Chapter 6

Fourier Transform

6.1 Definition and Properties of the Fourier Transform

6.1.1 Definitions

Definition 6.1 ***For a continuous signal*** $x(t)$ ***defined for all*** $t \in \mathbb{R}$ ***and absolutely integrable, the Fourier transform is defined as:***

$$X(f) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{+\infty} x(t) e^{-j2\pi f t}\, dt, \quad j^2 = -1$$

where:

- $X(f)$ ***is the Fourier transform of*** $x(t)$
- f ***is the frequency in Hz***
- j ***is the imaginary unit (***$j^2 = -1$***)***

The Fourier transform represents a time-domain signal as a superposition of sinusoidal components at different frequencies.

Examples 6.1 Example 1: Causal exponential
Consider the signal $x(t) = e^{-2t}u(t)$**, where** $u(t)$ **is the unit step function defined by:**

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$

and its derivative, is the Dirac delta function:

$$\frac{d}{dt}u(t) = \delta(t)$$

Then the Fourier transform is:

$$X(f) = \int_{0}^{\infty} e^{-2t} e^{-j2\pi f t}\, dt = \int_{0}^{\infty} e^{-(2+j2\pi f)t}\, dt = \frac{1}{2 + j2\pi f}.$$

Remark 6.1 ***The unit step function makes the signal*** **causal*, i.e., zero for*** $t < 0$***.***

Example 2: Dirac delta function
Consider $x(t) = \delta(t)$**, the Dirac delta function, defined by:**

$$\delta(t) = \begin{cases} 0, & t \neq 0 \\ \textbf{infinity}, & t = 0 \end{cases}, \quad \textbf{with} \int_{-\infty}^{\infty} \delta(t)dt = 1$$

Theorem 6.1 (Sifting Property of the Dirac Delta) ***Let*** $\delta(t)$ ***be the Dirac delta function, and let*** $\phi(t)$ ***be any continuous function. Then:***

$$\int_{-\infty}^{+\infty} \delta(t)\, \phi(t)\, dt = \phi(0)$$

More generally, for a shifted delta function $\delta(t - t_0)$***:***

$$\int_{-\infty}^{+\infty} \delta(t - t_0)\, \phi(t)\, dt = \phi(t_0)$$

Remark 6.2 This property means that the delta "samples" the value of the function at the point $t = t_0$***.***

The Fourier transform is:

$$\mathcal{F}\{\delta(t)\} = X(f) = \int_{-\infty}^{\infty} \delta(t) e^{-j2\pi f t}\, dt = 1$$

Remark 6.3 ***The Dirac delta contains all frequencies with equal amplitude.***

Example 3: Pure cosine
Consider $x(t) = \cos(3t)$**. Using Euler's formula:**

$$\cos(3t) = \frac{e^{j3t} + e^{-j3t}}{2}$$

Then the Fourier transform is:

$$X(f) = \frac{1}{2} \int_{-\infty}^{\infty} e^{-j(2\pi f - 3)t}\, dt + \frac{1}{2} \int_{-\infty}^{\infty} e^{-j(2\pi f + 3)t}\, dt = \frac{1}{2}\delta(2\pi f - 3) + \frac{1}{2}\delta(2\pi f + 3)$$

Remark 6.4 ***A pure cosine has two frequency components at*** $\pm 3/(2\pi)$ ***Hz.***

6.1.2 Properties of the Fourier Transform

1. Linearity

Theorem 6.2 (Linearity) ***For signals*** $x(t)$ ***and*** $y(t)$***, and constants*** $a, b \in \mathbb{C}$***:***

$$\mathcal{F}\{ax(t) + by(t)\} = aX(f) + bY(f).$$

Where $X(f)$ ***and*** $Y(f)$ ***are respectively the Fourier transform of*** $x(t)$ ***and*** $y(t)$***.***

Example 6.1 ***Compute the Fourier transform of*** $x(t) = 2e^{-t}u(t)$ ***and*** $y(t) = \cos(3t)$ ***combined as*** $x(t) + y(t)$***.***

$$\mathcal{F}\{x(t)\} = \frac{2}{1 + j2\pi f}, \quad \mathcal{F}\{y(t)\} = \frac{1}{2}\delta(2\pi f - 3) + \frac{1}{2}\delta(2\pi f + 3)$$

$$\mathcal{F}\{x(t) + y(t)\} = \frac{2}{1 + j2\pi f} + \frac{1}{2}\delta(2\pi f - 3) + \frac{1}{2}\delta(2\pi f + 3)$$

—

2. Time Shifting

Theorem 6.3 (Time Shifting) ***If*** $x(t)$ ***has Fourier transform*** $X(f)$***, then for any*** $t_0 \in \mathbb{R}$***:***

$$\mathcal{F}\{x(t - t_0)\} = X(f)e^{-j2\pi f t_0}$$

Example 6.2 ***Let*** $x(t) = e^{-t}u(t)$ ***and*** $t_0 = 1$***. Compute*** $\mathcal{F}\{x(t-1)\}$***.***
Solution:
Step 1: Write the shifted signal explicitly:

$$x(t - 1) = e^{-(t-1)}u(t - 1)$$

Here, $u(t-1)$ ***is the unit step function that ensures the signal is causal, i.e., zero for*** $t < 1$***.***
—
Step 2: Use the definition of the Fourier transform:

$$\mathcal{F}\{x(t-1)\} = \int_{-\infty}^{+\infty} x(t-1)e^{-j2\pi f t}\, dt = \int_{-\infty}^{+\infty} e^{-(t-1)}u(t-1)e^{-j2\pi f t}\, dt$$

—
Step 3: Adjust the limits using $u(t-1)$***:***

$$u(t-1) = 0 \textbf{ for } t < 1, \quad u(t-1) = 1 \textbf{ for } t \geq 1$$

So the integral becomes:

$$\mathcal{F}\{x(t-1)\} = \int_{1}^{\infty} e^{-(t-1)}e^{-j2\pi f t}\, dt$$

—
Step 4: Factor terms to simplify the exponent:

$$e^{-(t-1)}e^{-j2\pi f t} = e^{-(t-1)}e^{-j2\pi f(t-1)}e^{-j2\pi f} = e^{-(1+j2\pi f)(t-1)}e^{-j2\pi f}$$

—
Step 5: Change variable $\tau = t - 1 \implies t = \tau + 1, dt = d\tau$***:***

$$\mathcal{F}\{x(t-1)\} = \int_{0}^{\infty} e^{-(1+j2\pi f)\tau}e^{-j2\pi f}\, d\tau = e^{-j2\pi f}\int_{0}^{\infty} e^{-(1+j2\pi f)\tau}\, d\tau$$

—
Step 6: Evaluate the integral:

$$\int_0^\infty e^{-(1+j2\pi f)\tau}\,d\tau = \frac{1}{1+j2\pi f}$$

—

Step 7: Final result:

$$\mathcal{F}\{x(t-1)\} = \frac{1}{1+j2\pi f}\,e^{-j2\pi f}$$

Remark 6.5 The unit step $u(t-1)$ is included in the integral limits, which is why it does not appear explicitly in the final formula, but it is essential to make the signal causal.

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3. Frequency Shifting (Modulation)

Theorem 6.4 (Frequency Shifting) ***If $x(t)$ has Fourier transform $X(f)$, then***

$$\mathcal{F}\{x(t)e^{j2\pi f_0 t}\} = X(f-f_0)$$

Example 6.3 ***Let $x(t) = e^{-t}u(t)$ and $f_0 = 2$.***
Compute the Fourier transform of $x(t)e^{j2\pi(2)t} = x(t)e^{j4\pi t}$.
Solution:
Step 1: Write the signal explicitly with the unit step:

$$x(t)e^{j4\pi t} = e^{-t}u(t)\cdot e^{j4\pi t} = e^{-t}e^{j4\pi t}u(t)$$

Here, $u(t)$ ensures the signal is causal (zero for $t<0$).
—
Step 2: Use the definition of the Fourier transform:

$$\mathcal{F}\{x(t)e^{j4\pi t}\} = \int_{-\infty}^{+\infty} e^{-t}e^{j4\pi t}u(t)\,e^{-j2\pi ft}\,dt$$

Combine exponentials:

$$e^{j4\pi t}\cdot e^{-j2\pi ft} = e^{-j2\pi ft}\cdot e^{j2\pi(2)t} = e^{-j2\pi(f-2)t}$$

So the integral becomes:

$$\mathcal{F}\{x(t)e^{j4\pi t}\} = \int_0^\infty e^{-t}e^{-j2\pi(f-2)t}\,dt$$

- The lower limit 0 comes from $u(t)$. - The upper limit is $+\infty$ because the signal exists for all $t\geq 0$.
—
Step 3: Combine terms in the exponent:

$$e^{-t}\cdot e^{-j2\pi(f-2)t} = e^{-(1+j2\pi(f-2))t}$$

—

Step 4: Integrate:

$$\int_0^\infty e^{-(1+j2\pi(f-2))t}\,dt = \frac{1}{1+j2\pi(f-2)}$$

—

Step 5: Final result:

$$\mathcal{F}\{x(t)e^{j4\pi t}\} = \frac{1}{1+j2\pi(f-2)}$$

Remark: Multiplying by $e^{j2\pi f_0 t}$ in time shifts the Fourier transform in frequency by f_0, here $f_0 = 2$. The unit step $u(t)$ ensures the integral starts at 0.

—

4. Differentiation in Time

Theorem 6.5 (Time Differentiation) ***If $x(t)$ has Fourier transform $X(f)$, then***

$$\mathcal{F}\left\{\frac{dx}{dt}\right\} = j2\pi f X(f)$$

More generally,for the n-th derivative

$$\mathcal{F}\left\{\frac{d^n x(t)}{dt^n}\right\} = (j2\pi f)^n\, X(f), \quad n = 1, 2, 3, \ldots$$

Example 6.4 ***Let $x(t) = e^{-t}u(t)$.***
1. Compute the Fourier transform of $\frac{dx}{dt}(t)$ using the propertie of time differentiation.
2. Compute $\frac{dx}{dt}(t)$ and its Fourier transform using an other method.
Solution:
1. Recall the property:

$$\mathcal{F}\left\{\frac{dx}{dt}(t)\right\} = j2\pi f X(f)$$

with $\mathcal{F}\{e^{-t}u(t)\} = \dfrac{1}{1+j2\pi f}$. Then

$$\mathcal{F}\left\{\frac{dx}{dt}(t)\right\} = j2\pi f\frac{1}{1+j2\pi f} = \frac{j2\pi f}{1+j2\pi f}$$

2. Step 1: Recall that $x(t) = e^{-t}u(t)$. Using the product rule for derivatives:

$$\frac{d}{dt}[e^{-t}u(t)] = \frac{de^{-t}}{dt}u(t) + e^{-t}\frac{du(t)}{dt}$$

—

Step 2: Compute each term:

1. $\dfrac{de^{-t}}{dt}u(t) = -e^{-t}u(t)$
2. $\dfrac{du(t)}{dt} = \delta(t)$ ***by definition of the Dirac delta***
So the derivative is:

$$\frac{dx}{dt}(t) = -e^{-t}u(t) + e^{-t}\delta(t)$$

But notice that $e^{-t}\delta(t) = \delta(t)$ ***because*** $\delta(t)$ ***"samples" the function at*** $t = 0$***:***

$$\int_{-\infty}^{\infty} e^{-t}\delta(t)\phi(t)dt = \phi(0)e^{0} = \phi(0) = \int_{-\infty}^{\infty} \delta(t)\phi(t)dt$$

That is $e^{-t}\delta(t) = \delta(t)$ ***in distribution sens. Hence, we can write:***

$$\frac{dx}{dt}(t) = \delta(t) - e^{-t}u(t)$$

—

Step 3: Fourier transform of the derivative
Using linearity:

$$\mathcal{F}\left\{\frac{dx}{dt}(t)\right\} = \mathcal{F}\{\delta(t)\} - \mathcal{F}\{e^{-t}u(t)\}$$

—

Step 4: Fourier transforms of each term:
1. $\mathcal{F}\{\delta(t)\} = 1$
2. $\mathcal{F}\{e^{-t}u(t)\} = \dfrac{1}{1 + j2\pi f}$
So we get:

$$\mathcal{F}\left\{\frac{dx}{dt}\right\} = 1 - \frac{1}{1 + j2\pi f}$$

—

Step 5: Simplify the expression:

$$1 - \frac{1}{1 + j2\pi f} = \frac{1 + j2\pi f - 1}{1 + j2\pi f} = \frac{j2\pi f}{1 + j2\pi f}$$

—

Final Result:

$$\frac{dx}{dt} = \delta(t) - e^{-t}u(t), \quad \mathcal{F}\left\{\frac{dx}{dt}\right\} = \frac{j2\pi f}{1 + j2\pi f}$$

Remark 6.6 The delta function appears because of the derivative of the unit step $u(t)$***, and it is crucial to include it for a correct Fourier transform.***

—

5. Convolution in Time

Theorem 6.6 (Convolution) ***If*** $x(t)$ ***and*** $y(t)$ ***have Fourier transforms*** $X(f)$ ***and*** $Y(f)$***, then***

$$\mathcal{F}\{x(t) * y(t)\} = X(f) \cdot Y(f)$$

Example 6.5 ***Let*** $x(t) = e^{-t}u(t)$ ***and*** $y(t) = u(t)$***. Compute the convolution*** $x(t) * y(t)$ ***and its Fourier transform.***
Solution:
Step 1: Recall the definition of convolution:

$$(x * y)(t) = \int_{-\infty}^{+\infty} x(\tau)\, y(t-\tau)\, d\tau$$

Since both $x(t)$ ***and*** $y(t)$ ***are causal (***$u(t)$***), the integral limits reduce to:***

$$(x * y)(t) = \int_{0}^{t} x(\tau)\, y(t-\tau)\, d\tau$$

—

Step 2: Substitute $x(\tau) = e^{-\tau}u(\tau)$ ***and*** $y(t-\tau) = u(t-\tau)$***:***
- $u(\tau) = 1$ ***for*** $\tau \geq 0$
- $u(t-\tau) = 1$ ***for*** $0 \leq \tau \leq t$

$$(x * y)(t) = \int_{0}^{t} e^{-\tau} \cdot 1\, d\tau$$

—

Step 3: Evaluate the integral:

$$\int_0^t e^{-\tau} d\tau = \Big[-e^{-\tau}\Big]_0^t = -e^{-t} + e^0 = 1 - e^{-t}$$

So the convolution result is:

$$x(t) * y(t) = 1 - e^{-t}$$

—

Step 4: Verify with the direct Fourier transform

$$\mathcal{F}\{1 - e^{-t}\} = \mathcal{F}\{1\} - \mathcal{F}\{e^{-t}u(t)\} = \frac{1}{j2\pi f} - \frac{1}{1 + j2\pi f}$$

- This matches $X(f)Y(f)$ ***by the convolution theorem. Indeed, Fourier transforms of*** $x(t)$ ***and*** $y(t)$***:***
1. $x(t) = e^{-t}u(t) \implies X(f) = \frac{1}{1+j2\pi f}$
2. $y(t) = u(t) \implies Y(f) = \frac{1}{j2\pi f} + \pi\delta(f)$
- Often in examples, the $\delta(f)$ ***term is ignored when focusing on*** $f \neq 0$***. Recall the convolution theorem:***

$$\mathcal{F}\{x(t) * y(t)\} = X(f)\, Y(f)$$

we have

$$X(f)Y(f) = \frac{1}{1 + j2\pi f} \cdot \frac{1}{j2\pi f} = \frac{1}{(j2\pi f)(1 + j2\pi f)}$$

Partial fraction decomposition: $\frac{1}{(j2\pi f)(1+j2\pi f)} = \frac{1}{j2\pi f} - \frac{1}{1+j2\pi f}$

$$\implies X(f)Y(f) = \frac{1}{j2\pi f} - \frac{1}{1+j2\pi f} = \mathcal{F}\{1-e^{-t}\}$$

Remark 6.7 - Convolution in the time domain corresponds to multiplication in the frequency domain.
- The unit step $u(t)$ determines the integration limits and ensures causality.

—

6. Multiplication in Time

Theorem 6.7 (Time Multiplication) ***If $x(t)$ and $y(t)$ have Fourier transforms $X(f)$ and $Y(f)$, then***

$$\mathcal{F}\{x(t)\,y(t)\} = X(f) * Y(f)$$

where $*$ denotes convolution in frequency.

Example 6.6 ***Let $x(t) = u(t)$ and $y(t) = u(t-1)$. Compute the Fourier transform of the product $x(t)y(t)$.***
Solution:
Step 1: Write the product explicitly using the unit step functions:

$$x(t)y(t) = u(t) \cdot u(t-1)$$

- $u(t) = 0$ for $t < 0$, 1 for $t \geq 0$ - $u(t-1) = 0$ for $t < 1$, 1 for $t \geq 1$
So the product is:

$$x(t)y(t) = \begin{cases} 0, & t < 1 \\ 1, & t \geq 1 \end{cases} = u(t-1)$$

—

Step 2: Recall the Fourier transform of a shifted unit step:

$$\mathcal{F}\{u(t-a)\} = \frac{1}{j2\pi f}e^{-j2\pi f a} + \pi\delta(f)$$

Here, $a = 1$. So:

$$\mathcal{F}\{u(t-1)\} = \frac{1}{j2\pi f}e^{-j2\pi f} + \pi\delta(f)$$

—

Step 3: Connect to the convolution property
- The Fourier transform of a product of two signals in time is equal to the convolution of their Fourier transforms:

$$\mathcal{F}\{x(t)y(t)\} = X(f) * Y(f)$$

- Here:

$$X(f) = \mathcal{F}\{u(t)\} = \frac{1}{j2\pi f} + \pi\delta(f)$$

$$Y(f) = \mathcal{F}\{u(t-1)\} = \frac{1}{j2\pi f}e^{-j2\pi f} + \pi\delta(f)$$

- The direct computation gave

$$X(f) * Y(f) = \int_{-\infty}^{\infty} X(\lambda)\, Y(f-\lambda)\, d\lambda = \mathcal{F}\{u(t-1)\} = \mathcal{F}\{x(t)y(t)\}$$

Remark 6.8 - This shows that multiplying two signals in time corresponds to convolution in frequency.
- This is useful for modulated signals or windowed signals, where one signal acts as a "window" for the other.

7. Fourier Transform: Frequency Derivative Property

Theorem 6.8 (Frequency Derivative Property) ***Let $x(t)$ be an absolutely integrable signal with Fourier transform***

$$X(f) = \mathcal{F}\{x(t)\}.$$

Then the derivative of $X(f)$ with respect to frequency f is given by:

$$\frac{dX(f)}{df} = \mathcal{F}\{-j2\pi t\, x(t)\}.$$

Proof.

1. **By definition, the Fourier transform of $x(t)$ is**

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt.$$

2. **Differentiating $X(f)$ with respect to f:**

$$\frac{dX(f)}{df} = \int_{-\infty}^{\infty} x(t)\, \frac{d}{df}\left(e^{-j2\pi f t}\right) dt.$$

3. **Since**

$$\frac{d}{df}e^{-j2\pi f t} = -j2\pi t\, e^{-j2\pi f t},$$

we have

$$\frac{dX(f)}{df} = \int_{-\infty}^{\infty} (-j2\pi t\, x(t))\, e^{-j2\pi f t}\, dt.$$

4. **Recognizing this as a Fourier transform:**

$$\frac{dX(f)}{df} = \mathcal{F}\{-j2\pi t\, x(t)\}.$$

■

Example 6.7 *Let*

$$x(t) = e^{-t}u(t),$$

where $u(t)$ ***is the unit step function. Compute*** $\frac{dX(f)}{df}$.
Solution:
Step 1: Fourier transform of $x(t)$

$$X(f) = \mathcal{F}\{x(t)\} = \int_0^\infty e^{-t}e^{-j2\pi ft}dt = \int_0^\infty e^{-(1+j2\pi f)t}dt = \frac{1}{1+j2\pi f}.$$

Step 2: Write the integral

$$\frac{dX(f)}{df} = \int_0^\infty \left(-j2\pi t\, e^{-t}\right) e^{-j2\pi ft}dt = -j2\pi \int_0^\infty t\, e^{-(1+j2\pi f)t}dt$$

Let $\alpha = 1 + j2\pi f$***, then:***

$$\frac{dX(f)}{df} = -j2\pi \int_0^\infty t\, e^{-\alpha t}dt$$

—

Step 3: Use the standard integral

$$\int_0^\infty t\, e^{-\alpha t}dt = \frac{1}{\alpha^2}, \quad \Re(\alpha) > 0$$

Hence:

$$\frac{dX(f)}{df} = -j2\pi \cdot \frac{1}{(1+j2\pi f)^2} = -\frac{j2\pi}{(1+j2\pi f)^2}$$

Step 4: Verification by direct differentiation

$$\frac{dX(f)}{df} = \frac{d}{df}\left(\frac{1}{1+j2\pi f}\right) = -\frac{j2\pi}{(1+j2\pi f)^2}.$$

Step 5: Conclusion

$$\frac{dX(f)}{df} = \mathcal{F}\{-j2\pi t\, e^{-t}u(t)\} = -\frac{j2\pi}{(1+j2\pi f)^2}.$$

This confirms that multiplication by t ***in the time domain corresponds to differentiation with respect to*** f ***in the frequency domain.***

8. Fourier Transform of the Complex Conjugate

Theorem 6.9 (Fourier Transform of Complex Conjugate) ***Let*** $x(t)$ ***be a signal with Fourier transform***

$$X(f) = \mathcal{F}\{x(t)\}.$$

Then the Fourier transform of the complex conjugate $\overline{x(t)}$ ***is given by:***

$$\mathcal{F}\{\overline{x(t)}\} = \overline{X(-f)}.$$

Proof.

1. **By definition, the Fourier transform of $\overline{x(t)}$ is**

$$\mathcal{F}\{\overline{x(t)}\} = \int_{-\infty}^{\infty} \overline{x(t)}\, e^{-j2\pi ft}\, dt$$

2. **Consider $X(-f)$:**

$$X(-f) = \int_{-\infty}^{\infty} x(t)\, e^{j2\pi ft}\, dt$$

3. **Take the complex conjugate:**

$$\overline{X(-f)} = \int_{-\infty}^{\infty} \overline{x(t)}\, e^{-j2\pi ft}\, dt$$

4. **Hence:**

$$\mathcal{F}\{\overline{x(t)}\} = \overline{X(-f)}$$

■

Example 6.8 ***Let***

$$x(t) = e^{-j2\pi f_0 t}.$$

Step 1: Fourier transform of $x(t)$

$$X(f) = \delta(f - f_0)$$

Step 2: Complex conjugate of $x(t)$

$$\overline{x(t)} = e^{j2\pi f_0 t}$$

Step 3: Fourier transform of $\overline{x(t)}$

$$\mathcal{F}\{\overline{x(t)}\} = \delta(f + f_0) = \overline{X(-f)}$$

This verifies the theorem.

Remark 6.9 ***- For real signals,*** $\overline{x(t)} = x(t)$***.***
- Therefore, $X(-f) = \overline{X(f)}$***, which is the conjugate symmetry property.***

6.2 Inverse Fourier Transform

6.2.1 Definition

If $X(f)$ is the Fourier transform of $x(t)$, then the original signal $x(t)$ can be recovered by the **inverse Fourier transform:**

$$x(t) = \mathcal{F}^{-1}\{X(f)\} = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi ft}\, df$$

- This integral recovers the time-domain signal from its frequency-domain representation.

Example 6.9 ***Given***

$$X(f) = \frac{1}{1 + j2\pi f},$$

find $x(t) = \mathcal{F}^{-1}\{X(f)\}$***.***

Proof. We recognize this as a standard Fourier pair:

$$\mathcal{F}\{e^{-at}u(t)\} = \frac{1}{a + j2\pi f}, \quad a > 0$$

Comparing with $X(f) = \frac{1}{1+j2\pi f}$***, we have*** $a = 1$***.***
Therefore, the inverse Fourier transform is:

$$x(t) = e^{-t}u(t)$$

■

6.2.2 Properties of the Inverse Fourier Transform

Theorem 6.10 (Linearity)

$$\mathcal{F}^{-1}\{aX_1(f) + bX_2(f)\} = ax_1(t) + bx_2(t)$$

Proof. Follows directly from the linearity of the integral:

$$\int (aX_1(f) + bX_2(f))e^{j2\pi ft}df = a\int X_1(f)e^{j2\pi ft}df + b\int X_2(f)e^{j2\pi ft}df$$

■

Example 6.10

$$X(f) = 2X_1(f) - X_2(f) \implies x(t) = 2x_1(t) - x_2(t)$$

Theorem 6.11 (Time Shifting)

$$\mathcal{F}^{-1}\{X(f)e^{-j2\pi f f_0}\} = x(t - f_0)$$

Proof.

$$\mathcal{F}^{-1}\{X(f)e^{-j2\pi f f_0}\} = \int_{-\infty}^{\infty} X(f)e^{-j2\pi f f_0}e^{j2\pi ft}df = \int_{-\infty}^{\infty} X(f)e^{j2\pi f(t-f_0)}df = x(t - f_0)$$

■

Example 6.11 ***If*** $X(f) = \frac{1}{1+j2\pi f}$ ***and*** $f_0 = 1$***, then***

$$x(t) = e^{-(t-1)}u(t-1)$$

Theorem 6.12 (Frequency Shifting)

$$\mathcal{F}^{-1}\{X(f-f_0)\} = x(t)e^{j2\pi f_0 t}$$

Proof.

$$\mathcal{F}^{-1}\{X(f-f_0)\} = \int X(f-f_0)e^{j2\pi ft}df$$

Change of variable $u = f - f_0$***,*** $du = df$***:***

$$= \int X(u)e^{j2\pi(u+f_0)t}du = e^{j2\pi f_0 t}\int X(u)e^{j2\pi ut}du = x(t)e^{j2\pi f_0 t}$$

■

Example 6.12 ***If*** $X(f) = \frac{1}{1+j2\pi f}$ ***and*** $f_0 = 2$***, then***

$$x(t) = e^{-t}u(t)e^{j4\pi t}$$

Theorem 6.13 (Scaling)

$$\mathcal{F}^{-1}\{X(af)\} = \frac{1}{|a|}x\left(\frac{t}{a}\right), \quad a \neq 0$$

Proof.

$$\mathcal{F}^{-1}\{X(af)\} = \int X(af)e^{j2\pi ft}df$$

Substitute $u = af \implies du = adf$***:***

$$= \int X(u)e^{j2\pi\frac{u}{a}t}\frac{du}{a} = \frac{1}{|a|}\int X(u)e^{j2\pi u\frac{t}{a}}du = \frac{1}{|a|}x\left(\frac{t}{a}\right)$$

■

Example 6.13 ***If*** $X(f) = \frac{1}{1+j2\pi f}$ ***and*** $a = 2$***, then***

$$x(t) = \frac{1}{2}e^{-t/2}u(t/2)$$

—

Theorem 6.14 (Conjugation)

$$\mathcal{F}^{-1}\{\overline{X(f)}\} = \overline{x(-t)}$$

Proof.

$$\mathcal{F}^{-1}\{\overline{X(f)}\} = \int \overline{X(f)}e^{j2\pi ft}df$$

Change variable $u = -f$***,*** $du = -df$***:***

$$= \int \overline{X(-u)}e^{-j2\pi ut}(-du) = \int \overline{X(-u)}e^{-j2\pi ut}du = \overline{x(-t)}$$

■

6.2.3 Parseval's Theorem (Conservation of Energy)

Theorem 6.15 (Parseval's Theorem) ***Let $x(t)$ be a signal with Fourier transform***

$$X(f) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{+\infty} x(t)e^{-j2\pi ft}dt.$$

Then the total energy of the signal is conserved in the frequency domain:

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \int_{-\infty}^{+\infty} |X(f)|^2 df.$$

Proof.
Step 1: Energy in time domain
The energy of the signal in the time domain is defined as

$$E = \int_{-\infty}^{+\infty} |x(t)|^2 dt = \int_{-\infty}^{+\infty} x(t)\overline{x(t)}dt.$$

Step 2: Express $x(t)$ in terms of its Fourier transform
To connect the time-domain energy to the frequency domain, we use the inverse Fourier transform:

$$x(t) = \int_{-\infty}^{+\infty} X(f)e^{j2\pi ft}df.$$

- Writing $x(t)$ in terms of $X(f)$ allows us to express the time-domain energy as an integral over the frequency components. - Each frequency component $X(f)$ contributes to $x(t)$, and therefore to the total energy.
Similarly, the complex conjugate is:

$$\overline{x(t)} = \int_{-\infty}^{+\infty} \overline{X(\nu)}e^{-j2\pi\nu t}d\nu.$$

Step 3: Substitute into the energy integral

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} X(f)e^{j2\pi ft}df\right)\left(\int_{-\infty}^{+\infty} \overline{X(\nu)}e^{-j2\pi\nu t}d\nu\right) dt$$

- Now the energy integral is expressed entirely in terms of the frequency-domain representation.
- This is why we first wrote $x(t)$ in terms of $X(f)$.
Step 4: Rearrange as double integral

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \int_{-\infty}^{+\infty}\int_{-\infty}^{\infty} X(f)\overline{X(\nu)}\left(\int_{-\infty}^{\infty} e^{j2\pi(f-\nu)t}dt\right) df\, d\nu$$

- The inner integral over t gives the Dirac delta:

$$\int_{-\infty}^{+\infty} e^{j2\pi(f-\nu)t}dt = \delta(f-\nu)$$

Step 5: Apply the sifting property of the Dirac delta

$$\int_{-\infty}^{+\infty} X(f)\overline{X(\nu)}\delta(f-\nu)df = X(\nu)\overline{X(\nu)} = |X(\nu)|^2$$

- The delta function "picks out" the value at $f = \nu$, collapsing the double integral into a single integral.
Step 6: Integrate over ν

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(\nu)|^2 d\nu = \int_{-\infty}^{+\infty} |X(f)|^2 df$$

- This confirms that the total energy is conserved in the frequency domain.
■

Example 6.14 ***Let***

$$x(t) = e^{-t}u(t),$$

where $u(t)$ is the unit step function. Verify Parseval's theorem.
Step 1: Energy in time domain

$$\int_0^{\infty} |x(t)|^2 dt = \int_0^{\infty} e^{-2t} dt = \frac{1}{2}.$$

Step 2: Fourier transform of $x(t)$

$$X(f) = \int_0^{\infty} e^{-t}e^{-j2\pi ft} dt = \frac{1}{1+j2\pi f}.$$

Step 3: Energy in frequency domain

$$\int_{-\infty}^{\infty} |X(f)|^2 df = \int_{-\infty}^{\infty} \frac{1}{1+(2\pi f)^2} df$$

Change of variable $u = 2\pi f$, $df = du/(2\pi)$:

$$\int_{-\infty}^{\infty} |X(f)|^2 df = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{du}{1+u^2} = \frac{1}{2\pi}\cdot\pi = \frac{1}{2}.$$

Step 4: Verification

$$\int_0^{\infty} |x(t)|^2 dt = \frac{1}{2} = \int_{-\infty}^{\infty} |X(f)|^2 df$$

This confirms Parseval's theorem.

6.2.4 Fourier Series Expansion of a Non-Periodic Function

1. Fourier Series of a Periodic Function

Let $f_T(t)$ be a periodic function with period T. Its Fourier series is:

$$f_T(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}, \quad \omega_0 = \frac{2\pi}{T}$$

where the Fourier coefficients are:

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f_T(t)e^{-jn\omega_0 t} dt$$

- Each c_n gives the amplitude and phase of the harmonic of frequency $n\omega_0$.

2. Non-Periodic Functions

For a non-periodic function $f(t)$, we consider it as the limit of a periodic function with very large period $T \to \infty$.
- As T increases, $\omega_0 = \frac{2\pi}{T} \to 0$.
- The discrete frequencies $n\omega_0$ become continuous: $\omega = n\omega_0 \in \mathbb{R}$.
- The Fourier series sum transforms into an integral over all frequencies.

3. From Discrete Sum to Integral

Start from the Fourier series:

$$f_T(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}, \quad c_n = \frac{1}{T}\int_{-T/2}^{T/2} f_T(\tau)e^{-jn\omega_0\tau}d\tau$$

Define the frequency increment:

$$\Delta\omega = \omega_0 = \frac{2\pi}{T}$$

Then:

$$c_n = \frac{\Delta\omega}{2\pi}\int_{-T/2}^{T/2} f_T(\tau)e^{-jn\Delta\omega\tau}d\tau$$

$$f_T(t) = \sum_{n=-\infty}^{\infty} \frac{\Delta\omega}{2\pi}\int_{-T/2}^{T/2} f_T(\tau)e^{-jn\Delta\omega\tau}d\tau\, e^{jn\Delta\omega t}$$

4. Limit as $T \to \infty$

Taking $T \to \infty$ $(\Delta\omega \to 0)$:

$$f(t) = \lim_{T\to\infty}\sum_{n=-\infty}^{\infty} \frac{\Delta\omega}{2\pi}\int_{-T/2}^{T/2} f(\tau)e^{-jn\Delta\omega\tau}d\tau\, e^{jn\Delta\omega t}$$

$$\Rightarrow f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(\tau)e^{-j\omega\tau}d\tau\right]e^{j\omega t}d\omega$$

- The term in brackets is the Fourier transform:

$$\boxed{F(\omega) = \int_{-\infty}^{\infty} f(\tau)e^{-j\omega\tau}d\tau}$$

- Therefore, the inverse Fourier transform is:

$$\boxed{f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega}$$

—

5. **Interpretation**

- **Fourier series decomposes a periodic function into discrete harmonics.**
- **For a non-periodic function, the harmonics become continuous, forming a continuous spectrum.**
- **The Fourier transform is the continuous analogue of the Fourier series.**
- **The coefficients c_n become a continuous function $F(\omega)$ of frequency.**

Example 6.15 ***Let*** $f(t) = e^{-at}u(t)$***,*** $a > 0$***.***

$$\begin{aligned} F(\omega) &= \int_0^{\infty} e^{-at} e^{-j\omega t} dt \\ &= \int_0^{\infty} e^{-(a+j\omega)t} dt \\ &= \left[\frac{-1}{a+j\omega} e^{-(a+j\omega)t} \right]_0^{\infty} \\ &= 0 - \left(\frac{-1}{a+j\omega} \right) = \frac{1}{a+j\omega} \end{aligned}$$

- This result shows that the non-periodic function $e^{-at}u(t)$ ***has a **continuous spectrum**, which is the limit of the Fourier series coefficients as*** $T \to \infty$***.***

6.3 Application of the Fourier Transform to Solving Differential Equations

The Fourier transform is a powerful tool to solve linear differential equations. It converts derivatives in the time domain into algebraic multiplication in the frequency domain, simplifying the solution process.

6.3.1 General Principle

Consider a linear differential equation of order n with constant coefficients:

$$y^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + \cdots + a_1 y'(t) + a_0 y(t) = f(t),$$

where $f(t)$ is a known input signal and $y(t)$ is the unknown output.
Steps to solve using Fourier transform:

1. **Apply the Fourier transform to each term. For derivatives:**

$$\mathcal{F}\{y'(t)\} = j2\pi f\, Y(f), \quad \mathcal{F}\{y''(t)\} = (j2\pi f)^2\, Y(f), \quad \ldots$$

2. **The differential equation becomes an algebraic equation in the frequency domain:**

$$\Big((j2\pi f)^n + a_{n-1}(j2\pi f)^{n-1} + \cdots + a_1(j2\pi f) + a_0 \Big) Y(f) = F(f),$$

where $F(f) = \mathcal{F}\{f(t)\}$.

3. **Solve for $Y(f)$:**

$$Y(f) = \frac{F(f)}{(j2\pi f)^n + a_{n-1}(j2\pi f)^{n-1} + \cdots + a_1(j2\pi f) + a_0}.$$

4. **Apply the inverse Fourier transform to recover $y(t)$:**

$$y(t) = \mathcal{F}^{-1}\{Y(f)\}.$$

Remark 6.10 ***Differentiation in time corresponds to multiplication by** $(j2\pi f)^n$ **in frequency, simplifying the solution of linear differential equations.***

Example 6.16 ***1. Example: First-Order Differential Equation***
Solve:

$$y'(t) + y(t) = e^{-t}u(t),$$

where** $u(t)$ **is the unit step function.
Proof. Step 1: Fourier Transform

$$\mathcal{F}\{y'(t)\} + \mathcal{F}\{y(t)\} = \mathcal{F}\{e^{-t}u(t)\}$$

$$(j2\pi f)Y(f) + Y(f) = \frac{1}{1 + j2\pi f} \quad \Rightarrow \quad Y(f)(1 + j2\pi f) = \frac{1}{1 + j2\pi f}$$

$$\Rightarrow Y(f) = \frac{1}{(1 + j2\pi f)^2}.$$

Step 2: Inverse Fourier Transform
Using the known Fourier pair:

$$\mathcal{F}\{te^{-t}u(t)\} = \frac{1}{(1 + j2\pi f)^2} \quad \Rightarrow \quad y(t) = te^{-t}u(t).$$

Solution:

$$y(t) = te^{-t}u(t).$$

■

3. Advantages

- **Converts differential equations into algebraic equations in frequency.**
- **Handles signals defined for all $t \in (-\infty, \infty)$.**
- **Useful for analyzing linear time-invariant (LTI) systems and frequency response.**
- **Simplifies convolutions, since convolution in time corresponds to multiplication in frequency.**
