Chapter 5

Laplace Transform

5.1 Definition and Properties of the Laplace Transform

Definition 5.1 ***The Laplace transform of a function*** $f(t)$***, defined for*** $t \geq 0$***, is the function*** $F(s)$ ***given by***

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^{\infty} f(t)e^{-st}\,dt,$$

where s ***is a complex number such that the integral converges.***

Remark 5.1 ***The Laplace transform exists for all functions*** $f(t)$ ***of exponential order; that is, there exist constants*** $M, \alpha > 0$ ***such that*** $|f(t)| \leq Me^{\alpha t}$ ***for all*** $t \geq 0$***.***

Example 5.1 ***Find the Laplace transform of*** $f(t) = e^{2t}$***, for*** $t \geq 0$***.***

Solution 5.1 ***By definition of the Laplace transform:***

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{2t}e^{-st}\,dt = \int_0^{\infty} e^{-(s-2)t}\,dt$$

Step 1: Condition for convergence
Let $s = \sigma + i\omega$***, with*** $\sigma = \Re(s)$ ***and*** $\omega = \Im(s)$***. Then***

$$e^{-(s-2)t} = e^{-(\sigma-2)t}\,e^{-i\omega t}$$

The magnitude of the integrand is:

$$|e^{-(s-2)t}| = |e^{-(\sigma-2)t}e^{-i\omega t}| = e^{-(\sigma-2)t} \quad (\textbf{\textit{since}}\ |e^{-i\omega t}| = 1)$$

For the improper integral to converge as $t \to \infty$***, the exponent must be negative:***

$$-(\sigma - 2) < 0 \quad \Rightarrow \quad \sigma - 2 > 0 \quad \Rightarrow \quad \sigma > 2$$

Step 2: Compute the integral
If $\Re(s) > 2$***:***

$$\int_0^{\infty} e^{-(s-2)t}\,dt = \left[\frac{e^{-(s-2)t}}{-(s-2)}\right]_0^{\infty} = \frac{1}{s-2}$$

$$\Rightarrow \mathcal{L}\{e^{2t}\} = \frac{1}{s-2}, \quad \Re(s) > 2$$

5.1.1 Properties of the Laplace Transform

1. Linearity:

$$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$$

Example 5.2 ***Let*** $f(t) = e^t$ ***and*** $g(t) = \sin(t)$***, with*** $a = 2$***,*** $b = 3$***.***
Step 1: Compute $\mathcal{L}\{e^t\}$

$$\mathcal{L}\{e^t\} = \int_0^\infty e^t e^{-st} dt = \int_0^\infty e^{-(s-1)t} dt$$

Converges for $\Re(s) > 1$***:***

$$\int_0^\infty e^{-(s-1)t} dt = \frac{1}{s-1}$$

Step 2: Compute $\mathcal{L}\{\sin(t)\}$

$$\mathcal{L}\{\sin(t)\} = \int_0^\infty \sin(t) e^{-st} dt = \frac{1}{s^2+1}, \quad \Re(s) > 0$$

Step 3: Apply linearity:

$$\mathcal{L}\{2e^t + 3\sin(t)\} = 2 \cdot \frac{1}{s-1} + 3 \cdot \frac{1}{s^2+1} = \frac{2}{s-1} + \frac{3}{s^2+1}$$

2. First Derivative:

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$$

Example 5.3 ***Let*** $f(t) = t^2$***. Then*** $f'(t) = 2t$***,*** $f(0) = 0$***.***
Step 1: Compute $\mathcal{L}\{t^2\}$

$$\mathcal{L}\{t^2\} = \int_0^\infty t^2 e^{-st} dt = \frac{2}{s^3}, \quad s > 0$$

Step 2: Use derivative property:

$$\mathcal{L}\{2t\} = \mathcal{L}\{f'(t)\} = s \cdot \frac{2}{s^3} - 0 = \frac{2}{s^2}$$

Step 3: Verify directly:

$$\mathcal{L}\{2t\} = 2\int_0^\infty t e^{-st} dt = 2 \cdot \frac{1}{s^2} = \frac{2}{s^2}$$

3. Second Derivative:

$$\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$$

Example 5.4 ***Let*** $f(t) = t^2$***. Then*** $f''(t) = 2$***,*** $f(0) = 0$***,*** $f'(0) = 0$***.***
Step 1: Compute $\mathcal{L}\{t^2\}$

$$\mathcal{L}\{t^2\} = \frac{2}{s^3}, \quad s > 0$$

Step 2: Apply formula:

$$\mathcal{L}\{f''(t)\} = s^2 \cdot \frac{2}{s^3} - 0 - 0 = \frac{2}{s}$$

Step 3: Direct check:

$$\mathcal{L}\{2\} = \int_0^\infty 2e^{-st} dt = \frac{2}{s}, \quad s > 0$$

4. Integration:

$$\mathcal{L}\Big\{\int_0^t f(\tau)d\tau\Big\} = \frac{\mathcal{L}\{f(t)\}}{s}$$

Example 5.5 ***Let*** $f(t) = e^{2t}$***. Then*** $\int_0^t e^{2\tau}d\tau = \frac{1}{2}(e^{2t} - 1)$***.***
Step 1: Compute $\mathcal{L}\{e^{2t}\} = \frac{1}{s-2}, \quad \Re(s) > 2$
Step 2: Apply integration property:

$$\mathcal{L}\Big\{\int_0^t e^{2\tau}d\tau\Big\} = \frac{\mathcal{L}\{e^{2t}\}}{s} = \frac{1/(s-2)}{s} = \frac{1}{s(s-2)}, \quad \Re(s) > 2$$

5. Shifting Theorem:

The result of the Shifting Theorem comes directly from the definition of the Laplace transform and a simple change of variable in the integral. Formally:

Theorem 5.1 (Exponential Shift / First Shifting Theorem) ***Let*** $f(t)$ ***be a function whose Laplace transform exists, and let***

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt$$

be the Laplace transform of $f(t)$***. Then, for any real constant*** a***:***

$$\mathcal{L}\{f(t)e^{at}\} = \mathcal{L}\{e^{at}f(t)\} = F(s-a)$$

Proof. By definition of the Laplace transform:

$$\mathcal{L}\{e^{at}f(t)\} = \int_0^\infty e^{at}f(t)e^{-st}dt = \int_0^\infty f(t)e^{-(s-a)t}dt = F(s-a)$$

Convergence condition: If $F(s)$ converges for $\Re(s) > \alpha$, then $F(s-a)$ converges for $\Re(s) > \alpha + a$. ■

Example 5.6 ***Let*** $f(t) = t$***. Then***

$$F(s) = \mathcal{L}\{t\} = \int_0^\infty te^{-st}dt = \frac{1}{s^2}, \quad s > 0$$

Take $a = 3$***:***

$$\mathcal{L}\{te^{3t}\} = F(s-3) = \frac{1}{(s-3)^2}, \quad \Re(s) > 3$$

Step-by-step explanation:
i. Compute the Laplace transform of $f(t)$ ***without the exponential:*** $F(s) = 1/s^2$***.***
ii. Apply the exponential shift theorem: replace s ***by*** $s - a = s - 3$ ***in*** $F(s)$***.***
iii. Adjust the convergence region: originally $\Re(s) > 0$***, now*** $\Re(s) > 3$***.***

6. Time Shift (Translation in the Time Domain)

Definition 5.2 (Heaviside Step Function) ***The Heaviside step function*** $u(t-a)$ ***is defined as:***

$$u(t-a) = \begin{cases} 0, & t < a \\ 1, & t \geq a \end{cases}$$

It "turns on" a function at $t = a$ ***and ensures causality for the Laplace transform.***

Theorem 5.2 (Time Shift Property) ***Let*** $f(t)$ ***be a function whose Laplace transform exists. For a real constant*** $a > 0$***, the Laplace transform of the shifted function*** $f(t-a)$ ***multiplied by the Heaviside function*** $u(t-a)$ ***is:***

$$\mathcal{L}\{f(t-a)\,u(t-a)\} = e^{-as}\,F(s),$$

where

$$\mathcal{L}\{f(t)\}(s) = F(s) = \int_0^\infty f(t)e^{-st}\,dt, \quad \Re(s) > 0.$$

Remark 5.2 ***The Heaviside function ensures that the shifted function starts at*** $t = a$***:***

$$f(t-a)\,u(t-a) = \begin{cases} 0, & t < a \\ f(t-a), & t \geq a \end{cases}$$

Proof. **By definition,**

$$\mathcal{L}\{f(t-a)u(t-a)\}(s) = \int_0^\infty f(t-a)u(t-a)e^{-st}\,dt.$$

Since $u(t-a) = 0$ **for** $t < a$**, the integral reduces to**

$$= \int_a^\infty f(t-a)e^{-st}\,dt.$$

With the substitution $\tau = t - a$, $t = \tau + a$, $dt = d\tau$, **we obtain**

$$= \int_0^\infty f(\tau)e^{-s(\tau+a)}\,d\tau = e^{-as}\int_0^\infty f(\tau)e^{-s\tau}\,d\tau.$$

Therefore,

$$\mathcal{L}\{f(t-a)u(t-a)\}(s) = e^{-as}F(s).$$

■

Example 5.7 ***Let*** $f(t) = \sin t$***. Its Laplace transform is***

$$F(s) = \mathcal{L}\{\sin t\}(s) = \frac{1}{s^2+1}.$$

Now consider $g(t) = \sin(t-a)u(t-a)$***. By the time shifting property,***

$$\mathcal{L}\{g(t)\}(s) = e^{-as}F(s) = \frac{e^{-as}}{s^2+1}.$$

Particular Case

For $a = 2$, **we obtain**

$$\mathcal{L}\{\sin(t-2)u(t-2)\}(s) = \frac{e^{-2s}}{s^2+1}.$$

Example 5.8 ***Let*** $f(t) = t$***. Find*** $\mathcal{L}\{f(t-2)u(t-2)\}$***.***

Solution 5.2 We know that $\mathcal{L}\{f(t)\} = \mathcal{L}\{t\} = \dfrac{1}{s^2}$***.***

$$f(t-2)\,u(t-2) = \begin{cases} 0, & t < 2 \\ f(t-a), & t \geq 2 \end{cases}$$

By the time shift property:

$$\mathcal{L}\{f(t-2)u(t-2)\} = \mathcal{L}\{f(t-2)\} = e^{-2s}\mathcal{L}\{f(t)\} = e^{-2s} \cdot \frac{1}{s^2}, \qquad t \geq 2,$$

where

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{t\} = \frac{1}{s^2}$$

So the final result is:

$$\mathcal{L}\{(t-2)\} = \frac{e^{-2s}}{s^2}, \qquad t \geq 2.$$

Explanation:

- ***For*** $t < 2$***, the function is zero.***
- ***For*** $t \geq 2$***, the function grows linearly starting from zero at*** $t = 2$***.***
- ***The multiplication by*** e^{-2s} ***in the Laplace domain corresponds to the shift by 2 units in the time domain.***

7. Dilation and Concentration Property of the Laplace Transform

Let

$$F(s) = \mathcal{L}\{f(t)\}(s) = \int_0^\infty e^{-st} f(t)\,dt.$$

For $a > 0$,

$$\mathcal{L}\{f(at)\}(s) = \frac{1}{a}F\left(\frac{s}{a}\right).$$

- If $a > 1$**: the function** $f(at)$ **is concentrated in the time domain (compressed toward** $t = 0$**).**
- If $0 < a < 1$**: the function** $f(at)$ **is dilated (stretched in time).**
Proof.

$$\mathcal{L}\{f(at)\}(s) = \int_0^\infty e^{-st} f(at)\,dt.$$

Let $u = at \quad \Rightarrow \quad t = \frac{u}{a},\ dt = \frac{du}{a}.$

$$= \int_0^\infty e^{-s\frac{u}{a}} f(u) \cdot \frac{du}{a}.$$

$$= \frac{1}{a}\int_0^\infty e^{-\frac{s}{a}u} f(u)\,du.$$

$$= \frac{1}{a}F\left(\frac{s}{a}\right).$$

■

Example 5.9 Concentration ($a > 1$)
Take $f(t) = e^{-t}$. Its Laplace transform is:

$$F(s) = \frac{1}{s+1}.$$

Now compute for $f(2t) = e^{-2t}$:

$$\mathcal{L}\{e^{-2t}\}(s) = \frac{1}{2}F\Big(\frac{s}{2}\Big) = \frac{1}{2} \cdot \frac{1}{\frac{s}{2}+1} = \frac{1}{s+2}.$$

Dilation ($0 < a < 1$)
Take $f(t) = e^{-t}$. Again,

$$F(s) = \frac{1}{s+1}.$$

Now compute for $f\Big(\frac{t}{2}\Big) = e^{-\frac{t}{2}}$:

$$\mathcal{L}\{e^{-\frac{t}{2}}\}(s) = \frac{1}{\frac{1}{2}}F(2s) = 2 \cdot \frac{1}{2s+1} = \frac{2}{2s+1}.$$

Conclusion
- For $a > 1$, the signal is compressed in time (concentration).
- For $0 < a < 1$, the signal is stretched in time (dilation).

8. Derivative of the Laplace Transform

Let $f : [0, \infty) \to \mathbb{C}$ be a function of *exponential order*: there exist constants $M > 0$ and $k \in \mathbb{R}$ such that

$$|f(t)| \leq Me^{kt} \quad \textbf{for all } t \geq 0.$$

Then the Laplace transform

$$F(s) = \mathcal{L}\{f\}(s) = \int_0^\infty f(t)e^{-st}\,dt$$

exists for all s with $\Re(s) > k$. Under these hypotheses (and sufficient regularity conditions on f), we can differentiate $F(s)$ with respect to s by differentiating under the integral sign.

First derivative

Starting from the definition,

$$F(s) = \int_0^\infty f(t)e^{-st}\,dt,$$

we compute the derivative with respect to s:

$$\begin{aligned} F'(s) &= \frac{d}{ds}\int_0^\infty f(t)e^{-st}\,dt = \int_0^\infty f(t)\,\frac{\partial}{\partial s}\Big(e^{-st}\Big)\,dt \\ &= \int_0^\infty f(t)\,(-t)e^{-st}\,dt = -\int_0^\infty t\,f(t)\,e^{-st}\,dt. \end{aligned}$$

Thus we obtain the fundamental relation

$$\boxed{F'(s) = -\mathcal{L}\{t\, f(t)\}(s)} \qquad \big(\Re(s) > k\big).$$

General formula for the n-th derivative By iterating this process, for any integer $n \geq 1$ we have

$$\boxed{F^{(n)}(s) = (-1)^n \int_0^\infty t^n f(t)\, e^{-st}\, dt = (-1)^n\, \mathcal{L}\{t^n f(t)\}(s)}$$

or equivalently

$$\boxed{\mathcal{L}\{t^n f(t)\}(s) = (-1)^n F^{(n)}(s)\,.}$$

Example 5.10 ***Compute*** $\mathcal{L}\{t^2 \sin t\}(s)$ ***for*** $\Re(s) > 0$***.***

Solution 5.3 ***We use the general formula***

$$\mathcal{L}\{t^n f(t)\}(s) = (-1)^n F^{(n)}(s),$$

with $f(t) = \sin t$ ***and*** $F(s) = \mathcal{L}\{\sin t\}(s) = \dfrac{1}{s^2+1}$ ***(valid for*** $\Re(s) > 0$***).***
Compute derivatives of $F(s)$***:***

$$F(s) = (s^2+1)^{-1}.$$

First derivative:

$$F'(s) = -\frac{2s}{(s^2+1)^2}.$$

Second derivative:

$$\begin{aligned} F''(s) &= -2(s^2+1)^{-2} + 8s^2(s^2+1)^{-3} \\ &= \frac{-2(s^2+1) + 8s^2}{(s^2+1)^3} \\ &= \frac{-2+6s^2}{(s^2+1)^3} \\ &= -2\,\frac{1-3s^2}{(s^2+1)^3}. \end{aligned}$$

Finally, by the formula with $n = 2$***:***

$$\mathcal{L}\{t^2 \sin t\}(s) = (-1)^2 F''(s) = F''(s).$$

Therefore,

$$\boxed{\mathcal{L}\{t^2 \sin t\}(s) = -2\,\frac{1-3s^2}{(s^2+1)^3}, \qquad \Re(s) > 0.}$$

9. Initial and Final Value Theorems

Theorem 5.3 (Initial Value Theorem (IVT)) ***Let $f(t)$ be a causal function (i.e., $f(t) = 0$ for $t < 0$), piecewise continuous on $[0, \infty)$, and of exponential order (that is, there exist constants $M > 0$ and $\alpha \geq 0$ such that $|f(t)| \leq Me^{\alpha t}$ for all $t \geq 0$). If its Laplace transform is***

$$F(s) = \mathcal{L}\{f(t)\}(s) = \int_0^\infty f(t)e^{-st}\,dt,$$

then

$$\boxed{\lim_{t\to 0^+} f(t) = \lim_{s\to\infty} sF(s).}$$

Example 5.11 ***Let $f(t) = \sin t$. Then***

$$F(s) = \frac{1}{s^2+1}.$$

Compute:

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty} \frac{s}{s^2+1} = 0.$$

On the other hand,

$$\lim_{t\to 0^+} \sin t = 0.$$

Thus, the theorem is verified.

Theorem 5.4 (Final Value Theorem (FVT)) ***Let $f(t)$ satisfy the same conditions as above (causal, piecewise continuous, and of exponential order, i.e. $|f(t)| \leq Me^{\alpha t}$). If $\lim_{t\to\infty} f(t)$ exists and all poles of $sF(s)$ have negative real parts (except possibly at $s = 0$), then***

$$\boxed{\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s).}$$

Example 5.12 ***Let $f(t) = 1 - e^{-2t}$. Then***

$$F(s) = \frac{1}{s} - \frac{1}{s+2}.$$

Compute:

$$\lim_{s\to 0} sF(s) = \lim_{s\to 0}\left(1 - \frac{s}{s+2}\right) = 1.$$

On the other hand,

$$\lim_{t\to\infty}\left(1 - e^{-2t}\right) = 1.$$

Thus, the theorem is verified.

10. Laplace Transform of a Convolution

Definition 5.3 (Definition of Convolution) ***Let*** $f(t)$ ***and*** $g(t)$ ***be two causal functions (i.e.,*** $f(t) = g(t) = 0$ ***for*** $t < 0$***). Their convolution is defined by***

$$(f * g)(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau, \qquad t \geq 0.$$

Theorem 5.5 (Convolution Theorem) ***If*** $F(s) = \mathcal{L}\{f(t)\}(s)$ ***and*** $G(s) = \mathcal{L}\{g(t)\}(s)$***, then***

$$\boxed{\mathcal{L}\{(f * g)(t)\}(s) = F(s)\, G(s).}$$

Proof. Start from the definition of the Laplace transform:

$$\mathcal{L}\{(f * g)(t)\}(s) = \int_0^\infty \left(\int_0^t f(\tau) g(t-\tau)\, d\tau \right) e^{-st}\, dt.$$

By Fubini's theorem (changing the order of integration), this becomes

$$\int_0^\infty f(\tau) \left(\int_0^\infty g(u) e^{-s(u+\tau)}\, du \right) d\tau = \left(\int_0^\infty f(\tau) e^{-s\tau}\, d\tau \right) \left(\int_0^\infty g(u) e^{-su}\, du \right).$$

Thus,

$$\mathcal{L}\{(f * g)(t)\}(s) = F(s)G(s).$$

■

Example 5.13 ***Let*** $f(t) = 1$ ***and*** $g(t) = t$ ***for*** $t \geq 0$***. Then their convolution is***

$$(f * g)(t) = \int_0^t 1 \cdot (t-\tau)\, d\tau = \int_0^t (t-\tau)\, d\tau = \frac{t^2}{2}.$$

Now apply the Laplace transform:

$$\mathcal{L}\{(f * g)(t)\}(s) = \mathcal{L}\left\{\tfrac{t^2}{2}\right\}(s) = \frac{1}{s^3}.$$

On the other hand, by the theorem:

$$F(s) = \mathcal{L}\{1\}(s) = \frac{1}{s}, \quad G(s) = \mathcal{L}\{t\}(s) = \frac{1}{s^2},$$

so

$$F(s)G(s) = \frac{1}{s} \cdot \frac{1}{s^2} = \frac{1}{s^3}.$$

Both results agree, verifying the theorem.

5.2 Common Laplace Transforms

Time-domain function $f(t)$	**Laplace transform** $F(s) = \mathcal{L}\{f(t)\}(s)$
1 **(constant)**	$\frac{1}{s}, \quad \Re(s) > 0$
t	$\frac{1}{s^2}, \quad \Re(s) > 0$
$t^n,\ n \in \mathbb{N}$	$\frac{n!}{s^{n+1}}, \quad \Re(s) > 0$
e^{at}	$\frac{1}{s-a}, \quad \Re(s) > a$
$\cos(at)$	$\frac{s}{s^2+a^2}, \quad \Re(s) > 0$
$\sin(at)$	$\frac{a}{s^2+a^2}, \quad \Re(s) > 0$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}, \quad \Re(s) > a$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}, \quad \Re(s) > a$
Unit step $u(t)$	$\frac{1}{s}, \quad \Re(s) > 0$
Shifted step $u(t-a)$	$\frac{e^{-as}}{s}, \quad a > 0,\ \Re(s) > 0$
Dirac impulse $\delta(t)$	1
Shifted impulse $\delta(t-a)$	$e^{-as}, \quad a \geq 0$
$\frac{\sin(at)}{t}$	$\arctan\left(\frac{a}{s}\right), \quad \Re(s) > 0$
$\cosh(at)$	$\frac{s}{s^2-a^2}, \quad \Re(s) > \lvert a\rvert$
$\sinh(at)$	$\frac{a}{s^2-a^2}, \quad \Re(s) > \lvert a\rvert$

5.3 Inverse Laplace Transform

Definition 5.4 ***The inverse Laplace transform is the operation that allows us to recover the original function*** $f(t)$ ***from its Laplace transform*** $F(s)$***. We denote it by***

$$f(t) = \mathcal{L}^{-1}\{F(s)\}.$$

Formally, if $F(s) = \mathcal{L}\{f(t)\}$***, then***

$$\mathcal{L}^{-1}\{F(s)\} = f(t).$$

Remark 5.3 ***The inverse Laplace transform is unique for functions of exponential order, meaning that each Laplace transform corresponds to a single original function*** $f(t)$ ***(up to piecewise continuity).***

Example 5.14 ***Find the inverse Laplace transform of***

$$F(s) = \frac{1}{s-2}, \quad \Re(s) > 2.$$

Solution 5.4 By the definition of the Laplace transform:

$$\mathcal{L}\{e^{2t}\} = \int_0^\infty e^{2t}e^{-st}dt = \frac{1}{s-2}, \quad \Re(s) > 2.$$

Hence, by matching $F(s)$ with a known Laplace transform:

$$\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} = e^{2t}.$$

5.3.1 Properties of the Inverse Laplace Transform

Let $(\mathcal{L}f(t) = F(s))$. Then the inverse Laplace transform is defined by

$$f(t) = \mathcal{L}^{-1}F(s).$$

1. Linearity

$$\mathcal{L}^{-1}aF(s) + bG(s) = a, \mathcal{L}^{-1}F(s) + b, \mathcal{L}^{-1}G(s).$$

2. Inverse of a Shift in the (s)-domain

$$\mathcal{L}^{-1}F(s-a) = e^{at}f(t), \quad t \geq 0.$$

3. Inverse of Multiplication by (s)

$$\mathcal{L}^{-1}sF(s) = f'(t) + f(0), \delta(t).$$

4. Inverse of Division by (s)

$$\mathcal{L}^{-1}(\frac{F(s)}{s}) = \int_0^t f(\tau), d\tau.$$

5. Inverse of Multiplication by (e^{-as})

$$\mathcal{L}^{-1}(e^{-as}F(s)) = u(t-a).f(t-a).$$

6. Inverse of Multiplication by (s^n)

$$\mathcal{L}^{-1}(s^nF(s)) = f^{(n)}(t) + \textbf{initial conditions (Dirac terms)}.$$

7. Inverse of Differentiation in (s)

$$\mathcal{L}^{-1}(\frac{dF}{ds}) = -tf(t).$$

8. Inverse of a Product

$$\mathcal{L}^{-1}F(s)G(s) = (f*g)(t) = \int_0^t f(\tau)g(t-\tau), d\tau.$$

Examples 5.1 Example 1: Linearity

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2}+\frac{1}{s^2+1}\right\}=t+\sin t.$$

Example 2: Shift in the s-domain

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-2)^2+1}\right\}=e^{2t}\sin t.$$

Example 3: Time shift

$$\mathcal{L}^{-1}\left\{e^{-3s}\,\frac{1}{s^2}\right\}=u(t-3)(t-3).$$

Example 4: Multiplication by s

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2}\right\}=1.$$

Example 5: Division by s

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s+1)}\right\}=1-e^{-t}.$$

Example 6: Differentiation in s

$$\mathcal{L}^{-1}\left\{-\frac{1}{s^2}\right\}=-t.$$

Example 7: Convolution

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s+1)}\right\}=\int_0^t e^{-(t-\tau)}\,d\tau=1-e^{-t}.$$

5.4 Application of Laplace Transforms to Differential Equations

Laplace transforms are a powerful tool for solving linear differential equations with given initial conditions. The main idea is to transform the differential equation into an algebraic equation in the variable s, solve it, and then apply the inverse Laplace transform to find the solution in the time domain.

General Method

1. **Apply the Laplace transform to both sides of the differential equation.**
2. **Use the properties of the Laplace transform to handle derivatives:**

$$\mathcal{L}\{y'(t)\}=sY(s)-y(0),\quad \mathcal{L}\{y''(t)\}=s^2Y(s)-sy(0)-y'(0).$$

3. **Insert the initial conditions into the equation.**
4. **Solve the resulting algebraic equation for $Y(s)$.**
5. **Apply the inverse Laplace transform to obtain $y(t)$.**