Chapter 4

Series

4.1 Numerical Series

4.1.1 Introduction to Numerical Series

In many problems of physics, we encounter quantities that are obtained by adding infinitely many contributions:

- the total distance traveled by an oscillating spring with decreasing amplitudes,
- the total intensity emitted by a radioactive particle in successive pulses,
- the energy stored in an electric circuit with repeated damping,
- the Fourier expansion of a periodic signal.

In such cases, it is essential to understand how to give a precise meaning to an "infinite sum".

4.1.2 Sequences and Numerical Series

Definition 4.1 (Partial sum) ***Let*** $(u_n)_{n\in\mathbb{N}}$ ***be a sequence of real numbers. The partial sum of order*** n ***is defined as***

$$S_n = \sum_{k=0}^{n} u_k = u_0 + u_1 + u_2 + \cdots + u_n.$$

Definition 4.2 (Numerical series) ***The series associated with*** (u_n) ***is the sequence of partial sums*** (S_n)***. We write***

$$\sum_{n=0}^{+\infty} u_n = \lim_{n\to+\infty} S_n,$$

whenever this limit exists.

Definition 4.3 (Convergence and divergence) ***If the sequence of partial sums*** (S_n) ***has a finite limit*** S***, we say that the series*** converges ***and we write***

$$\sum_{n=0}^{+\infty} u_n = S.$$

If (S_n) ***does not have a finite limit, the series is said to*** diverge***.***

Example 4.1 (Distance traveled by a moving object) ***A moving object first travels a distance*** d***, then half of the previous distance, then half again, and so on.***
The successive displacements are:

$$d, \quad \frac{d}{2}, \quad \frac{d}{4}, \quad \frac{d}{8}, \quad \dots$$

The n***-th partial sum is:***

$$S_n = d + \frac{d}{2} + \frac{d}{4} + \cdots + \frac{d}{2^n}.$$

This is a geometric sum:

$$S_n = d \cdot \frac{1 - \left(\frac{1}{2}\right)^{n+1}}{1 - \frac{1}{2}} = 2d\left(1 - \frac{1}{2^{n+1}}\right).$$

As $n \to +\infty$***,***

$$S_n \to 2d.$$

Interpretation: ***even though the object makes infinitely many steps, the total distance traveled is finite and equal to*** $2d$***.***

Theorem 4.1 ***If the series*** $\sum_{n=1}^{\infty} u_n$ ***is convergent, then its general term*** u_n ***tends to zero:***

$$\lim_{n\to\infty} u_n = 0.$$

Corollary 4.1 ***If the general term*** u_n ***of the series*** $\sum_{n=1}^{\infty} u_n$ ***does not tend to zero, then the series diverges.***

Example 4.2 ***Consider the series***

$$\sum_{n=1}^{\infty} \frac{n+1}{n}.$$

Here, the general term is

$$u_n = \frac{n+1}{n} = 1 + \frac{1}{n}.$$

Since

$$\lim_{n\to\infty} u_n = 1 \neq 0,$$

the general term does not tend to zero.
By the contrapositive of the Theorem 4.1 this series diverges.

4.1.3 Geometric Series

Definition 4.4 ***A*** geometric series ***is a series of the form***

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \cdots,$$

where a ***is the first term and*** r ***is the common ratio.***

Study of the Convergence:

We compute the partial sum of order n**:**

$$S_n = \sum_{k=0}^{n} ar^k = \begin{cases} a\underbrace{(1+1+1+\cdots+1)}_{n+1 \textbf{ times}} = a(n+1), & \textbf{if } r = 1, \\ \dfrac{a(1-r^{n+1})}{1-r}, & \textbf{if } r \neq 1. \end{cases}$$

Now, let us study the convergence of S_n **as** $n \to \infty$**:**

$$\lim_{n\to\infty} S_n = \begin{cases} \lim_{n\to\infty} a(n+1) = +\infty, & \textbf{if } r = 1, \\ \lim_{n\to\infty} \dfrac{a(1-r^{n+1})}{1-r}, & \textbf{if } r \neq 1. \end{cases}$$

For the case $r \neq 1$**, we analyze according to** $|r|$**:**

$$\lim_{n\to\infty} \frac{a(1-r^{n+1})}{1-r} = \begin{cases} \dfrac{a(1-\lim\limits_{n\to\infty} r^{n+1})}{1-r} = \dfrac{a}{1-r}, & \textbf{if } |r| < 1, \\ \dfrac{a(1-\lim\limits_{n\to\infty} r^{n+1})}{1-r} = \dfrac{a(1-\infty)}{1-r}, \quad \textbf{diverges}, & \textbf{if } |r| > 1. \end{cases}$$

Example 4.3 ***(successive reflections of light)***
Consider a ray of light (rayon de lumiére) entering a system of two parallel mirrors with a reflection coefficient r***, where*** $0 < r < 1$***. At the first reflection, the intensity of the ray is***

$$I_0 = I.$$

At the second reflection,

$$I_1 = Ir.$$

At the third reflection,

$$I_2 = Ir^2,$$

and so on.
The total intensity after infinitely many reflections is

$$I_{total} = I + Ir + Ir^2 + Ir^3 + \cdots = \sum_{n=0}^{\infty} Ir^n.$$

This is a geometric series with first term $a = I$ ***and common ratio*** r***. Since*** $0 < r < 1$***, the series converges and***

$$I_{total} = \frac{I}{1-r}.$$

4.1.4 Exponential Series

Definition 4.5 (Taylor expansion of a general function) ***Let $f(x)$ be a function infinitely differentiable at $x = 0$. Its*** Taylor expansion ***of order N around $x = 0$ is***

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(N)}(0)}{N!}x^N + R_{N+1}(x),$$

where $R_{N+1}(x)$ is the remainder term. As N becomes very large, $R_{N+1}(x) \to 0$.

Example 4.4 (Application to the exponential function) ***Consider $f(x) = e^x$. All derivatives satisfy***

$$f^{(n)}(x) = e^x \quad \Rightarrow \quad f^{(n)}(0) = 1.$$

Thus, the Taylor expansion of order N is

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^N}{N!} + R_{N+1}(x),$$

with $R_{N+1}(x) \to 0$ as $N \to \infty$.

Definition 4.6 ***The exponential series is defined as***

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} := \lim_{N\to\infty} \sum_{n=0}^{N} \frac{x^n}{n!}.$$

Study of the Convergence

We can write the finite sum explicitly up to order N:

$$\sum_{n=0}^{N} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^N}{N!} = \left(e^x - R_{N+1}(x)\right).$$

Taking the limit on both sides, we have

$$\lim_{N\to\infty} \sum_{n=0}^{N} \frac{x^n}{n!} = \lim_{N\to\infty} \left(e^x - R_{N+1}(x)\right),$$

where $R_{N+1}(x)$ is the remainder term from the Taylor expansion of e^x.
For N sufficiently large, the remainder $R_{N+1}(x)$ tends to zero. Therefore,

$$\lim_{N\to\infty} \sum_{n=0}^{N} \frac{x^n}{n!} = e^x.$$

Hence, the exponential series is convergent and converges to e^x.

Example 4.5 ***Consider the exponential series with $x = 1$:***

$$\sum_{n=0}^{\infty} \frac{1}{n!} = \lim_{N\to\infty} \sum_{n=0}^{N} \frac{1}{n!} = e^1 = e \approx 2.71828.$$

Example 4.6 (Small displacement approximation) ***Consider a mass m attached to a spring with spring constant k. For a very small displacement x, the potential energy is***

$$U(x) = \frac{1}{2}kx^2.$$

We want to compute $e^{-kx/m}$ for small x.

Step 1: Use the exponential series

$$e^x = \lim_{N\to\infty} \sum_{n=0}^{N} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Step 2: Replace x by $-kx/m$

$$e^{-kx/m} = \lim_{N\to\infty} \sum_{n=0}^{N} \frac{(-kx/m)^n}{n!} = 1 - \frac{kx}{m} + \frac{(kx)^2}{2m^2} - \frac{(kx)^3}{6m^3} + \cdots$$

Step 3: Approximation for small x For very small x, higher order terms are negligible:

$$e^{-kx/m} \approx 1 - \frac{kx}{m} + \frac{(kx)^2}{2m^2}.$$

This is widely used in physics for linear approximations.

4.1.5 Series With Non-Negative Terms

Definition 4.7 ***A series $\sum_{n=0}^{\infty} u_n$ is called a*** **series with non-negative terms** ***if***

$$u_n \geq 0 \quad \textit{for all } n \geq 0.$$

1. Key Property: Monotonicity of the Partial Sums

Proposition 4.1 ***Let $\sum_{n=0}^{\infty} u_n$ be a series with non-negative terms, and let***

$$S_N = \sum_{n=0}^{N} u_n$$

be its sequence of partial sums.
Since each term $u_n \geq 0$, we have

$$S_{N+1} = S_N + u_{N+1} \geq S_N.$$

Hence, the sequence (S_N) is increasing.
Therefore, the convergence of the series reduces to checking whether the sequence (S_N) is bounded above:

$$\textit{The series converges} \iff \textit{the partial sums } S_N \textit{ are bounded.}$$

Example 4.7 ***Consider the series***

$$\sum_{n=0}^{\infty} \frac{1}{2^n}.$$

Here, each term is non-negative:

$$u_n = \frac{1}{2^n} \geq 0 \quad \textbf{\textit{for all}}\ n \geq 0.$$

The partial sums are

$$S_N = \sum_{n=0}^{N} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^N}.$$

Since this is a geometric series with first term $a = 1$ ***and ratio*** $r = \frac{1}{2}$***, the sum of the first*** $N+1$ ***terms is***

$$S_N = \frac{1 - r^{N+1}}{1 - r} = \frac{1 - (1/2)^{N+1}}{1 - 1/2} = 2\left(1 - \frac{1}{2^{N+1}}\right).$$

Because $\frac{1}{2^{N+1}} > 0$***, we have***

$$S_N = 2\left(1 - \frac{1}{2^{N+1}}\right) < 2.$$

Thus, the sequence of partial sums (S_N) ***is increasing and bounded above by 2. Therefore, the series converges.***

2. Riemann Series

Definition 4.8 ***A Riemann series is a series of the form***

$$\sum_{n=1}^{\infty} \frac{1}{n^p}, \qquad p > 0.$$

- ***If*** $p > 1$***, the series*** $\sum \frac{1}{n^p}$ ***converges.***
- ***If*** $0 < p \leq 1$***, the series*** $\sum \frac{1}{n^p}$ ***diverges.***

Example 4.8 ***we present two cases:***
a. Convergent series:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \quad \textbf{\textit{(here}}\ p = 2 > 1,\ \textbf{\textit{so the series converges).}}$$

b. Divergent series:

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}} \quad \textbf{\textit{(here}}\ p = 1/2 \leq 1,\ \textbf{\textit{so the series diverges).}}$$

3. Bertrand Series

Definition 4.9 (Bertrand Series) ***A Bertrand series is a series of positive terms defined by***

$$\sum_{n=2}^{\infty} \frac{1}{n^{\alpha}(\ln n)^{\beta}},$$

where

$$n \geq 2, \quad \alpha > 0, \quad \beta \in \mathbb{R}.$$

Theorem 4.2 (Convergence of Bertrand Series) ***Consider the Bertrand series***

$$\sum_{n=2}^{\infty} \frac{1}{n^{\alpha}(\ln n)^{\beta}}, \quad \alpha > 0,\ \beta \in \mathbb{R}.$$

The convergence depends on α ***and*** β ***as follows:***

1. $\alpha > 1$***: The series converges for all*** $\beta \in \mathbb{R}$***.***
2. $\alpha < 1$***: The series diverges for all*** $\beta \in \mathbb{R}$***.***
3. $\alpha = 1$***: The series reduces to***

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{\beta}}.$$

In this case:

- ***If*** $\beta > 1$***, the series converges.***
- ***If*** $\beta \leq 1$***, the series diverges.***

Remark 4.1 ***Let us consider the following cases:***

- ***Case*** $\alpha > 1$***: Compare with the convergent Riemann series*** $\sum 1/n^{\alpha}$***. Since***

$$0 < \frac{1}{n^{\alpha}(\ln n)^{\beta}} \leq \frac{1}{n^{\alpha}}, \quad n \geq 2,$$

the series converges by the comparison test for any β***.***

- ***Case*** $\alpha < 1$***: Compare with the divergent Riemann series*** $\sum 1/n^{\alpha}$***. Since***

$$\frac{1}{n^{\alpha}(\ln n)^{\beta}} \geq \frac{1}{n^{\alpha}(\ln 2)^{\beta}} > 0, \quad n \geq 2,$$

the series diverges by the comparison test for any β***.***

- ***Case*** $\alpha = 1$***: Use the integral test with***

$$f(x) = \frac{1}{x(\ln x)^{\beta}}, \quad x \geq 2.$$

Substituting $t = \ln x$ ***(***$dx = x\,dt$***), we get***

$$\int_{2}^{\infty} \frac{dx}{x(\ln x)^{\beta}} = \int_{\ln 2}^{\infty} \frac{dt}{t^{\beta}}.$$

A p-integral $\int_A^{\infty} t^{-p} dt$ ***converges if and only if*** $p > 1$***. Hence:***

- $\beta > 1 \implies$ ***series converges.***
- $\beta \leq 1 \implies$ ***series diverges.***

α	β	Convergence
$\alpha > 1$	any β	Convergent
$\alpha < 1$	any β	Divergent
$\alpha = 1$	$\beta > 1$	Convergent
$\alpha = 1$	$\beta \leq 1$	Divergent

Example 4.9 (Simple Bertrand Series) ***Consider the series***

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}.$$

- This is a Bertrand series with $\alpha = 1$ ***and*** $\beta = 2$.
- Since $\alpha = 1$ ***and*** $\beta > 1$, ***the series converges by the integral test.***

4. Comparison Convergence Criterion

Theorem 4.3 ***Let*** $\sum u_n$ ***and*** $\sum v_n$ ***be two series with non-negative terms, i.e.,*** $u_n \geq 0$ ***and*** $v_n \geq 0$ ***for all*** $n \geq 1$.

- ***If there exists a constant*** $C > 0$ ***such that*** $u_n \leq C\, v_n$ ***for all sufficiently large*** n ***and if*** $\sum v_n$ ***converges, then*** $\sum u_n$ ***also converges.***
- ***If*** $u_n \geq v_n \geq 0$ ***for all sufficiently large*** n ***and if*** $\sum v_n$ ***diverges, then*** $\sum u_n$ ***also diverges.***

Example 4.10 ***Study the convergence of***

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + 3n + 1}.$$

Solution: For all $n \geq 1$,

$$n^2 + 3n + 1 \geq n^2 \quad \Rightarrow \quad \frac{1}{n^2 + 3n + 1} \leq \frac{1}{n^2}.$$

Since $\sum \frac{1}{n^2}$ ***converges, by the Theorem 4.3, the given series also converges.***

5. Equivalence Convergence Criterion

Definition 4.10 (Equivalent Series) ***Let***

$$\sum_{n=1}^{\infty} u_n \quad \text{and} \quad \sum_{n=1}^{\infty} v_n$$

be two series with positive terms. We say that the series are equivalent if and only if ***their terms are asymptotically equal, that is,***

$$\lim_{n \to \infty} \frac{u_n}{v_n} = 1,$$

and we write $u_n \sim v_n$ ***as*** $n \to \infty$.

Theorem 4.4 (Equivalence Test) ***If***

$$\sum_{n=1}^{\infty} u_n \quad \textbf{\textit{and}} \quad \sum_{n=1}^{\infty} v_n$$

are equivalent series with positive terms, then

$$\sum_{n=1}^{\infty} u_n \ \textbf{\textit{converges}} \iff \sum_{n=1}^{\infty} v_n \ \textbf{\textit{converges}}.$$

In other words, equivalent series are simultaneously convergent or divergent.

Example 4.11 ***Study the convergence of the series***

$$\sum_{n=1}^{\infty} u_n, \quad \textbf{\textit{where}}\ u_n = \frac{n+1}{n^3+n}.$$

Solution: We compare u_n ***with*** $v_n = \frac{1}{n^2}$.
Compute the limit:

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\frac{n+1}{n^3+n}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{(n+1)n^2}{n^3+n} = \lim_{n\to\infty} \frac{n^3+n^2}{n^3+n} = 1.$$

Since $\sum v_n = \sum \frac{1}{n^2}$ ***converges and the limit equals 1, by the equivalence criterion, the series*** $\sum u_n$ ***also converges.***

6. D'Alembert's Convergence Criterion (D'Alembert's Ratio Test)

Theorem 4.5 (D'Alembert's Ratio Test) ***Let*** $\sum u_n$ ***be a series with positive terms. Suppose the limit***

$$L = \lim_{n\to\infty} \frac{u_{n+1}}{u_n}$$

exists. Then:

- ***If*** $L < 1$***, the series*** $\sum u_n$ ***converges.***
- ***If*** $L > 1$***, the series*** $\sum u_n$ ***diverges.***
- ***If*** $L = 1$***, the test is inconclusive.***

Example 4.12 ***Study the convergence of the series***

$$\sum_{n=1}^{\infty} \frac{2^n}{n!}.$$

Solution: Let $u_n = \frac{2^n}{n!}$***. Compute the ratio:***

$$\frac{u_{n+1}}{u_n} = \frac{2^{n+1}/(n+1)!}{2^n/n!} = \frac{2^{n+1} \cdot n!}{2^n \cdot (n+1)!} = \frac{2}{n+1}.$$

Take the limit as $n \to \infty$:

$$L = \lim_{n\to\infty} \frac{2}{n+1} = 0 < 1.$$

Since $L < 1$, by the ratio test, the series $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ converges.

5. Cauchy's Convergence Criterion

Theorem 4.6 (Cauchy's Root Test for Positive-Term Series) ***Let $\sum_{n=1}^{\infty} u_n$ be a series with positive terms ($u_n \geq 0$). Suppose the limit***

$$L = \lim_{n\to\infty} \sqrt[n]{u_n}\,.$$

- ***If $L < 1$, the series $\sum_{n=1}^{\infty} u_n$ converges.***
- ***If $L > 1$, the series $\sum_{n=1}^{\infty} u_n$ diverges.***
- ***If $L = 1$, the test is inconclusive.***

Example 4.13 ***Study the convergence of the series***

$$\sum_{n=1}^{\infty} \frac{3^n}{2^n+1}.$$

Solution: Let

$$u_n = \frac{3^n}{2^n+1}.$$

We apply Cauchy's root test and compute the n-th root of u_n:

$$\sqrt[n]{u_n} = \sqrt[n]{\frac{3^n}{2^n+1}} = \frac{\sqrt[n]{3^n}}{\sqrt[n]{2^n+1}}.$$

Step by step:
a. Numerator: $\sqrt[n]{3^n} = 3$.
b. Denominator:

$$\sqrt[n]{2^n+1} = \sqrt[n]{2^n\left(1+\frac{1}{2^n}\right)} = \sqrt[n]{2^n} \cdot \sqrt[n]{1+\frac{1}{2^n}} = 2 \cdot \sqrt[n]{1+\frac{1}{2^n}}.$$

c. Limit of the second factor:

$$\lim_{n\to\infty} \sqrt[n]{1+\frac{1}{2^n}} = 1.$$

d. Hence, the n-th root limit:

$$\lim_{n\to\infty} \sqrt[n]{u_n} = \frac{3}{2} > 1.$$

Since the limit is greater than 1, by Cauchy's root test, the series

$$\sum_{n=1}^{\infty} \frac{3^n}{2^n+1}$$

diverges.

7. Cauchy's Integral Convergence Criterion

Theorem 4.7 (Cauchy's Integral Test) ***Let*** $f : [1, \infty) \to \mathbb{R}$ ***be a continuous, positive, and decreasing function. Consider the series***

$$\sum_{n=1}^{\infty} u_n \quad \textbf{\textit{with}} \quad u_n = f(n).$$

Then the series $\sum_{n=1}^{\infty} u_n$ ***converges if and only if the improper integral***

$$\int_1^{\infty} f(x)\,dx$$

converges.

Example 4.14 ***Study the convergence of the series***

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Solution: Consider the function

$$f(x) = \frac{1}{x^2}, \qquad x \geq 1.$$

- $f(x)$ ***is continuous, positive, and decreasing on*** $[1, \infty)$***.***
- ***By Cauchy's integral test, we compare the series with the improper integral***

$$\int_1^{\infty} \frac{dx}{x^2}.$$

Compute the integral:

$$\int_1^{\infty} \frac{dx}{x^2} = \lim_{t\to\infty} \int_1^{t} x^{-2}dx = \lim_{t\to\infty} \left[-\frac{1}{x}\right]_1^t = \lim_{t\to\infty} \left(-\frac{1}{t} + 1\right) = 1.$$

Since the integral converges, by Cauchy's integral test, the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges.

4.1.6 Alternating Series

Definition 4.11 (Alternating Series) ***A series***

$$\sum_{n=1}^{\infty} u_n$$

is called alternating if the terms alternate in sign, i.e.,

$$u_n \cdot u_{n+1} < 0 \quad \textbf{\textit{for all}}\ n \geq 1,$$

or equivalently,

$$u_n = (-1)^n a_n \quad \textbf{\textit{or}} \quad u_n = (-1)^{n+1} a_n$$

with $a_n \geq 0$ ***for all*** n***.***

Leibniz Convergence Criterion for Alternating Series

Theorem 4.8 (Leibniz Criterion for Alternating Series) ***Let*** $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ ***be an alternating series with*** $a_n \geq 0$***. If the sequence*** (a_n) ***satisfies***

$$a_{n+1} \leq a_n \quad \textbf{\textit{(decreasing)}} \quad \textbf{\textit{and}} \quad \lim_{n\to\infty} a_n = 0,$$

then the series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ ***converges.***

Example 4.15 ***Study the convergence of the alternating series***

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}.$$

Solution: Let

$$a_n = \frac{1}{n} \geq 0.$$

Check the conditions of Leibniz Criterion:
- Sequence (a_n) ***is decreasing:*** $a_{n+1} = \frac{1}{n+1} \leq \frac{1}{n} = a_n$***.***
- Limit: $\lim_{n\to\infty} a_n = 0$***.***
Since both conditions are satisfied, by Leibniz criterion, the series

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$$

converges.

4.1.7 Series with Arbitrary Terms

Definition 4.12 ***A series with arbitrary terms is a series of the form***

$$\sum_{n=0}^{\infty} u_n,$$

where the terms u_n ***can be positive, negative, or zero.***

Absolute Convergence Criterion

Proposition 4.2 ***A series***

$$\sum_{n=0}^{\infty} u_n$$

is said to be absolutely convergent if the series of the absolute values

$$\sum_{n=0}^{\infty} |u_n|$$

converges.

Remark 4.2 ***If a series $\sum u_n$ is absolutely convergent, then it is also convergent.***
However, the converse is not true: a series may converge conditionally (convergent but not absolutely convergent).

Example 4.16 ***Consider the series***

$$\sum_{n=1}^{\infty} \frac{(-1)^n n}{(n^2+1)^2}.$$

Step 1: Consider the absolute value series

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n n}{(n^2+1)^2} \right| = \sum_{n=1}^{\infty} \frac{n}{(n^2+1)^2}.$$

Step 2: Compare with a simpler series

$$\frac{n}{(n^2+1)^2} < \frac{n}{n^4} = \frac{1}{n^3}, \quad \textit{for all } n \geq 1.$$

Step 3: Use the comparison test
- The series $\sum \frac{1}{n^3}$ is convergent (Riemann series with $p = 3 > 1$). - Therefore, by the comparison test, the series $\sum n/(n^2+1)^2$ converges.
Step 4: Conclude absolute convergence
- Since $\sum |u_n| = \sum n/(n^2+1)^2$ converges, the original series

$$\sum_{n=1}^{\infty} \frac{(-1)^n n}{(n^2+1)^2}$$

is absolutely convergent. Then it converges.

Definition 4.13 ***A series***

$$\sum_{n=1}^{\infty} u_n$$

is said to be conditionally convergent if:

- ***The series $\sum u_n$ converges, but***
- ***The series of absolute values $\sum |u_n|$ diverges.***

Remark 4.3 ***Sometimes, a series converges only because of the cancellation of positive and negative terms, not because the absolute values form a convergent series. In this case,***

$$\sum u_n \quad \textit{converges but} \quad \sum |u_n| = +\infty,$$

and we call it conditional convergence.

Example 4.17 *The* alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

is conditionally convergent:

- ***It converges by the alternating series test.***
- ***But the harmonic series***

  $$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$$

  diverges.

Proof of divergence (integral test). Indeed, consider the function $f(x) = \frac{1}{x}$**, which is positive and decreasing for** $x \geq 1$**. We compute**

$$\int_1^{+\infty} \frac{dx}{x} = \lim_{R \to +\infty} \ln(R) = +\infty.$$

By the integral test (or Cauchy's criterion for positive series), if $\int_1^{\infty} f(x)\,dx = +\infty$**, then the series** $\sum_{n=1}^{\infty} f(n)$ **diverges. Hence,**

$$\sum_{n=1}^{\infty} \frac{1}{n} \quad \textbf{diverges}.$$

Therefore, the alternating harmonic series is conditionally convergent.

Remark 4.4 ***When we say that a series converges "by cancellation", we mean that its convergence is due to the compensation between positive and negative terms. Individually, the sums of the positive terms and the negative terms both diverge, but taken together they balance each other and the whole series converges.***

Consider the alternating harmonic series:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

- **The sum of the positive terms**

  $$1 + \frac{1}{3} + \frac{1}{5} + \cdots$$

  diverges.
- **The sum of the negative terms**

  $$-\frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \cdots$$

  also diverges.
- **However, when positive and negative terms are combined, they compensate each other, and the series converges.**

Thus, the convergence of the alternating harmonic series is due to cancellation of terms.

4.1.8 Series with Known Exact Sums

Some series have sums that can be calculated exactly, giving a finite value in closed form. These series are useful examples in analysis.
Geometric Series The geometric series

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}, \quad |r| < 1,$$

has a known exact sum.
Riemann Zeta Series for Even Integers The series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

is convergent and its exact sum is

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Similarly,

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

Alternating Harmonic Series The alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

converges and its sum is

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln 2.$$

4.2 Sequences and Series of Functions

4.2.1 Sequences of Functions

- We consider sequences of functions $f_n(x)$ and are interested in their behavior as $n \to \infty$.
- In physics, common examples include:

$$f_n(x) = \left(1 + \frac{x}{n}\right)^n \quad \textbf{or} \quad f_n(x) = x^n.$$

Example:

$$f_n(x) = x^n \quad \textbf{on } [0, 1].$$

- For $0 \le x < 1$, $f_n(x) \to 0$. - For $x = 1$, $f_n(1) = 1$.
- Practically, it is sufficient to examine the behavior of powers or exponentials.

4.2.2 Series of Functions

1. Definition and Difference with Numerical Series

- A numerical series is a sum of numbers:

$$\sum_{n=0}^{\infty} a_n, \quad a_n \in \mathbb{R} \textbf{ or } \mathbb{C}.$$

- A series of functions is a sum of functions depending on a variable x:

$$\sum_{n=0}^{\infty} u_n(x), \quad u_n(x) \textbf{ is a function of } x.$$

Main difference:
- In numerical series, convergence concerns numbers a_n.
- In series of functions, convergence can depend on x.
- We distinguish pointwise convergence (each x separately) and uniform convergence (all x simultaneously).

2. Examples of Function Series in Physics

1. Exponential series:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

2. Trigonometric series:

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \quad \cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

3. Geometric series:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x| < 1$$

Remark: In all these examples, each term $u_n(x)$ is a function of x, unlike in numerical series where a_n is a fixed number.

4.2.3 Convergence of Function Series

3. Pointwise Convergence

- The series $\sum u_n(x)$ converges pointwise if, for each x, the partial sums

$$S_N(x) = \sum_{n=0}^{N} u_n(x)$$

have a finite limit as $N \to \infty$.
Example:

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \dots$$

- Converges pointwise for $|x| < 1$.
- Limit: $S(x) = \frac{1}{1-x}$.

4. Uniform Convergence

- The series $\sum u_n(x)$ converges uniformly on an interval I if the partial sums $S_N(x)$ approach the limit uniformly for all $x \in I$.
Practical example:

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$

- Converges uniformly on any bounded segment $[a, b]$.
- Practical consequence: integration or differentiation term by term is allowed.

5. Practical Tips for Convergence

- If terms $u_n(x)$ decrease very fast (factorials, high powers, etc.), the series converges quickly.
- For small x, often only $2-3$ terms are needed for a good approximation.
Example: Small x approximations

$$e^x \approx 1 + x + \frac{x^2}{2}, \quad \sin x \approx x - \frac{x^3}{6}, \quad \cos x \approx 1 - \frac{x^2}{2}$$

- These approximations are widely used in physics:
- Small oscillations (harmonic motion)
- Electrodynamics: weak field expansions
- Thermodynamics and statistical mechanics: exponential expansions

4.3 Power Series (*Entire Series*)

Definition 4.14 ***A power series (or*** entire series***) centered at a point $x_0 \in \mathbb{R}$ is a series of the form***

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n,$$

where $a_n \in \mathbb{R}$ or $\mathbb{C}$ are coefficients.
- If $x_0 = 0$, the series is called a Maclaurin series:

$$\sum_{n=0}^{\infty} a_n x^n.$$

- The general term is

$$u_n(x) = a_n (x - x_0)^n.$$

-The set of values of x for which the series converges is called the domain of convergence of the series and is denoted by D

Example 4.18 ***Consider the exponential series***

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Solution 4.1 ***We study the absolute convergence of the series. The general term is***

$$u_n(x) = \frac{x^n}{n!}.$$

Applying the ratio test to $|u_n(x)|$***:***

$$\lim_{n\to\infty} \frac{|u_{n+1}(x)|}{|u_n(x)|} = \lim_{n\to\infty} \frac{|x|^{n+1}/(n+1)!}{|x|^n/n!} = \lim_{n\to\infty} \frac{|x|}{n+1} = 0.$$

Since this limit is less than 1 for all $x \in \mathbb{R}$***, the series converges absolutely for all*** x***. Then the domain of convergence denoted by*** D ***is*** $\mathbb{R}$ ***and we write*** $D = \mathbb{R}$.

4.3.1 Abel's Lemma for Power Series

Lemma 4.1 (Abel's Lemma for Power Series) ***Let*** $\sum_{n=0}^{\infty} a_n x^n$ ***be a power series. If there exists a real number*** $r > 0$ ***such that the sequence*** $(a_n r^n)_{n\geq 0}$ ***is bounded, then the power series***

$$\sum_{n=0}^{\infty} a_n x^n$$

converges absolutely for all $x \in \mathbb{R}$ ***such that*** $|x| < r$***.***

Radius of Convergence of a Power Series

Definition 4.15 (Radius of Convergence of a Power Series) ***Let*** $\sum_{n=0}^{\infty} a_n x^n$ ***be a power series. The radius of convergence*** R ***is the non-negative real number (or*** $+\infty$***) such that:***

- ***The series converges absolutely for all*** $x \in \mathbb{R}$ ***such that*** $|x| < R$***.***
- ***The series diverges for all*** $x \in \mathbb{R}$ ***such that*** $|x| > R$***.***

The set

$$\{x \in \mathbb{R} \mid |x| < R\}$$

is called the interval of convergence of the series. Convergence at the endpoints $x = \pm R$ ***must be studied separately.***

Determination of the Radius of Convergence

Lemma 4.2 (Hadamard's Lemma) ***For the power series*** $\sum_{n=0}^{\infty} a_n x^n$***, the radius of convergence*** R ***is***

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}}.$$

Definition 4.16 (Radius of Convergence) ***Let***

$$\sum_{n=0}^{\infty} a_n(x - x_0)^n$$

be a power series. According to Hadamard's Lemma, its radius of convergence R ***is given by***

$$R = \frac{1}{\limsup\limits_{n\to\infty} \sqrt[n]{|a_n|}}.$$

This value R ***can be finite, zero, or infinite.***

Remark 4.5

- ***If*** $0 < R < \infty$***, the series converges absolutely for*** $|x - x_0| < R$ ***and diverges for*** $|x - x_0| > R$***.***
- ***If*** $R = \infty$***, the series converges absolutely for all*** $x \in \mathbb{R}$***.***
- ***If*** $R = 0$***, the series converges only at*** $x = x_0$***.***

Remark 4.6 ***If the limit exists:***
- The radius of convergence can be calculated using the root test***:***

$$R = \frac{1}{\lim\limits_{n\to\infty} \sqrt[n]{|a_n|}}.$$

- Or using the ratio test ***if it applies:***

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|,$$

Example 4.19 ***Determine the radius of convergence of the power series***

$$\sum_{n=1}^{\infty} \frac{n!\, x^n}{n^n}.$$

Hypothesis: We use the Stirling approximation

$$n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \quad \textit{as } n \to \infty.$$

Solution 4.2 ***Let*** $a_n = \dfrac{n!}{n^n}$***. By Hadamard's Lemma, the radius of convergence is***

$$R = \frac{1}{\limsup\limits_{n\to\infty} \sqrt[n]{|a_n|}}.$$

Compute $\sqrt[n]{a_n}$ ***using the Stirling approximation:***

$$\sqrt[n]{a_n} = \sqrt[n]{\frac{n!}{n^n}} \sim \frac{\sqrt[n]{\sqrt{2\pi n}} \cdot \dfrac{n}{e}}{n} = \frac{1}{e} \quad \textit{as } n \to \infty.$$

Thus,

$$R = \frac{1}{1/e} = e.$$

The series converges absolutely for $|x| < e$***.***
Convergence at the endpoints $|x| = e$ ***must be checked separately.***
(i) Endpoint $x = e$***:***
The series becomes

$$\sum_{n=1}^{\infty} \frac{n!\, e^n}{n^n} = \sum_{n=1}^{\infty} \frac{n!}{(n/e)^n}.$$

Using Stirling:

$$\frac{n!}{(n/e)^n} \sim \sqrt{2\pi n} \quad \textbf{\textit{as }} n \to \infty.$$

Since $\sqrt{2\pi n} \to \infty$***, the general term does not tend to zero.*** $\implies$ ***The series diverges at*** $x = e$***.***
(ii) Endpoint $x = -e$***:***
The series becomes

$$\sum_{n=1}^{\infty} \frac{n!\, (-e)^n}{n^n} = \sum_{n=1}^{\infty} (-1)^n \frac{n!\, e^n}{n^n}.$$

Again,

$$\frac{n!\, e^n}{n^n} \sim \sqrt{2\pi n} \to \infty.$$

The alternating sign does not help because the terms do not tend to zero.
Then the series diverges at $x = -e$***.***
Conclusion:
The series converges absolutely for $|x| < e$ ***and diverges for*** $|x| = e$.

4.3.2 Derivatives and Integrals of a Power Series

Definition 4.17 (Term-by-Term Derivative) Let

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

be a power series with radius of convergence $R > 0$***. Then,*** $f(x)$ ***is differentiable for*** $|x - x_0| < R$***, and its derivative can be computed term by term:***

$$f'(x) = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1}.$$

Definition 4.18 (Term-by-Term Integral) Similarly, the indefinite integral of $f(x)$ ***can be computed term by term:***

$$\int f(x)\, dx = \sum_{n=0}^{\infty} \frac{a_n}{n+1} (x - x_0)^{n+1} + C,$$

where C ***is an arbitrary constant. This series has the same radius of convergence*** R ***as the original series.***

Example 4.20 Consider the power series

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Solution 4.3 Derivative: Term-by-term differentiation gives

$$f'(x) = \sum_{n=1}^{\infty} \frac{nx^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!}.$$

Change the index $m = n - 1$***:***

$$f'(x) = \sum_{m=0}^{\infty} \frac{x^m}{m!} = f(x).$$

Radius of convergence:
- Coefficients of the derivative series are $b_m = 1/m!$***.***
- Using Hadamard's formula:

$$R = \frac{1}{\limsup_{m\to\infty} \sqrt[m]{|b_m|}} = \frac{1}{\lim_{m\to\infty} \sqrt[m]{1/m!}}.$$

- Since $\lim_{m\to\infty} \sqrt[m]{1/m!} = 0$***, we get***

$$R = \infty.$$

Integral: Term-by-term integration gives

$$\int f(x)\,dx = C + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)n!} = C + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!}.$$

- Coefficients of the integral series are $u_n = 1/(n+1)!$***.***
- Applying Hadamard's formula:

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|u_n|}} = \frac{1}{\lim_{n\to\infty} \sqrt[n]{1/(n+1)!}}.$$

- Using the fact that $\sqrt[n]{(n+1)!} \to \infty$***, we get***

$$\lim_{n\to\infty} \sqrt[n]{|u_n|} = 0 \qquad \Longrightarrow \qquad R = \infty.$$

Conclusion:
- The derivative of $f(x)$ ***is*** $f(x)$ ***itself. - The integral can be expressed as a new power series. - In both cases, the radius of convergence remains*** $R = \infty$***, meaning the series converges absolutely for all*** $x \in \mathbb{R}$***.***

4.3.3 Operations on Power Series

Definition 4.19 (Sum and Difference of Power Series) ***Let***

$$f(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n \quad \textbf{\textit{and}} \quad g(x) = \sum_{n=0}^{\infty} b_n(x - x_0)^n$$

be two power series with radii of convergence R_1 ***and*** R_2***, respectively. Then, for*** $|x - x_0| < \min(R_1, R_2)$***, we can define***

$$f(x) \pm g(x) = \sum_{n=0}^{\infty}(a_n \pm b_n)(x - x_0)^n.$$

Definition 4.20 (Scalar Multiplication) ***For any scalar*** $\lambda \in \mathbb{R}$ ***(or*** $\mathbb{C}$***), we have***

$$\lambda f(x) = \sum_{n=0}^{\infty}(\lambda a_n)(x - x_0)^n \quad \textbf{\textit{for}}\ |x - x_0| < R,$$

where R ***is the radius of convergence of*** $f(x)$***.***

Definition 4.21 (Term-by-Term Multiplication (Cauchy Product)) ***Let***

$$f(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n \quad \textbf{\textit{and}} \quad g(x) = \sum_{n=0}^{\infty} b_n(x - x_0)^n.$$

The Cauchy product of f ***and*** g ***is***

$$(f \cdot g)(x) = \sum_{n=0}^{\infty} c_n(x - x_0)^n, \quad \textbf{\textit{where}}\ c_n = \sum_{k=0}^{n} a_k b_{n-k}.$$

The series converges at least for $|x - x_0| < \min(R_1, R_2)$***.***

Example 4.21 ***Consider***

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \textbf{\textit{and}} \quad g(x) = \sum_{n=0}^{\infty} x^n.$$

Compute $f(x) + g(x)$***,*** $2f(x)$***, and the Cauchy product*** $f(x) \cdot g(x)$ ***as power series.***

Solution 4.4 ***Sum:***

$$f(x) + g(x) = \sum_{n=0}^{\infty} \left(\frac{1}{n!} + 1\right) x^n.$$

Scalar multiplication:

$$2f(x) = \sum_{n=0}^{\infty} \frac{2x^n}{n!}.$$

Cauchy product:

$$(f \cdot g)(x) = \sum_{n=0}^{\infty} c_n x^n, \quad \textbf{\textit{with}}\ c_n = \sum_{k=0}^{n} \frac{1}{k!}.$$

Remark: The first series has radius $R_1 = \infty$ ***and the second series has radius*** $R_2 = 1$***.***
By the general property of the Cauchy product, the product series converges at least for

$$|x| < \min(R_1, R_2) = \min(\infty, 1) = 1.$$

Thus, we are guaranteed that the Cauchy product converges for $|x| < 1$***.***

4.3.4 Taylor Series

Definition 4.22 (Taylor Series) ***Let $f(x)$ be a function that is infinitely differentiable at a point $x_0 \in \mathbb{R}$. The Taylor series of f centered at x_0 is the power series***

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n,$$

where $f^{(n)}(x_0)$ denotes the n-th derivative of f evaluated at x_0.

Remark 4.7 ***- If $x_0 = 0$, the series is called a Maclaurin series.***
- The Taylor series may converge only for $|x - x_0| < R$, where R is the radius of convergence.
- If the series converges to $f(x)$ for all $|x - x_0| < R$, we say $f(x)$ is analytic at x_0.

Example 4.22 ***Find the Taylor series of $f(x) = e^x$ centered at $x_0 = 0$.***

Solution 4.5 ***- Compute derivatives: $f^{(n)}(x) = e^x \implies f^{(n)}(0) = 1$. - Apply the Taylor formula:***

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!}x^n = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

- The radius of convergence is $R = \infty$, so the series converges for all $x \in \mathbb{R}$.

Taylor Series of Common Functions

- **Exponential function:** $f(x) = e^x$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots, \quad R = \infty$$

- **Sine function:** $f(x) = \sin x$

$$\sin x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots, \quad R = \infty$$

- **Cosine function:** $f(x) = \cos x$

$$\cos x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots, \quad R = \infty$$

- **Natural logarithm:** $f(x) = \ln(1 + x)$, $|x| < 1$

$$\ln(1 + x) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots, \quad R = 1$$

- **Geometric series:** $f(x) = \frac{1}{1-x}$, $|x| < 1$

$$\frac{1}{1 - x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots, \quad R = 1$$

- **Arctangent:** $f(x) = \arctan x$, $|x| \leq 1$

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots, \qquad R = 1$$

- **Binomial series:** $(1+x)^\alpha, \quad |x| < 1, \ \alpha \in \mathbb{R}$

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \dots, \qquad R = 1$$

where $\binom{\alpha}{n} = \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!}$.

4.3.5 Differential Equations and Power Series

Definition 4.23 (Power Series Method for Solving Differential Equations) ***Let us consider a differential equation of the form***

$$y'' + P(x)y' + Q(x)y = 0,$$

where $P(x)$ ***and*** $Q(x)$ ***are analytic functions at*** x_0.
We look for a solution $y(x)$ ***in the form of a power series:***

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

Procedure:

1. ***Substitute the series for*** $y(x)$, $y'(x)$, ***and*** $y''(x)$ ***into the differential equation.***
2. ***Align terms with the same powers of*** $(x - x_0)$.
3. ***Obtain a recurrence relation for the coefficients*** a_n.
4. ***Solve the recurrence to find*** a_n ***and write the solution as a series.***

Example 4.23 ***Solve the differential equation***

$$y' - y = 0$$

using the power series method around $x_0 = 0$.

Solution 4.6 ***Step 1: Assume a power series solution:***

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Step 2: Compute the derivative:

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n.$$

Step 3: Substitute into the differential equation:

$$y' - y = \sum_{n=0}^{\infty} \left[(n+1)a_{n+1} - a_n\right] x^n = 0.$$

Step 4: Recurrence relation:

$$(n+1)a_{n+1} - a_n = 0 \quad \Longrightarrow \quad a_{n+1} = \frac{a_n}{n+1}.$$

Step 5: Solve the recurrence:

$$a_1 = \frac{a_0}{1}, \quad a_2 = \frac{a_1}{2} = \frac{a_0}{2!}, \quad a_3 = \frac{a_2}{3} = \frac{a_0}{3!}, \ldots$$

Step 6: Series solution:

$$y(x) = \sum_{n=0}^{\infty} \frac{a_0}{n!} x^n = a_0 \sum_{n=0}^{\infty} \frac{x^n}{n!} = a_0 e^x.$$

Radius of convergence: $R = \infty$***, series converges for all*** $x \in \mathbb{R}$***.***

Example 4.24 ***Solve the differential equation***

$$y'' - xy = 0$$

using a power series around $x_0 = 0$***.***

Solution 4.7 ***Step 1: Assume a power series solution:***

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Step 2: Compute derivatives:

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Step 3: Substitute into the equation:

$$\begin{aligned} y'' - xy &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - x \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1}. \end{aligned}$$

Step 4: Shift indices to align powers:
- For the first sum, let $k = n - 2 \implies n = k + 2$***:***

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k.$$

- For the second sum, let $k = n + 1 \implies n = k - 1$:

$$\sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{k=1}^{\infty} a_{k-1} x^k.$$

Step 5: Combine sums:

$$\sum_{k=0}^{\infty} \Big[(k+2)(k+1)a_{k+2} - a_{k-1}\Big] x^k = 0,$$

where we set $a_{-1} = 0$.
Step 6: Recurrence relation:

$$a_{k+2} = \frac{a_{k-1}}{(k+2)(k+1)}, \qquad k \geq 0.$$

Step 7: Compute first coefficients:
- a_0, a_1 are arbitrary constants. - $a_2 = a_{-1}/(2 \cdot 1) = 0$, $a_3 = a_0/(3 \cdot 2) = a_0/6$, $a_4 = a_1/(4 \cdot 3) = a_1/12$, etc.
Step 8: Series solution:

$$y(x) = a_0 \left(1 + \frac{x^3}{6} + \dots\right) + a_1 \left(x + \frac{x^4}{12} + \dots\right).$$

Remark: This gives a non-trivial solution expressed as a power series around $x = 0$. The series converges for all $x \in \mathbb{R}$ (infinite radius of convergence).

4.4 Fourier Series

Introduction to Fourier Series

A Fourier series is a way to represent a periodic function as an infinite sum of sines and cosines:

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx)\right),$$

where a_n and b_n are called the Fourier coefficients.
Importance in Physics:

- **Fourier series allow us to decompose complex periodic signals into simple harmonic components.**
- **They are widely used in electrodynamics, quantum mechanics, and wave propagation.**
- **They are essential in solving partial differential equations such as the heat equation, wave equation, and Laplace equation.**
- **They help analyze vibrations, sound waves, and signal processing.**

4.4.1 Fourier Series for a Function of Period T

Definitions 4.1 ***Let $f(t)$ be a periodic function with period $T > 0$, and define the fundamental frequency***

$$\omega_0 = \frac{2\pi}{T}.$$

1. Trigonometric Fourier Series (TFS):

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big(a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t) \Big),$$

with coefficients

$$a_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \cos(n\omega_0 t)\, dt, \quad b_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \sin(n\omega_0 t)\, dt, \quad n \geq 1,$$

and

$$a_0 = \frac{2}{T} \int_{-T/2}^{T/2} f(t)\, dt.$$

2. Exponential Fourier Series (EFS):

$$f(t) \sim \sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 t},$$

with coefficients

$$c_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-in\omega_0 t}\, dt, \quad n \in \mathbb{Z}.$$

Remark 4.8 ***: Both series are equivalent. The EFS is often more convenient in engineering, while the TFS is more intuitive in physics.***

Example 4.25 ***Find the Fourier series of the periodic function***

$$f(t) = t, \quad -1 < t < 1,$$

with period $T = 2$.

Solution 4.8 ***Step 1: Fundamental frequency***

$$\omega_0 = \frac{2\pi}{T} = \pi.$$

Step 2: Compute a_0:

$$a_0 = \frac{2}{T} \int_{-1}^{1} t\, dt = \frac{2}{2} \int_{-1}^{1} t\, dt = 0.$$

Step 3: Compute a_n:

$$a_n = \frac{2}{2} \int_{-1}^{1} t \cos(n\pi t)\, dt = \int_{-1}^{1} t \cos(n\pi t)\, dt = 0 \quad (\textbf{\textit{odd function}}).$$

Step 4: Compute b_n***:***

$$b_n = \int_{-1}^{1} t\sin(n\pi t)\,dt = 2\int_0^1 t\sin(n\pi t)\,dt$$

$$b_n = 2\left[-\frac{t\cos(n\pi t)}{n\pi}\Big|_0^1 + \frac{1}{n\pi}\int_0^1 \cos(n\pi t)\,dt\right] = \frac{2(-1)^{n+1}}{n\pi}.$$

Step 5: Fourier series:

$$f(t) \sim \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi}\sin(n\pi t).$$

Step 6: Exponential form (optional):

$$f(t) \sim \sum_{n=-\infty}^{\infty} c_n e^{in\pi t}, \quad c_n = \frac{(-1)^{n+1}}{in\pi}, \; n \neq 0, \quad c_0 = 0.$$

4.4.2 Polar (Amplitude-Phase) Form of Fourier Series

Let $f(t)$ have period T and fundamental frequency $\omega_0 = \frac{2\pi}{T}$. If the Fourier series of $f(t)$ is

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n\cos(n\omega_0 t) + b_n\sin(n\omega_0 t)\right),$$

we can rewrite each term as a single cosine with amplitude and phase:

$$a_n\cos(n\omega_0 t) + b_n\sin(n\omega_0 t) = A_n\cos(n\omega_0 t - \varphi_n),$$

where

$$A_n = \sqrt{a_n^2 + b_n^2}, \quad \varphi_n = \arctan\left(\frac{b_n}{a_n}\right).$$

Thus, the Fourier series in polar form becomes:

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} A_n\cos(n\omega_0 t - \varphi_n).$$

Remark 4.9

- A_n ***represents the amplitude of the*** n***-th harmonic.***
- φ_n ***represents the phase shift of the*** n***-th harmonic.***
- ***This form is particularly useful in physics and engineering to analyze signals in terms of **amplitude and phase** rather than separate sine and cosine components.***

Example 4.26 ***Consider the periodic function $f(t) = t$ on $[-1, 1]$ with period $T = 2$. We already know its Fourier series (trigonometric form):***

$$f(t) \sim \sum_{n=1}^{\infty} b_n\sin(n\pi t), \quad b_n = \frac{2(-1)^{n+1}}{n\pi}.$$

Solution 4.9 ***Step 1: Identify a_n and b_n: Since $f(t)$ is an odd function, all $a_n = 0$. Thus,***

$$a_n = 0, \quad b_n = \frac{2(-1)^{n+1}}{n\pi}.$$

Step 2: Compute amplitude A_n and phase φ_n:

$$A_n = \sqrt{a_n^2 + b_n^2} = |b_n| = \frac{2}{n\pi},$$

$$\varphi_n = \arctan\left(\frac{b_n}{a_n}\right) = \arctan(\infty) = \frac{\pi}{2} \quad (\textbf{\textit{since }} b_n > 0 \textbf{\textit{ for odd }} n).$$

Step 3: Write polar form:

$$f(t) \sim \sum_{n=1}^{\infty} A_n \cos(n\pi t - \varphi_n) = \sum_{n=1}^{\infty} \frac{2}{n\pi} \cos\left(n\pi t - \frac{\pi}{2}\right).$$

4.4.3 Computation of Fourier Coefficients for Even and Odd Functions

Let $f(t)$ be a periodic function with period T and fundamental frequency $\omega_0 = \frac{2\pi}{T}$.

1. Trigonometric Fourier Series (TFS)

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)\right),$$

with

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(n\omega_0 t)\,dt, \quad b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin(n\omega_0 t)\,dt.$$

Case 1: $f(t)$ is even ($f(-t) = f(t)$)

$$b_n = 0, \quad a_n = \frac{4}{T}\int_0^{T/2} f(t)\cos(n\omega_0 t)\,dt.$$

Case 2: $f(t)$ is odd ($f(-t) = -f(t)$)

$$a_n = 0, \quad b_n = \frac{4}{T}\int_0^{T/2} f(t)\sin(n\omega_0 t)\,dt.$$

2. Exponential Fourier Series (EFS)

$$f(t) \sim \sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 t}, \quad c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-in\omega_0 t}\,dt.$$

Simplifications:

- **If $f(t)$ is even: $c_{-n} = c_n$ and c_n is real.**
- **If $f(t)$ is odd: $c_{-n} = -c_n$ and c_n is purely imaginary.**

Remark 4.10 *: **Exploiting symmetry reduces the integration interval to*** $[0, T/2]$ ***and simplifies the calculation of Fourier coefficients.***

Example 4.27 ***1. Even function:*** $f(t) = |t|$***, defined on*** $[-1, 1]$ ***with period*** $T = 2$***.***
2. Odd function: $f(t) = t$***, defined on*** $[-1, 1]$ ***with period*** $T = 2$***.***

Solution 4.10 ***Case 1: Even function*** $f(t) = |t|$
Step 1: Compute a_0***:***

$$a_0 = \frac{2}{T}\int_{-1}^{1} |t|\, dt = \int_{-1}^{1} |t|\, dt = 1.$$

Step 2: Compute a_n ***(***$b_n = 0$ ***since function is even):***

$$a_n = 2\int_0^1 t\cos(n\pi t)\, dt = 2\int_0^1 t\cos(n\pi t)\, dt$$

$$a_n = 2\left[\frac{\sin(n\pi t)}{n\pi} + \frac{\cos(n\pi t)}{(n\pi)^2}\bigg|_0^1\right] = \frac{2((-1)^n - 1)}{(n\pi)^2}, \quad b_n = 0.$$

Step 3: Exponential coefficients c_n***:***

$$c_0 = \frac{a_0}{2} = \frac{1}{2}, \quad c_n = \frac{a_n}{2},\ n \neq 0 \quad (\textbf{\textit{real since function is even}}).$$

Case 2: Odd function $f(t) = t$
Step 1: Compute a_0 ***and*** a_n ***(***$a_0 = a_n = 0$***)***
Step 2: Compute b_n***:***

$$b_n = 2\int_0^1 t\sin(n\pi t)\, dt = 2\left[-\frac{t\cos(n\pi t)}{n\pi} + \frac{\sin(n\pi t)}{(n\pi)^2}\bigg|_0^1\right] = \frac{2(-1)^{n+1}}{n\pi}.$$

Step 3: Exponential coefficients c_n***:***

$$c_n = \frac{-ib_n}{2} = \frac{(-1)^n i}{n\pi}, \quad c_{-n} = -c_n, \quad c_0 = 0.$$

Conclusion:
- Even functions: only cosine terms (a_n***) are nonzero.***
- Odd functions: only sine terms (b_n***) are nonzero.***
- The exponential coefficients c_n ***reflect this symmetry (real for even, imaginary for odd).***

3. Conditions for the Existence of a Fourier Series

Let $f(t)$ be a function of period T. A Fourier series (trigonometric or exponential) exists if $f(t)$ satisfies the following Dirichlet conditions:

1. **$f(t)$ is periodic with period T.**
2. **$f(t)$ is piecewise continuous on one period $[0, T]$ (or $[-T/2, T/2]$), i.e., it has a finite number of finite discontinuities.**