesian coordinates.



Explanation of Cylindrical Coordinates:

In cylindrical coordinates, a point P in space is represented by (r, θ, z), where:

- r is the distance from the z-axis to the projection of the point onto the xy-plane.
- θ is the angle between the positive x-axis and the line connecting the origin to the projection of the point on the xy-plane.
- z is the height of the point along the z-axis.

b. Jacobian of the Transformation

When changing variables from (x, y, z) to (r, θ, z), we need the Jacobian determinant:

$$J = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The determinant of J is:

$$\det(J) = r$$

c. Triple Integral in Cylindrical Coordinates

A triple integral over a region V becomes:

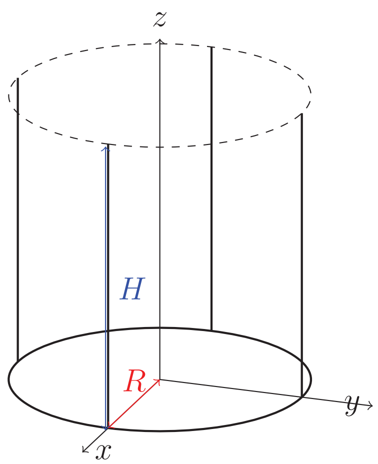
$$\iiint_V f(x, y, z)\, dx\, dy\, dz = \iiint_V f(r\cos\theta, r\sin\theta, z)\, r\, dr\, d\theta\, dz$$

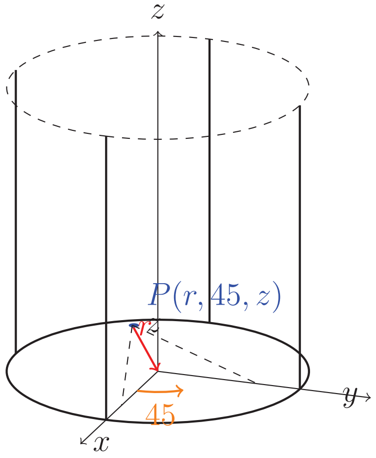
Example 1.27 ***Calculate the Integral $\iiint_V z\, dx\, dy\, dz$ over a Cylinder***

a. Representation of the Cylinder

Consider a vertical cylinder with radius R and height H. In Cartesian coordinates, the cylinder is defined by:

$$x^2 + y^2 \leq R^2, \quad 0 \leq z \leq H$$





b. Transformation to Cylindrical Coordinates
Recall the cylindrical coordinates transformation:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \quad \textbf{with } r \geq 0,\ 0 \leq \theta \leq 2\pi$$

c. Determining the Bounds

1. **Radius r: The distance from the center of the cylinder in the xy-plane ranges from 0 (the center) to R (the edge of the cylinder):**

$$0 \leq r \leq R$$

2. **Angle θ: To go around the full circle in the xy-plane, the angle varies from 0 to 2π:**

$$0 \leq \theta \leq 2\pi$$

3. **Height z: The vertical position ranges from the bottom of the cylinder to the top:**

$$0 \leq z \leq H$$

Solution:

$$\iiint_D z\,dx\,dy\,dz = \int_0^H \int_0^{2\pi} \int_0^R z \cdot r\,dr\,d\theta\,dz \qquad \textbf{(Cylindrical coordinates)}$$
$$= \int_0^H \int_0^{2\pi} \left[z\frac{r^2}{2} \right]_0^R d\theta\,dz$$
$$= \int_0^H \int_0^{2\pi} z\frac{R^2}{2}\,d\theta\,dz$$
$$= \int_0^H \left[z\frac{R^2}{2}\theta \right]_0^{2\pi} dz$$
$$= \int_0^H z\frac{R^2}{2} \cdot 2\pi\,dz$$
$$= \int_0^H \pi R^2 z\,dz$$
$$= \left[\pi R^2 \frac{z^2}{2} \right]_0^H$$
$$= \frac{\pi R^2 H^2}{2}$$

Remarks:

- **We used the Jacobian r in the integrand.**
- **The calculation applied Fubini's theorem to separate the integral into r, θ, and z integrals.**
- **All steps are written in a single equation for clarity.**

Example 1.28 ***Calculate***

$$I = \int\int\int_D dx\,dy\,dz, \quad D = \left\{(x,y,z) \in \mathbb{R}^3 \ \ /x^2+y^2+z^2 \le r^2\right\}.$$

$$I = \int\int_D \left(\int_{-\sqrt{r^2-x^2-y^2}}^{\sqrt{r^2-x^2-y^2}} dz \right) dx\,dy = 2\int\int_{D'} \sqrt{r^2-x^2-y^2}\,dx\,dy,$$

$D^{'}$ ***the disk of center*** $O=(0,0)$ ***and radius*** r***. Using the change of variables to polar coordinates***

$$x = \rho\cos(\theta),\ y = \rho\sin(\theta),$$

we have,

$$x^2+y^2 = \rho^2,$$

and

$$I = 2\int\int_{D'} \sqrt{r^2-\rho^2}\rho\,d\rho\,d\theta = -\frac{4\pi}{3}\left[\left(r^2-\rho^2\right)^{\frac{3}{2}}\right]_0^r = \frac{4\pi r^3}{3}.$$

3. Triple Integrals in Spherical Coordinates

a. In spherical coordinates, a point P in space is represented by three numbers:

$$(r, \theta, \phi)$$

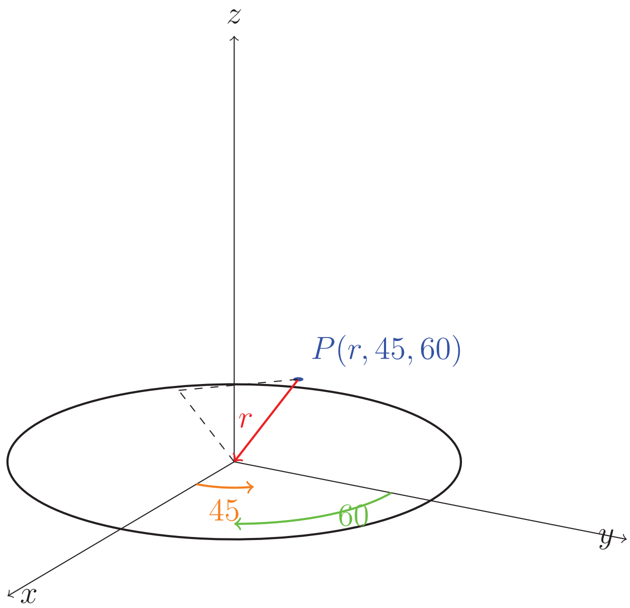
- $r \geq 0$ **is the distance from the origin to the point (radius),**
- $\theta \in [0, 2\pi)$ **is the angle in the xy-plane measured from the positive x-axis (azimuthal angle),**
- $\phi \in [0, \pi]$ **is the angle between the positive z-axis and the line connecting the origin to the point (polar angle or colatitude).**

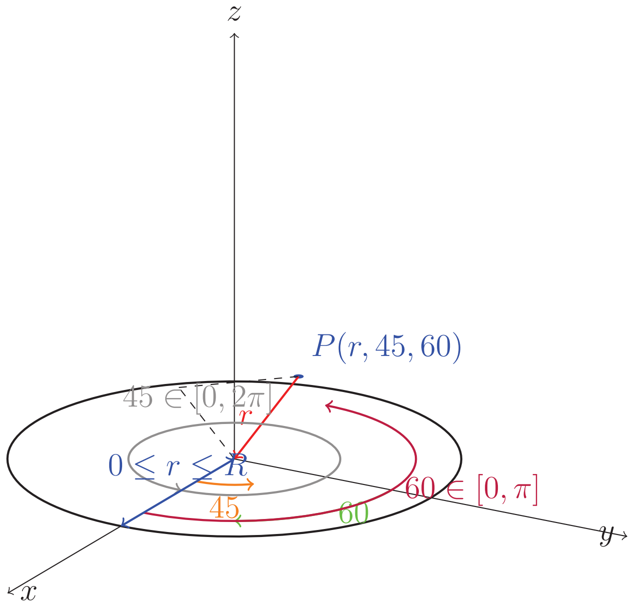
b. The relation between Cartesian coordinates (x, y, z) and spherical coordinates (r, θ, ϕ) is:

$$\begin{cases} x = r\, \sin\phi\, \cos\theta, \\ y = r\, \sin\phi\, \sin\theta, \\ z = r\, \cos\phi. \end{cases}$$

Spherical Coordinates Representation
Consider a sphere of radius R centered at the origin:

$$x^2 + y^2 + z^2 \leq R^2$$





c. Spherical Coordinates Transformation and Jacobian

The transformation from Cartesian coordinates (x, y, z) **to spherical coordinates** (r, θ, ϕ) **is:**

$$\begin{cases} x = r \sin\phi \cos\theta, \\ y = r \sin\phi \sin\theta, \\ z = r \cos\phi, \end{cases} \qquad r \geq 0, \;\; \theta \in [0, 2\pi), \;\; \phi \in [0, \pi].$$

The Jacobian matrix J **is**

$$J = \frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = \begin{bmatrix} \sin\phi\cos\theta & -r\sin\phi\sin\theta & r\cos\phi\cos\theta \\ \sin\phi\sin\theta & r\sin\phi\cos\theta & r\cos\phi\sin\theta \\ \cos\phi & 0 & -r\sin\phi \end{bmatrix}.$$

We compute the partial derivatives:

$$\begin{aligned} \frac{\partial x}{\partial r} &= \sin\phi\cos\theta, & \frac{\partial x}{\partial \theta} &= -r\sin\phi\sin\theta, & \frac{\partial x}{\partial \phi} &= r\cos\phi\cos\theta, \\ \frac{\partial y}{\partial r} &= \sin\phi\sin\theta, & \frac{\partial y}{\partial \theta} &= r\sin\phi\cos\theta, & \frac{\partial y}{\partial \phi} &= r\cos\phi\sin\theta, \\ \frac{\partial z}{\partial r} &= \cos\phi, & \frac{\partial z}{\partial \theta} &= 0, & \frac{\partial z}{\partial \phi} &= -r\sin\phi. \end{aligned}$$

Hence, the Jacobian matrix is:

$$J = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{bmatrix} = \begin{bmatrix} \sin\phi\cos\theta & -r\sin\phi\sin\theta & r\cos\phi\cos\theta \\ \sin\phi\sin\theta & r\sin\phi\cos\theta & r\cos\phi\sin\theta \\ \cos\phi & 0 & -r\sin\phi \end{bmatrix}.$$

d. The Jacobian determinant of this transformation is

$$\det J(\rho, \varphi, \theta) = r^2 \sin\phi$$

Triple Integral in Spherical Coordinates
Let $f(x, y, z)$ be a function defined in a region $\mathcal{D} \subset \mathbb{R}^3$. In spherical coordinates:

$$\begin{cases} x = r \sin \phi \cos \theta, \\ y = r \sin \phi \sin \theta, \\ z = r \cos \phi, \end{cases}$$

where $r \geq 0$, $0 \leq \phi \leq \pi$ (polar angle), $0 \leq \theta < 2\pi$ (azimuthal angle).
e. The triple integral of f over $\mathcal{D}$ becomes:

$$\iiint\limits_{\mathcal{D}} f(x, y, z)\, dx\, dy\, dz = \iiint\limits_{\mathcal{D}'} f(r \sin \phi \cos \theta,\ r \sin \phi \sin \theta,\ r \cos \phi)\ r^2 \sin \phi\, dr\, d\phi\, d\theta$$

where $r^2 \sin \phi$ is the Jacobian determinant of the transformation:

Example 1.29 ***Let*** $B = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq R^2\}$***. Taking*** $f(x, y, z) = 1$***, we obtain***

$$\mathrm{I} = \iiint_B 1\, dx\, dy\, dz = \int_0^{2\pi} \int_0^{\pi} \int_0^{R} r^2 \sin \varphi\, dr\, d\varphi\, d\theta.$$

First, integrate with respect to ρ***:***

$$\int_0^R \rho^2\, d\rho = \left[\tfrac{\rho^3}{3}\right]_0^R = \tfrac{R^3}{3}.$$

Next, integrate with respect to φ***:***

$$\int_0^{\pi} \sin \varphi\, d\varphi = [-\cos \varphi]_0^{\pi} = 2.$$

Finally, integrate with respect to θ***:***

$$\int_0^{2\pi} d\theta = 2\pi.$$

Thus,

$$\mathrm{I} = \frac{R^3}{3} \cdot 2 \cdot 2\pi = \frac{4}{3}\pi R^3.$$

Example 1.30 ***Let*** $B = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq R^2\}$***. Taking*** $f(x, y, z) = 1$***,and***

$$x = \rho \cos(\theta) \cos(\alpha), \quad y = \rho \sin(\theta) \cos(\alpha), \quad z = \rho \sin(\alpha),$$

with,

$$0 \leq \rho \leq r, \quad \theta \in [0 \;\; 2\pi]\,, \quad \alpha \left[-\frac{\pi}{2} \;\; \frac{\pi}{2}\right].$$

Calculate $I = \int \int \int_B f\,(x, y, z)\ dx\, dy\, dz.$

Solution:
We have

$$J = \begin{pmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \alpha} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \alpha} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \alpha} \end{pmatrix} = \begin{pmatrix} \cos(\alpha)\cos(\theta) & -\rho\sin(\theta)\cos(\alpha) & -\rho\sin(\alpha)\cos(\theta) \\ \cos(\alpha)\sin(\theta) & \rho\cos(\alpha)\cos(\theta) & -\rho\sin(\alpha)\sin(\theta) \\ \sin(\alpha) & 0 & \rho\cos(\theta) \end{pmatrix}$$

$$detJ = \rho^2\cos(\alpha).$$

Then

$$\begin{aligned} I &= \int\int\int_B f(x,y,z)\ dx\,dy\,dz \\ &= \int\int\int_B \rho^2\cos(\alpha)\,d\rho\,d\theta\,d\alpha \\ &= \left(\int_0^r \rho^2\,d\rho\right)\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(\alpha)\,d\alpha\right)\left(\int_0^{2\pi} d\theta\right) \\ &= \frac{4\pi r^3}{3}. \end{aligned}$$

Coordinate System	Coordinates	Integration Element	Use Case
Cartesian (2D)	(x, y)	$dx\,dy$	Double integrals over general regions
Polar (2D)	(r, θ)	$r\,dr\,d\theta$	Double integrals with circular symmetry
Cylindrical (3D)	(r, θ, z)	$r\,dr\,d\theta\,dz$	Triple integrals with axial symmetry
Spherical (3D)	(ρ, ϕ, θ)	$\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$	Triple integrals with point symmetry

Table 1.1: Summary of coordinate systems and their integration use.

1.3 Area and Volume Calculation

1.3.1 Area Calculation Using Double Integrals

The area of a surface projected onto the xy-plane, bounded by a region D, can be calculated using a double integral.

Definition 1.5 ***Let D be a region in the xy-plane and let $f(x,y)$ be a non-negative function defined on D. Then the area A under the surface $z = f(x,y)$ over the region D is given by:***

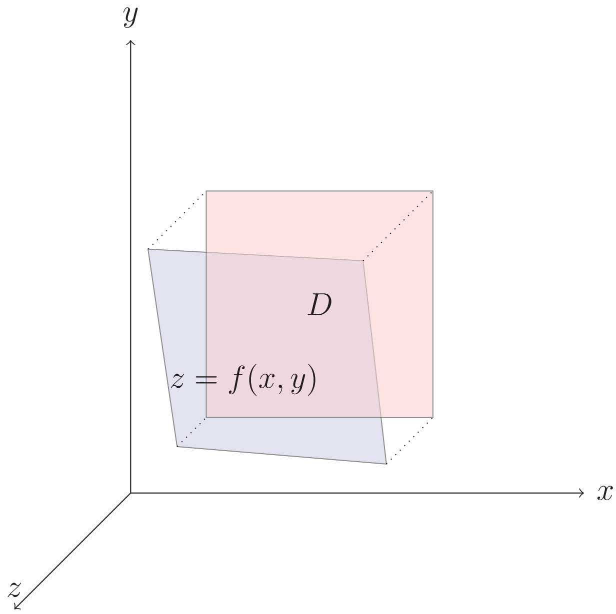
$$A = \iint_D f(x,y)\,dx\,dy$$

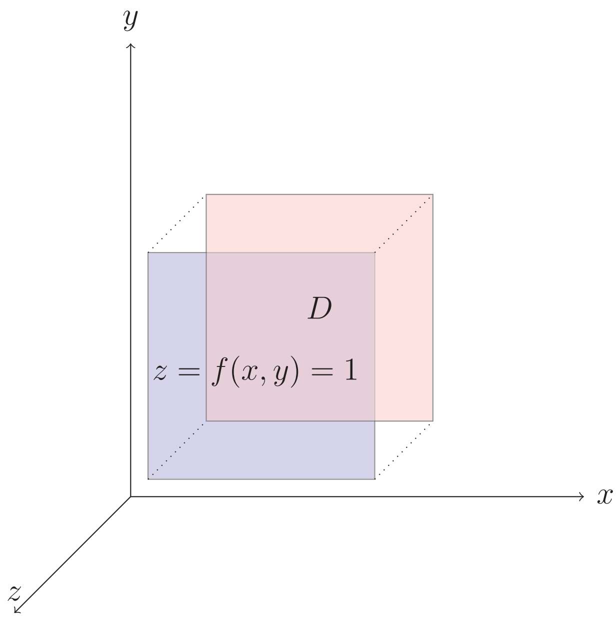
Here:

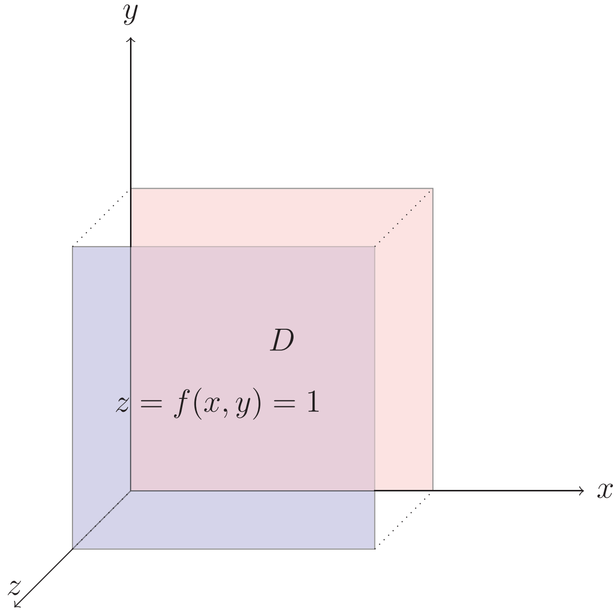
- ***D is the projection of the surface onto the xy-plane.***
- ***$f(x,y)$ represents the height of the surface above the point (x,y) in D.***

Remark: If $f(x, y) = 1$, the double integral simply gives the area of the region D itself:

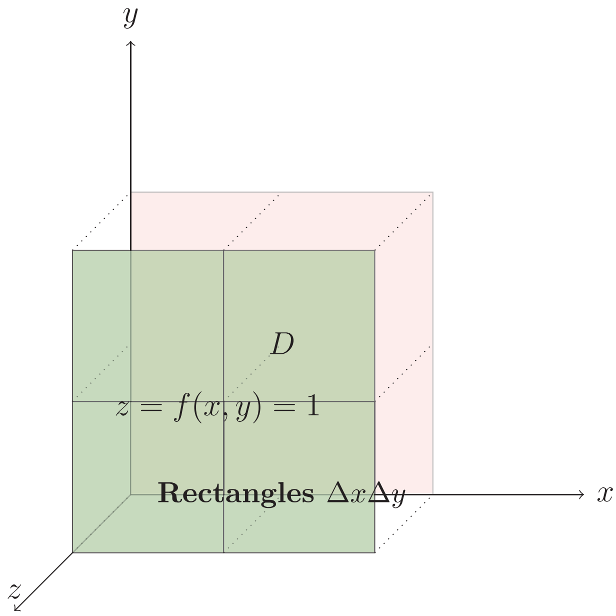
$$A = \iint_D 1 \, dx \, dy$$



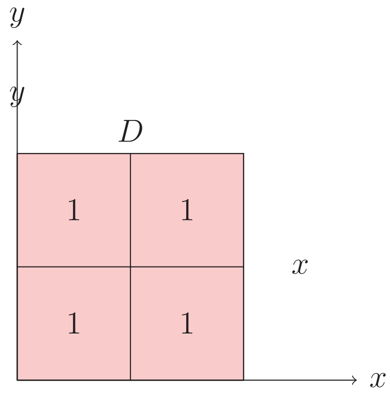




Visualisation of a Double Integral for $f(x, y) = 1$

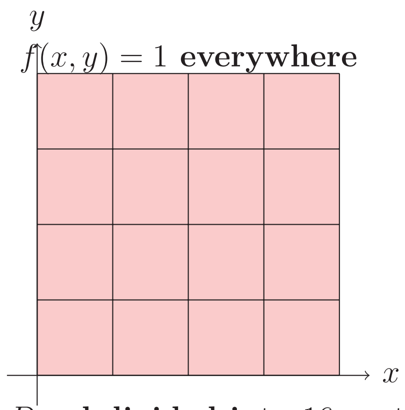


Visualisation of a Double Integral for $f(x, y) = 1$



Visualisation of a Double Integral for $f(x, y) = 1$

The domain $D = [0, 2] \times [0, 2]$ is subdivided into 16 small rectangles to illustrate the calculation of a double integral.



Domain D subdivided into 16 rectangles

1. Each small rectangle has an area $\Delta x \Delta y = 0.5 \times 0.5 = 0.25$. - The sum of all rectangle areas $= 16 \times 0.25 = 4$, which corresponds to the double integral:

$$\iint_D f(x, y)\, dx\, dy = 4.$$

2. Calculating the Area of Domain D Using a Double Integral

We want to compute the area of the domain

$$D = \{(x, y) \mid 0 \leq x \leq 2,\ 0 \leq y \leq 2\}.$$

Since the function is constant $f(x, y) = 1$, the double integral over D gives the area:

$$\textbf{Area}(D) = \iint_D 1\, dx\, dy.$$

We can write this as an iterated integral:

$$\textbf{Area}(D) = \int_0^2 \int_0^2 1\, dx\, dy.$$

First, integrate with respect to x:

$$\int_0^2 1\, dx = [x]_0^2 = 2 - 0 = 2.$$

Then, integrate with respect to y:

$$\int_0^2 2\, dy = [2y]_0^2 = 2 \times 2 - 0 = 4.$$

Hence, the exact area of the domain D is:

$$\boxed{4}.$$

Example 1.31 (Calculating Surface Using a Double Integral) *Let us compute the surface under the function*

$$f(x, y) = x + y$$

over the domain

$$D = \{(x, y) \mid 0 \leq x \leq 1,\ 0 \leq y \leq 1\}.$$

The surface is given by the double integral:

$$\textbf{\textit{Surface}} = \iint_D f(x, y)\, dx\, dy = \int_0^1 \int_0^1 (x + y)\, dx\, dy.$$

Step 1: Integrate with respect to x:

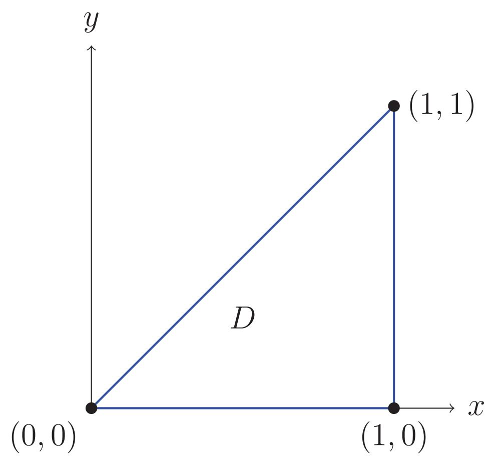
$$\int_0^1 (x + y)\, dx = \int_0^1 x\, dx + \int_0^1 y\, dx = \left[\frac{x^2}{2}\right]_0^1 + y[x]_0^1 = \frac{1}{2} + y.$$

Step 2: Integrate with respect to y:

$$\int_0^1 \left(\frac{1}{2} + y\right) dy = \int_0^1 \frac{1}{2}\, dy + \int_0^1 y\, dy = \left[\frac{y}{2}\right]_0^1 + \left[\frac{y^2}{2}\right]_0^1 = \frac{1}{2} + \frac{1}{2} = 1.$$

Hence, the exact surface under the function $f(x, y) = x + y$ over the domain D is: $\boxed{1}$.

Example 1.32 *Let's calculate the area of the triangle D defined by the points* $(0, 0), (1, 0), (1, 1)$. *1. Drawing the domain:*



2. Define the domain: The triangle can be described as:

$$D = \{(x, y) \mid 0 \leq x \leq 1,\ 0 \leq y \leq x\}.$$

3. Write the double integral:

$$\textbf{\textit{Area}}(D) = \iint_D 1\, dA = \int_{x=0}^1 \int_{y=0}^x 1\, dy\, dx.$$

4. Compute the integral:

$$\int_{x=0}^1 \int_{y=0}^x 1\, dy\, dx = \int_0^1 [y]_{y=0}^{y=x}\, dx = \int_0^1 x\, dx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2}.$$

Conclusion: The area of the triangle is therefore $\frac{1}{2}$.

1.3.2 Volume Calculation Using Triple Integrals

Definition 1.6 ***The volume of a three-dimensional solid*** V ***can be calculated using a triple integral if the solid occupies a domain*** D ***in*** $\mathbb{R}^3$***.***

Formally, if the solid is described by a continuous function $f(x, y, z)$ ***over a domain*** D***, the volume is given by:***

$$\textbf{\textit{Volume}} = \iiint_D f(x, y, z)\, dV$$

where $dV = dx\, dy\, dz$ ***in Cartesian coordinates.***

There are two situations:

1. **Pure volume of a solid: If we want only the geometrical volume of a solid occupying a region** $D \subset \mathbb{R}^3$**, then the integral is**

$$V = \iiint_D 1\, dV.$$

Here, the integrand is $f(x, y, z) = 1$ **because we only sum up infinitesimal volume elements.**

2. **Physical quantities (mass, charge, etc.): If the solid has a density function** $\rho(x, y, z)$**, then the mass is given by**

$$M = \iiint_D \rho(x, y, z)\, dV.$$

In this case, the function $\rho(x, y, z)$ **replaces the** 1 **because we are summing up weighted infinitesimal volumes.**

Remark 1.9 ***The choice of coordinates (Cartesian, cylindrical, spherical) depends on the symmetry of the region*** D***. Depending on the symmetry of the solid, other coordinate systems can be used, such as cylindrical coordinates*** (r, θ, z) ***or spherical coordinates*** (r, ϕ, θ)***:***

$$dV = r\, dr\, d\theta\, dz \quad \textbf{\textit{(cylindrical)}}, \quad dV = r^2 \sin\phi\, dr\, d\phi\, d\theta \quad \textbf{\textit{(spherical)}}.$$

Example 1.33 ***1.*** **Volume of a Sphere (using spherical coordinates):**

Let $D \subset \mathbb{R}^3$ ***be the ball of radius*** $R > 0$ ***centered at the origin:***

$$D = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \le R^2\}.$$

In spherical coordinates we use the variables (r, θ, ϕ) ***with the convention:***

$$\begin{aligned} x &= r \sin\phi \cos\theta, \\ y &= r \sin\phi \sin\theta, \qquad r \ge 0,\ \theta \in [0, 2\pi),\ \phi \in [0, \pi]. \\ z &= r \cos\phi, \end{aligned}$$

The volume element (Jacobian) is

$$dV = r^2 \sin\phi \, dr \, d\phi \, d\theta.$$

2. **Geometric volume:** ***The geometric volume*** V ***of the ball is***

$$V = \iiint_D 1 \, dV = \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} \int_{r=0}^{R} r^2 \sin\phi \, dr \, d\phi \, d\theta.$$

Evaluate the innermost integral in r***:***

$$\int_0^R r^2 \, dr = \frac{R^3}{3}.$$

Thus

$$V = \int_0^{2\pi} \int_0^{\pi} \frac{R^3}{3} \sin\phi \, d\phi \, d\theta = \frac{R^3}{3} \left(\int_0^{2\pi} d\theta \right) \left(\int_0^{\pi} \sin\phi \, d\phi \right).$$

Compute the angular integrals:

$$\int_0^{2\pi} d\theta = 2\pi, \qquad \int_0^{\pi} \sin\phi \, d\phi = \Big[-\cos\phi \Big]_0^{\pi} = 2.$$

Therefore

$$\boxed{V = \frac{R^3}{3} \cdot 2\pi \cdot 2 = \frac{4}{3}\pi R^3}.$$

3. **Mass with a density:** ***If the body has a density function*** $\delta(x, y, z)$ ***(note: we use*** δ ***to avoid conflict with the radial variable*** r***), then the mass is***

$$M = \iiint_D \delta(x, y, z) \, dV.$$

In spherical coordinates this becomes

$$M = \int_0^{2\pi} \int_0^{\pi} \int_0^{R} \delta\Big(r \sin\phi \cos\theta, r \sin\phi \sin\theta, r \cos\phi\Big) \, r^2 \sin\phi \, dr \, d\phi \, d\theta,$$

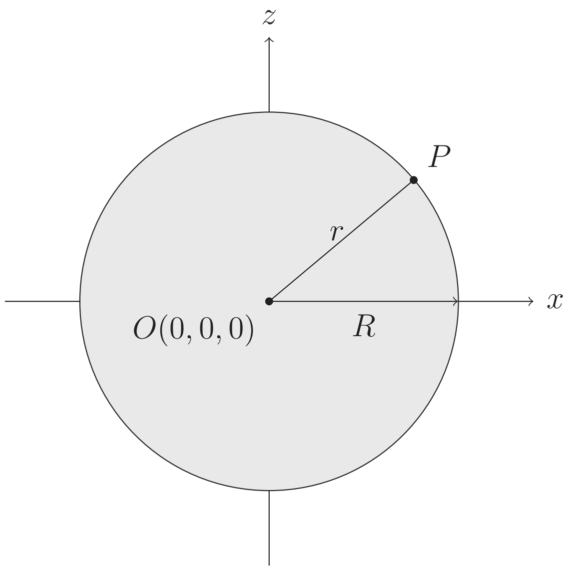
then

$$M = \int_0^{2\pi} \int_0^{\pi} \int_0^{R} \tilde{\delta}(r, \phi, \theta) \, r^2 \sin\phi \, dr \, d\phi \, d\theta.$$

where $\delta(r, \theta, \phi)$ ***denotes the same density expressed in spherical coordinates. If the density is constant,*** $\delta(x, y, z) = \delta_0$ ***(a constant), then***

$$M = \delta_0 \, V = \delta_0 \cdot \frac{4}{3}\pi R^3.$$

4. **Illustration (2D cross-section):**



This 2D figure is the vertical cross-section of the sphere by the xz-plane (it shows a circle of radius R centered at the origin). In full 3D the sphere is obtained by rotating this disk around the z-axis.

Example 1.34 ***Compute the volume of the rectangular box***

$$D = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \le x \le 2,\ 0 \le y \le 1,\ 0 \le z \le 3\}$$

by a triple integral.

Solution 1.4 ***The geometric volume of the region D is obtained by integrating the function 1 over D:***

$$V = \iiint_D 1\, dV.$$

Step 1: Set up the triple integral.
Since the bounds are constant and independent, we may write the integral in Cartesian coordinates as

$$V = \int_{x=0}^{2} \int_{y=0}^{1} \int_{z=0}^{3} 1\, dz\, dy\, dx.$$

Step 2: Integrate with respect to z.

$$\int_{z=0}^{3} 1\, dz = z\Big|_0^3 = 3.$$

Hence the integral reduces to

$$V = \int_{x=0}^{2} \int_{y=0}^{1} 3\, dy\, dx.$$

Step 3: Integrate with respect to y.

$$\int_{y=0}^{1} 3\, dy = 3y\Big|_0^1 = 3.$$

Thus

$$V = \int_{x=0}^{2} 3\,dx.$$

Step 4: Integrate with respect to x***.***

$$\int_{x=0}^{2} 3\,dx = 3x\Big|_0^2 = 6.$$

Answer:

$$\boxed{V = 6}$$

The volume of the rectangular box is 6 ***cubic units.***

Example 1.35 ***Compute the volume of the solid bounded by the cylinder*** $x^2+y^2 \leq 4$ ***and the planes*** $z = 0$ ***and*** $z = 1 + x^2 + y^2$***. Use cylindrical coordinates.***

Solution 1.5 ***The solid*** D ***is given by***

$$D = \{(x, y, z) \mid x^2 + y^2 \leq 4,\ 0 \leq z \leq 1 + x^2 + y^2\}.$$

We compute the geometric volume

$$V = \iiint_D 1\,dV.$$

Step 1: Change to cylindrical coordinates.
Set

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z,$$

with $r \in [0, 2]$***,*** $\theta \in [0, 2\pi)$***, and*** z ***between the planes. The Jacobian is*** $dV = r\,dr\,d\theta\,dz$***. The upper bound for*** z ***becomes*** $z = 1 + r^2$***.***

Step 2: Set up the triple integral.

$$V = \int_{\theta=0}^{2\pi} \int_{r=0}^{2} \int_{z=0}^{1+r^2} r\,dz\,dr\,d\theta.$$

Step 3: Integrate with respect to z***.***

$$\int_{z=0}^{1+r^2} r\,dz = r\Big(1 + r^2\Big).$$

So

$$V = \int_0^{2\pi} \int_0^2 r\Big(1 + r^2\Big)\,dr\,d\theta = \int_0^{2\pi} \left(\int_0^2 (r + r^3)\,dr\right) d\theta.$$

Step 4: Integrate with respect to r***.***

$$\int_0^2 (r + r^3)\,dr = \left[\frac{r^2}{2} + \frac{r^4}{4}\right]_0^2 = \frac{4}{2} + \frac{16}{4} = 2 + 4 = 6.$$

Step 5: Integrate with respect to θ***.***

$$V = \int_0^{2\pi} 6\,d\theta = 6 \cdot 2\pi = 12\pi.$$

Answer:

$$\boxed{V = 12\pi}$$

The volume of the solid is 12π ***cubic units.***