Chapter 3

Differential Equations

3.1 Ordinary Differential Equations

3.1.1 Generalities

Definition 3.1 ***A differential equation is an equation in which the unknown is a function and where some derivatives of the unknown function appear.***

Example 3.1 ***Let*** u ***be a function, the following equations are differential equations.***

1. $u^{'} = 2u$
2. $u^{''} - 3u^{'} + 1 = 0$
3. $u^{(3)} = u$

Definition 3.2 ***Let*** $u = u(x)$ ***be an unknown function of the variable*** x***. An equation of the form***

$$F\Big(x, u, u', u'', \ldots, u^{(n)}\Big) = 0, \tag{3.1}$$

with $n \in \mathbb{N}$, ***is called a differential equation of order*** n. ***Here*** $u', u\text{"}, \ldots, u^{(n)}$ ***denote the derivatives of*** u ***of orders*** $1, 2, \ldots, n$***, respectively.***

Remarks 3.1

1. ***The equation*** (3.1) ***involves*** $(n+2)$ ***variables.***
2. ***The unknown function may also be denoted by*** $y, t, \ldots$
3. ***In a differential equation, when we write*** $u, u', u'', \ldots, u^{(n)}$***, it is understood that we mean*** $u(x), u'(x), u''(x), \ldots, u^{(n)}(x)$.

Definition 3.3 ***A solution of equation*** (3.1) ***on the interval*** I ***is a function that is*** n ***times differentiable on*** I ***and satisfies*** (3.1)***.***

Example 3.2 ***It can be easily verified that the function*** $u(x) = ce^{4x},\ c \in \mathbb{R}$, ***is a solution of the differential equation*** $u' = 4u$. ***Indeed, it is clear that if*** $u(x) = ce^{4x}$ ***then*** $u'(x) = 4ce^{4x} = 4u(x)$.

Definition 3.4 *A differential equation of the type $u'f(u) = g(x)$ is called a* separable variables equation*.*

Example 3.3 *The equation $u' = \dfrac{e^{-u}}{x^2}$ can be rewritten as $u'e^u = \dfrac{1}{x^2}$. We can easily find the solutions of this equation. By integrating both sides, we obtain*

$$e^u = -\frac{1}{x} + k, \qquad k \in \mathbb{R}.$$

This yields

$$u(x) = \ln\Big| -\frac{1}{x} + k\Big|, \qquad k \in \mathbb{R}.$$

Definition 3.5 *The* order *of a differential equation is the order of the highest derivative appearing in the equation.*

Example 3.4

1. $2xu' + u = 0$ *is a first-order differential equation.*
2. $u'' + u' - 3u = \ln(x)$ *is a second-order differential equation.*

Definition 3.6

1. *A differential equation of order n is said to be* linear *if it has the form*

$$f_0(x)u + f_1(x)u' + f_2(x)u'' + \cdots + f_n(x)u^{(n)} = f(x), \tag{3.2}$$

where $f_0, f_1, f_2, \ldots, f_n, f$ are real continuous functions on an interval $I \subset \mathbb{R}$.

2. *If $f(x) = 0$ for all $x \in I$, then equation (3.2) is called* homogeneous*, and it has the form*

$$f_0(x)u + f_1(x)u' + f_2(x)u'' + \cdots + f_n(x)u^{(n)} = 0.$$

3. *Equation (3.2) is said to have* constant coefficients *if the functions $f_0, f_1, f_2, \ldots, f_n$ are constants on I. In other words, equation (3.2) can be written as*

$$a_0u + a_1u' + a_2u'' + \cdots + a_nu^{(n)} = f(x),$$

where $a_0, a_1, a_2, \ldots, a_n$ are real constants.

Remark 3.1 *In a linear differential equation, none of the terms $u, u', u'', \ldots, u^{(n)}$ are raised to a power.*

Example 3.5

1. $e^xu + x^{\frac{1}{2}}u'' = x^2 + 1$ *is a linear differential equation, and* $e^xu + x^{\frac{1}{2}}u'' = 0$ *is the associated homogeneous equation.*
2. $2u' - 3u'' + \frac{1}{5}u^{(3)} = x$ *is a linear differential equation with constant coefficients.*
3. *The equation $(u')^2 + u'' + 3u = 0$ is* not *a linear differential equation.*

Proposition 3.1 ***If*** u_1, u_2 ***are two solutions of the linear homogeneous equation***

$$f_0(x)u + f_1(x)u' + f_2(x)u'' + \cdots + f_n(x)u^{(n)} = 0, \tag{3.3}$$

then $\alpha u_1 + \beta u_2$ ***is also a solution of*** (3.3)***, for any constants*** $\alpha, \beta \in \mathbb{R}$***.***

Consider the linear differential equation (3.2) and its associated homogeneous equation (3.3). The following proposition allows us to find the general solution of equation (3.2).

Proposition 3.2 ***If*** u_0 ***is a particular solution of*** (3.2) ***and*** u_1 ***is a solution of the homogeneous equation*** (3.3)***, then***

$$u = u_1 + u_0$$

is a general solution of (3.2)***.***

Remark 3.2 ***Recall that a particular solution of equation*** (3.2) ***is a function that is*** n ***times differentiable and satisfies*** (3.2)***.***

3.2 First-Order Differential Equation

3.2.1 First-Order Linear Differential Equation Without a Nonhomogeneous Term

Consider the equation

$$a_1(x)u' + a_0(x)u = 0, \quad \textbf{with} \quad a_1(x) \neq 0, \tag{3.4}$$

which is equivalent to the separable equation

$$u' + a(x)u = 0, \quad \textbf{where} \quad a(x) = \frac{a_0(x)}{a_1(x)}. \tag{3.5}$$

To solve equation (3.5), we follow the steps:

$$\begin{aligned} u' + a(x)u = 0 &\iff u' = -a(x)u, \\ &\iff \frac{du}{dx} = -a(x)u, \\ &\iff \frac{du}{u} = -a(x)\,dx, \\ &\iff \int \frac{du}{u} = -\int a(x)\,dx, \\ &\iff \ln|u| = -A(x) + k, \\ &\iff |u| = e^{-A(x)+k}, \\ &\iff u = Ke^{-A(x)}, \end{aligned}$$

where $A(x)$ is an antiderivative of $a(x)$ and $K = \pm e^k,\ k \in \mathbb{R}$.

Example 3.6 ***The solution of the equation***

$$u' - \sqrt{x}u = 0, \quad x > 0,$$

is given by

$$u(x) = Ke^{-A(x)},$$

with $K = \pm e^k$ ***and***

$$-A(x) = \int \sqrt{x}\, dx = \frac{2}{3}\sqrt{x^3}.$$

3.2.2 First-Order Linear Differential Equation with a Nonhomogeneous Term

Consider the equation

$$a_1(x)u' + a_0(x)u = f_1(x), \quad \text{with} \quad a_1(x) \neq 0, \tag{3.6}$$

which is equivalent to

$$u' + a(x)u = f(x), \quad \text{where} \quad a(x) = \frac{a_0(x)}{a_1(x)} \quad \text{and} \quad f(x) = \frac{f_1(x)}{a_1(x)}. \tag{3.7}$$

According to Proposition 3.2, the solution of equation (3.7) is of the form

$$u(x) = u_0(x) + u_1(x),$$

where $u_1(x) = Ke^{-A(x)}$ is the solution of the homogeneous equation associated with (3.7), and $u_0(x)$ is a particular solution of (3.7).

Example 3.7 ***Consider the equation***

$$u' - \sqrt{x}u = 1 - x\sqrt{x}, \quad x > 0. \tag{3.8}$$

From the previous example, $u_1(x) = Ke^{\frac{2}{3}\sqrt{x^3}}$ ***is a solution of the homogeneous equation associated with*** (3.8)***. On the other hand, it can be easily verified that*** $u_0(x) = x$ ***is a particular solution of*** (3.8)***.***
Therefore, the general solution of equation (3.8) ***is***

$$u(x) = x + Ke^{\frac{2}{3}\sqrt{x^3}} = x + Ke^{\frac{2}{3}x\sqrt{x}}, \quad K \in \mathbb{R}.$$

The question now is: how can we find a particular solution?

3.2.3 Finding a Particular Solution: The Method of Variation of the Constant

We know that the solution of the homogeneous equation associated with (3.7) is of the form

$$u_1(x) = Ke^{-A(x)}, \quad K \in \mathbb{R}.$$

The *method of variation of the constant* consists in looking for a particular solution of (3.7) **of the form**

$$u_0(x) = K(x)e^{-A(x)},$$

where $K(x)$ **is a function of the variable** x **instead of a constant.**
Saying that $u_0(x) = K(x)e^{-A(x)}$ **is a solution of** (3.7) **means that**

$$u_0'(x) + a(x)u_0(x) = f(x), \quad \textbf{with} \quad A'(x) = a(x). \tag{3.9}$$

$$\begin{aligned}
u_0^{'}(x) + a(x)u_0(x) = f(x) &\iff \left(K(x)e^{-A(x)}\right)^{'} + a(x)K(x)e^{-A(x)} = f(x)\\
&\iff K^{'}(x)e^{-A(x)} - K(x)A^{'}(x)e^{-A(x)} + a(x)K(x)e^{-A(x)} = f(x)\\
&\iff K^{'}(x)e^{-A(x)} - K(x)a(x)e^{-A(x)} + a(x)K(x)e^{-A(x)} = f(x)\\
&\iff K^{'}(x)e^{-A(x)} = f(x)\\
&\iff K^{'}(x) = f(x)e^{A(x)}\\
&\iff K(x) = \int f(x)e^{A(x)}\,dx.
\end{aligned}$$

Thus, a particular solution of (3.7) **can be written in the form**

$$u_0(x) = \left(\int f(x)e^{A(x)}\,dx\right)e^{-A(x)}.$$

Exercise 3.1 ***Find the solutions of the equation***

$$u' - 2u = e^{3x+1}. \tag{3.10}$$

Proof. Finding the solution of the homogeneous equation:
The solution of the homogeneous equation

$$u' - 2u = 0$$

associated with equation (3.10) **is given by**

$$u_1(x) = Ke^{2x}, \quad K \in \mathbb{R}.$$

$$\begin{aligned}
u^{'} = 2u &\iff \frac{du}{dx} = 2u\\
&\iff \frac{du}{u} = 2dx\\
&\iff \ln|u| = 2x + k, \;\; k \in \mathbb{R}\\
&\iff u = Ke^{2x}, \; K = \pm e^{k}.
\end{aligned}$$

Then

$$u_1(x) = Ke^{2x}$$

Finding the particular solution:

We look for a function $K(x)$ such that a particular solution of equation (3.10) is of the form

$$u_0(x) = K(x)e^{2x}.$$

$u_0(x) = K(x)e^{2x}.$

$$\begin{aligned}
u_0'(x) - 2u_0(x) = e^{3x+1} &\iff \left(K(x)e^{2x}\right)' - 2K(x)e^{2x} = e^{3x+1}\\
&\iff K'(x)e^{2x} + 2K(x)e^{2x} - 2K(x)e^{2x} = e^{3x+1}\\
&\iff K'(x)e^{2x} = e^{3x+1}\\
&\iff K'(x) = e^{3x+1}e^{-2x}\\
&\iff K(x) = \int e^{x+1}\,dx\\
&\Longrightarrow K(x) = e\int e^{x}\,dx\\
&\Longrightarrow K(x) = e^{x+1}.
\end{aligned}$$

The solution of equation (3.10) is of the form

$$u(x) = e^{x+1}e^{2x} + Ke^{2x} = \left(e^{x+1} + K\right)e^{2x}, \quad K \in \mathbb{R}.$$

■

3.2.4 First-Order Linear Differential Equation with Constant Coefficients

Consider equations of the form

$$a_1u' + a_0u = f_1(x). \tag{3.11}$$

This is a special case of equations (3.6). Equation (3.11) is solved in the same way as (3.6).

3.2.5 Bernoulli Differential Equation

Definition 3.7 ***Any equation of the form***

$$u' + a(x)u + b(x)u^n = 0 \tag{3.12}$$

is called a Bernoulli equation.

Solving the Bernoulli Equation

1. **If $n = 0$, equation (3.12) becomes of the form (3.7).**
2. **If $n = 1$, equation (3.12) becomes of the form (3.5).**

3. **If $n \neq 0$ and $n \neq 1$, we try to transform equation (3.12) into a first-order linear differential equation. To do this, we follow the following method: we divide by u^n and equation (3.12) becomes**

$$u^{-n}u^{'} + a(x)u^{1-n} + b(x) = 0 \tag{3.13}$$

We set $y = u^{1-n}$, so that

$$\frac{1}{1-n}y' = u^{-n}u'.$$

Therefore, equation (3.13) becomes

$$\frac{1}{1-n}y' + a(x)y + b(x) = 0. \tag{3.14}$$

Equation (3.14) is of the form (3.6).

Example 3.8 ***Solve the following equation:***

$$u' + e^x u + e^x u^3 = 0. \tag{3.15}$$

Proof. Dividing by u^3 we obtain

$$u^{-3}u' + e^x u^{-2} = -e^x. \tag{3.16}$$

By making the change of variable $y = u^{-2}$ and differentiating both sides, we obtain

$$y' = -2u'u^{-3}.$$

In other words,

$$u'u^{-3} = -\frac{1}{2}y.$$

This change of variable allows us to write equation (3.16) in the form

$$-\frac{1}{2}y' + e^x y = -e^x, \tag{3.17}$$

which is a first-order linear equation with a nonhomogeneous term whose solution follows the previous steps. ■

3.2.6 Homogeneous Differential Equation

Let H be a numerical function defined and continuous on a domain $D \subset \mathbb{R}$.

Definition 3.8 ***A differential equation is called*** **homogeneous** ***if it is of the form***

$$F(x, u, u') = 0$$

and remains unchanged when x is replaced by αx and u by αu, while leaving u' unchanged. These equations are of the form

$$u' = H\left(\frac{u}{x}\right). \tag{3.18}$$

The solution of equation (3.18) generally reduces to solving a simple equation using the change of variable $t = \frac{u}{x}$, with $u = tx$ and $u' = t'x + t$. The solutions are in the form (x, u).

Example 3.9 ***Solve the equation***

$$2xuu' = u^2 - x^2.$$

Proof. When x is replaced by αx and u by αu, leaving u' unchanged, we obtain

$$2\alpha^2 xuu' = \alpha^2(u^2 - x^2),$$

which is exactly

$$2xuu' = u^2 - x^2.$$

Thus, the equation is homogeneous.
We use the change of variable $t = \frac{u}{x}$, with $u = tx$ and $u' = t'x + t$. The given equation becomes

$$\begin{aligned}
2xuu^{'} = u^2 - x^2 &\iff 2uu^{'} = \frac{u^2}{x} - x \\
&\iff 2tx\left(t^{'}x + t\right) = \left(\frac{u}{x}\right)u - x \\
&\iff 2x^2tt^{'} + 2t^2x = t^2x - x \\
&\iff 2x^2tt^{'} = -\left(t^2 + 1\right) \\
&\iff \frac{2t}{t^2+1}\,dt = -\frac{1}{x}\,dx \\
&\iff \ln\left(t^2+1\right) = -\ln|x| + k, \quad k \in \mathbb{R} \\
&\iff \ln\left(t^2+1\right)|x| = k \\
&\iff x = \frac{K}{t^2+1},\ u = \frac{Kt}{t^2+1},\ K = \pm e^k.
\end{aligned}$$

■

3.3 Second-Order Differential Equations

We consider equations of the form

$$F(x, u, u', u'') = 0.$$

To solve these equations, we distinguish several cases.

3.3.1 Equations of the form $F(x, u', u'') = 0$

In this type of equation, the function u does not appear in the equation. A technique for solving it consists in using the change of variable $y = u'$, and the equation becomes of the form

$$F(x, y, y') = 0.$$

Example 3.10 ***Find the solutions of the differential equation***

$$xu'' - u' = 1.$$

Proof. Using the change of variable $y = u'$, the previous equation becomes

$$xy' = 1 + y.$$

We then have:

$$\begin{aligned}
xy^{'} = 1 + y \iff & x\frac{dy}{dx} = (1 + y) \\
\iff & \frac{dy}{1+y} = \frac{1}{x}dx \\
\iff & \ln|y+1| = \ln|x| + \ln|k| \quad k \in \mathbb{R}^* \\
\iff & 1 + y = kx, \quad k \in \mathbb{R}^* \\
\iff & u^{'} = kx - 1 \\
\iff & u = \frac{k}{2}x^2 - x + c, \quad c \in \mathbb{R}.
\end{aligned}$$

■

3.3.2 Equations of the form $F(x, u'') = 0$

In this equation, u and u' do not both appear in the equation. We have a relation linking x and u''. The technique for solving it is to integrate u'' to find u', then integrate u' to find u.

Example 3.11 ***Solve the equation***

$$(1 + x^2)u'' = 1.$$

Proof.

$$\begin{aligned}
(1 + x^2)u'' = 1 \iff & u'' = \frac{1}{1+x^2} \\
\iff & u' = \arctan(x) + k, \quad k \in \mathbb{R}. \\
\iff & u = \int \arctan(x)\,dx + kx + c, \quad c \in \mathbb{R}.
\end{aligned}$$

We use integration by parts to calculate $\int \arctan(x)\,dx$. ■

3.3.3 Equations of the form $F(u, u', u'') = 0$

The variable x does not appear in these equations of the form $F(u, u', u'') = 0$. The technique for solving them is to use the change of variable $u' = y$, reducing the problem to a first-order equation.
Proof. Substitution $u' = y$ and Expression for u''
Let us consider a second-order differential equation in which we set

$$u' = y.$$

Step 1: Expressing the second derivative.
By definition, the second derivative is

$$u'' = \frac{d}{dx}(u').$$

Since $u' = y$, we have

$$u'' = \frac{dy}{dx}.$$

Step 2: Using the chain rule when y is a function of u.
Sometimes it is convenient to consider y as a function of u. Then, by the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

But $\frac{du}{dx} = u' = y$, so that

$$u'' = \frac{dy}{dx} = y\frac{dy}{du}.$$

Step 3: Summary.

- **If $y = u'$ is treated as a function of x, then $u'' = \frac{dy}{dx}$.**
- **If $y = u'$ is treated as a function of u, then**

$$u'' = y\frac{dy}{du}.$$

This substitution is particularly useful in second-order differential equations where x does not appear explicitly, as it reduces the problem to a first-order differential equation in y as a function of u. ■

Example 3.12 ***Solve the equation***

$$2uu^{''} - (u^{'})^2 = 1$$

Proof. Using the change of variable $y = u'$, we have $u'' = y\frac{dy}{du}$. It follows that

$$\begin{aligned} 2uu^{''} - (u^{'})^2 = 1 &\iff 2uy\frac{dy}{du} = 1 + y^2 \\ &\iff \frac{2y}{1+y^2}dy = \frac{1}{u}du \\ &\iff u = k\left(1 + y^2\right), \end{aligned}$$

On the other hand, we have $y = u'$, so that

$$\begin{aligned} y &= \frac{du}{dx} \\ &= \frac{d}{dx}\left(k + ky^2\right) \\ &= 2k\frac{dy}{dx}y, \end{aligned}$$

It follows that

$$\begin{aligned} 2k\frac{dy}{dx} = 1 &\iff 2kdy = dx \\ &\iff y = \frac{x}{2k} + k^{'} \\ &\iff u = k\left(1 + \left(\frac{x}{2k} + k^{'}\right)^2\right), \quad k, k^{'} \in \mathbb{R}. \end{aligned}$$

■

3.3.4 Second-Order Linear Differential Equations

I/ The case where the coefficients are non-constant
We consider the equation

$$a_2(x)u^{''} + a_1(x)u^{'} + a_0(x)u = f_1(x) \tag{3.19}$$

There are two techniques to solve equation (3.19):

(i) If a particular solution of equation (3.19) is known, then the solution of this equation is of the form $u = u_0 + u_1$, where u_0 is the particular solution and u_1 is the solution of the homogeneous equation associated with (3.19).

(ii) If the particular solution of equation (3.19) is not known, the method of variation of constants is used when we have two linearly independent solutions g_1, g_2 of the homogeneous equation associated with (3.19).

As we have just seen, to find a solution of equation (3.19), we must first solve the homogeneous equation

$$a_2(x)u^{''} + a_1(x)u^{'} + a_0(x)u = 0$$

which can be written in the form

$$u^{''} + b_1(x)u^{'} + b_0(x)u = 0 \tag{3.20}$$

where b_1, b_0, f are continuous functions on a domain $D \subset \mathbb{R}$.

(a) Finding solutions of the homogeneous equation $u^{''} + b_1(x)u^{'} + b_0(x)u = 0$
The general rule is as follows:

Lemma 3.1 ***If g_1 and g_2 are two linearly independent solutions of (3.20), then the general solution of (3.20) is $u_1 = \lambda_1 g_1 + \lambda_2 g_2$, where λ_1, λ_2 are arbitrary real constants.***

Remark 3.3 ***g_1 and g_2 being linearly independent means that they are not proportional. In other words, there is no real λ such that $g_1 = \lambda g_2$.***

We distinguish several possible cases:
First case: We know g_1 and g_2
If g_1 and g_2 are two particular solutions of (3.20), then the general solution of (3.20) is $u_1 = \lambda_1 g_1 + \lambda_2 g_2, \quad \lambda_1, \lambda_2 \in \mathbb{R}$.
Second case: We know only one particular solution of the homogeneous equation
If there exists a non-zero function g on D such that

$$g^{''} + b_1(x)g^{'} + b_0(x)g = 0, \tag{3.21}$$

then we can reduce the solution of (3.20) to solving a first-order equation by setting $u = gy$. Indeed, when we set $u = gy$ we have $u^{'} = g^{'}y + y^{'}g$ and $u^{''} = g^{''}y + y^{'}g^{'} + y^{''}g + g^{'}y^{'}$, and thus equation (3.20) becomes

$$gy^{''} + \Big(2g^{'} + b_1(x)g\Big)\, y^{'} + \Big(g^{''} + b_1(x)g^{'} + b_0(x)g\Big)\, y = 0,$$

using (3.21), we find

$$gy^{''} + \Big(2g^{'} + b_1(x)g\Big)\, y^{'} = 0,$$

which is indeed a first-order differential equation that can be easily solved.

$$gy^{''} + \Big(2g^{'} + b_1(x)g\Big)\, y^{'} = 0 \Longleftrightarrow \frac{y^{''}}{y^{'}} = -2\frac{g^{'}}{g} - b_1(x)$$

Another change of variable is necessary: we set $v = y^{'}$, and thus $\dfrac{dv}{dx} = y^{''}$. It follows that

$$\frac{dv}{v} = -2\frac{g^{'}}{g}\, dx - b_1(x)\, dx.$$

Integrating both sides of the equation gives

$$\ln|v| = -2\ln|g| - B(x) + k.$$

Thus,

$$v = Ke^{-2\ln|g| - B(x)},$$

where $B(x)$ is a primitive of $b_1(x)$. If $G(x)$ is a primitive of $e^{-2\ln|g|-B(x)}$, then

$$v = y^{'} = Ke^{-2\ln|g|-B(x)} \Longrightarrow y = KG(x) + \eta.$$

Now, it is enough to replace y to find

$$u = gy = KgG(x) + \eta g.$$

The general solution of (3.20) is

$$u_1 = \lambda_1 g_1 + \lambda_2 g_2, \quad \lambda_1, \lambda_2 \in \mathbb{R},$$

with $g_1 = g$ and $g_2 = u = gy = KgG(x) + \eta g$.

Third case: We do not know any particular solution of the homogeneous equation

In the case where we do not know any particular solution of (3.20), we look for the general solution u_1 in the form of a product of two unknown functions. We set $u_1 = vg$, and we choose $v(x)$ so that the factor of $g^{'}$ is zero. Starting from these conditions, we have: $u_1 = vg \Longrightarrow u_0^{'} = v^{'}g + g^{'}v$ and thus $u_0^{''} = v^{''}g + g^{''}v + 2v^{'}g^{'}$. Substituting these results into equation (3.20), we find:

$$g^{''}v + \Big(b_1(x)v + 2v^{'}\Big)\,g^{'} + \Big(v^{''} + b_1(x)v^{'} + b_0(x)v\Big)\,g = 0. \tag{3.22}$$

By choosing

$$\Big(b_1(x)v + 2v^{'}\Big) = 0, \tag{3.23}$$

we have

$$g^{''}v + \Big(v^{''} + b_1(x)v^{'} + b_0(x)v\Big) = 0. \tag{3.24}$$

Solving equation (3.23) gives v, and solving (3.24) gives g. Thus the general solution of (3.20) is fully determined.

Last case: We reduce to a homogeneous equation with constant coefficients

The technique is to make a suitable change of variable to transform equation (3.20) into a homogeneous equation with constant coefficients, which is simpler to solve.

Example 3.13 ***Consider the equation***

$$ax^2u^{''} + bxu^{'} + cu = 0 \tag{3.25}$$

where a, b, c are real constants and $a \neq 0$.

We perform the change of variable $x = \alpha e^t$, with $\alpha = 1$ if $x > 0$, and $\alpha = -1$ if $x < 0$. It follows that

$$\frac{dt}{dx} = \frac{1}{\alpha e^t},$$

hence

$$u^{'} = \frac{du}{dx} = \frac{du}{dt}\frac{dt}{dx} = \frac{1}{\alpha e^t}\frac{du}{dt}.$$

$$\begin{aligned} u'' &= \frac{d^2u}{dx^2} = \frac{du'}{dx} = \frac{du'}{dt}\frac{dt}{dx} = \frac{1}{\alpha e^t}\frac{du'}{dt} \\ &= \frac{1}{\alpha e^t}\frac{d}{dt}\left(\frac{1}{\alpha e^t}\frac{du}{dt}\right) \\ &= \frac{1}{\alpha^2 e^t}\left(-\frac{du}{dt} + \frac{d^2u}{dt^2}\right) \\ &= \frac{1}{e^t}\left(-\frac{du}{dt} + \frac{d^2u}{dt^2}\right). \end{aligned}$$

Substituting these results into* (3.25)*, this equation becomes

$$\frac{d^2u}{dt^2} + \left(\frac{b}{a} - 1\right)\frac{du}{dt} + \frac{c}{a}u = 0,$$

which is a linear homogeneous differential equation with constant coefficients.

(b)

Finding a particular solution of the equation $u'' + b_1(x)u' + b_0(x)u = f(x)$

If g_1 and g_2 are two non-zero and linearly independent solutions of equation (3.20), the solution of equation (3.19) is of the form $u = g_1u_1 + g_2u_2$, where u_1, u_2 are unknown functions that can be determined if certain additional conditions are imposed. We use the method of variation of constants.
A simple calculation gives

$$u' = g_1'u_1 + g_1u_1' + g_2'u_2 + g_2u_2',$$

and

$$u'' = g_1''u_1 + g_1'u_1' + g_1'u_1' + g_1u_1'' + g_2''u_2 + g_2'u_2' + g_2'u_2' + g_2u_2''.$$

Considering that g_1 and g_2 are solutions of equation (3.20), and that $u = g_1u_1 + g_2u_2$ is a solution of equation (3.19), we obtain

$$2\left(g_1'u_1' + g_2'u_2'\right) + b_1(x)\left(g_1u_1' + g_2u_2'\right) + g_1u_1'' + g_2u_2'' = f(x). \tag{3.26}$$

By imposing the additional condition

$$g_1u_1' + g_2u_2' = 0,$$

and differentiating both sides, we obtain

$$g_1u_1'' + g_2u_2'' = -\left(g_1'u_1' + g_2'u_2'\right).$$

Substituting this result into equation (3.26), we get

$$g_1'u_1' + g_2'u_2' = f(x).$$

In conclusion, to find $u_1^{'}$ and $u_2^{'}$, we solve the system

$$\begin{cases} g_1 u_1^{'} + g_2 u_2^{'} = 0, \\ g_1^{'} u_1^{'} + g_2^{'} u_2^{'} = f(x). \end{cases}$$

As soon as we have $u_1^{'}$ and $u_2^{'}$, we compute their antiderivatives to obtain u_1 and u_2.

II/ The case where the coefficients are constant
We consider the equation

$$a_2 u^{''} + a_1 u^{'} + a_0 u = f_1(x),$$

which can be written in the form

$$u^{''} + b_1 u^{'} + b_0 u = f(x), \tag{3.27}$$

where a_0, a_1, a_2, b_0, b_1 are real numbers with $a_2 \neq 0$ and f_1, f are continuous functions on a domain $D \subset \mathbb{R}$. Let

$$u^{''} + b_1 u^{'} + b_0 u = 0 \tag{3.28}$$

be the homogeneous equation associated with (3.27). It is clear that the solutions of (3.27) are of the form $u = u_0 + u_1$, where u_0 is the general solution of (3.28) and u_1 is a particular solution of (3.27). Therefore, from now on, we will focus on the determination of u_0 and u_1.

1/ Calculation of the general solution of the homogeneous equation
To determine the solutions of the homogeneous equation (3.28), we define the characteristic equation associated with (3.28), which is given by

$$r^2 + b_1 r + b_0 = 0. \tag{3.29}$$

The solutions of equation (3.29) depend on the sign of the discriminant $\Delta = b_1^2 - 4b_0$. Indeed,

- **If $\Delta > 0$, equation (3.29) has two distinct real solutions:**

$$r_1 = \frac{-b_1 - \sqrt{\Delta}}{2}, \quad r_2 = \frac{-b_1 + \sqrt{\Delta}}{2}.$$

- **If $\Delta < 0$, equation (3.29) has two complex solutions:**

$$r_1 = \frac{-b_1 - i\sqrt{\Delta'}}{2} = \alpha - i\beta, \quad r_2 = \frac{-b_1 + i\sqrt{\Delta'}}{2} = \alpha + i\beta, \quad -\Delta' = \Delta,\ i^2 = -1.$$

- **If $\Delta = 0$, equation (3.29) has a double solution:**

$$r_1 = r_2 = \frac{-b_1}{2}.$$

Let λ_1, λ_2 be two arbitrary real numbers. The following table summarizes the methods for calculating the general solution of the homogeneous equation (3.29):

$\Delta > 0$	$g_1(x) = e^{r_1 x},\ g_2(x) = e^{r_2 x}$	$u_1 = \lambda_1 e^{r_1 x} + \lambda_2 e^{r_2 x}$
$\Delta < 0$	$g_1(x) = e^{\alpha x}\cos(\beta x),\ g_2(x) = e^{\alpha x}\sin(\beta x)$	$u_1 = \lambda_1 e^{\alpha x}\cos(\beta x) + \lambda_2 e^{\alpha x}\sin(\beta x)$
$\Delta = 0$	$g_1(x) = e^{r_1 x},\ g_2(x) = x e^{r_1 x}$	$u_1 = (\lambda_1 + \lambda_2 x)e^{r_1 x}$

2/ Calculation of a particular solution
Now, we aim to determine a particular solution of equation (3.27). Generally, the form of f guides us in choosing the particular solution u_0. Several situations are possible:

- **If $f(x)$ is a polynomial of degree n, we look for u_0 in the form of a polynomial of degree n that satisfies (3.27).**

  Example 3.14 ***Find a particular solution of the equation***

$$u'' - 3u' - 1 = 4x^2 + 1 \tag{3.30}$$

  We set $u_0 = ax^2 + bx + c$ with $a \neq 0$. We have $u_0' = 2ax + b$, $u_0'' = 2a$. Substituting these results into (3.30) and performing identification, we find the values of a, b, c.

- **If $f(x) = h(x)e^{rx}$, where $h(x)$ is a polynomial of degree n, we look for $u_0 = k(x)e^{rx}$, with $k(x)$ a polynomial of degree m such that**
  - $m = n$ **if** $r^2 + b_1 r + b_0 \neq 0$
  - $m = n + 1$ **if** $r^2 + b_1 r + b_0 = 0$
  - $m = n + 2$ **if** r **is a double root of** $r^2 + b_1 r + b_0$

  Example 3.15 ***Find a particular solution of***

$$u'' - 3u' + 1 = 2e^{3x} \tag{3.31}$$

  Here, $h(x) = 2$, $r = 3$, $n = 0$, $(3)^2 - 3(3) + 1 = 1 \neq 0$, hence $m = 0$, $k(x) = k$, $k \in \mathbb{R}$, and $u_0 = ke^{3x}$. We compute $u_0' = 3ke^{3x}$, $u_0'' = 9ke^{3x}$. Substituting these results into equation (3.31) and identifying both sides, we find the value of k.

- **If $f(x) = a\cos(rx) + b\sin(rx)$, we look for u_0 in one of the following forms:**
  - $u_0 = \alpha\cos(rx) + \beta\sin(rx)$,
  - $u_0 = x(\alpha\cos(rx) + \beta\sin(rx))$ **if** $\cos(rx)$ **is a solution of the homogeneous equation.**

  Using the same previous technique, we find α, β.

- **If $f(x) = f_1(x) + f_2(x) + \cdots + f_m(x)$ where $f_1, f_2, \cdots, f_m$ take one of the previous forms, then we look for $u_0 = v_1 + v_2 + \cdots + v_m$ where v_i is a particular solution of**

$$v_i'' + b_1 v_i' + b_0 v_i = f_i(x), \quad 1 \leq i \leq m. \tag{3.32}$$

- **If $f(x)$ cannot be written in any of the above forms, we use the method of variation of constants. If $u_1 = \lambda_1 g_1 + \lambda_2 g_2,\ (\lambda_1, \lambda_2 \in \mathbb{R})$ is the general solution of the homogeneous equation (3.28), we look for a particular solution of (3.27) in the form $u_0 = \lambda_1(x) g_1 + \lambda_2(x) g_2$, where $\lambda_1(x), \lambda_2(x)$ are unknown continuous functions to be determined. This returns us to the method already seen in the case of a second-order linear differential equation with non-constant coefficients and a non-homogeneous term (Ib).**

3.4 Multivariable Functions

3.4.1 Functions of Several Variables with Real Values

Let n be a non-zero natural number, and E a non-empty subset of $\mathbb{R}^n$.

Definition 3.9 ***A function of n real variables with real values is any function f defined as follows***

$$f : \left| \begin{array}{ccc} E \subset \mathbb{R}^n & \longrightarrow & \mathbb{R} \\ (x_1, x_2, x_3, \cdots, x_n) & \longmapsto & f(x_1, x_2, x_3, \cdots, x_n) = y \end{array} \right.$$

Examples 3.1 ***1. f is a function of two variables x and y***

$$f : \left| \begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ (x, y) & \longmapsto & f(x, y) = x^2 + y^2 \end{array} \right.$$

$f(1, 2) = 1^2 + 2^2 = 5.$

2. g is a function of three real variables.

$$f : \left| \begin{array}{ccc} \mathbb{R}^2 \times \mathbb{R}^* & \longrightarrow & \mathbb{R} \\ (x, y, z) & \longmapsto & f(x, y, z) = \dfrac{e^x + 2\sin(y)}{3z^2}. \end{array} \right.$$

1. Domain of Definition of a Function of Two Real Variables

Let f be a function of two real variables x, y defined by

$$f : \left| \begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ (x, y) & \longmapsto & f(x, y). \end{array} \right.$$

The domain of definition of f, denoted by D, is the set of pairs $(x, y) \in \mathbb{R}^2$ for which $f(x, y)$ exists.

Example 3.16

$$f : \left| \begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ (x, y) & \longmapsto & f(x, y) = \ln(x) + \sin(y). \end{array} \right.$$

For f to be well-defined, both $\ln(x)$ and $\sin(y)$ must be defined simultaneously. Therefore, $x > 0$ and $y \in \mathbb{R}$, so

$$D = \mathbb{R}_+^* \times \mathbb{R}.$$

2. Graphical Representation of a Function of Two Real Variables

Let f be a function defined on a domain D of $\mathbb{R}^2$ such that

$$f : \left| \begin{array}{ccc} D \subset \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ (x,y) & \longmapsto & f(x,y). \end{array} \right.$$

The graphical representation of f is a surface S_f in $\mathbb{R}^3$ defined by

$$S_f = \Big\{ (x,y,z) \in \mathbb{R}^3 \ / \ [z = f(x,y)] \wedge [(x,y) \in D] \Big\}.$$

In other words, S_f is the set of points in space with coordinates $M(x,y,f(x,y))$ for $(x,y) \in D$. To each point $(x,y) \in D$ corresponds a point in space lying on the surface S_f.

3.4.2 First-Order Partial Derivatives

In this section, we assume that the notion of the derivative of a function defined from $\mathbb{R}$ to $\mathbb{R}$ is known, and we want to provide its generalization for functions of several variables with values in $\mathbb{R}$. For simplicity, we start with functions of two real variables; the case of functions of three or more real variables follows easily.

Definition 3.10 ***Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$. We say that f has a first derivative at $x_0 \in \mathbb{R}^2$ along the vector $u = (u_1, u_2)$ if, and only if, $\psi_u : t \longmapsto f(x_0 + tu)$ is differentiable at 0. In this case, $\psi_u'(0)$ represents the derivative of f at the point x_0 in the direction of u, denoted by $D_u f(x_0)$, and we have***

$$D_u f(x_0) = \lim_{t \to 0} \frac{f(x_0 + tu) - f(x_0)}{t}$$

Example 3.17 $f(x,y) = xy,\ x_0 = (0,0),\ u = (1,1).$ ***Compute $D_u f(x_0)$.***
We have $x_0 + tu = (t,t)$ and $f(t,t) = t^2$. thus

$$\lim_{t \to 0} \frac{f(x_0 + tu) - f(x_0)}{t} = \lim_{t \to 0} \frac{t^2}{t} = \lim_{t \to 0} t = 0.$$

Hence,

$$D_u f(x_0) = 0.$$

Definition 3.11 ***The derivatives of a function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$, along the vectors $\overrightarrow{i}(1,0)$ and $\overrightarrow{j}(0,1)$, if they exist, correspond respectively to the partial derivatives with respect to x and y, denoted by $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$. We then have***

$$D_{\overrightarrow{i}} f(x,y) = \frac{\partial f}{\partial x}(x,y) \quad \textbf{\textit{and}} \quad D_{\overrightarrow{j}} f(x,y) = \frac{\partial f}{\partial y}(x,y).$$

Now, we generalize the notion of partial derivatives to functions defined on $\mathbb{R}^n$.

Definition 3.12 ***Let*** $f : D \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ ***and*** $a = (a_1, a_2, \cdots, a_n) \in D$. ***For*** $i = 1, 2, \cdots, n$, ***the partial derivative of*** f ***at*** a ***with respect to*** x_i ***is denoted*** $\frac{\partial f}{\partial x_i}(a)$ ***and is defined as the derivative of the partial function taken at*** a_i

$$\frac{\partial f}{\partial x_i}(a) = \lim_{x_i \to a_i} \frac{f(a_1, a_2, \cdots, x_i, \cdots, a_n) - f(a_1, a_2, \cdots, a_i, \cdots, a_n)}{x_i - a_i} = f'_{x_i}(a),$$

we can also write

$$\frac{\partial f}{\partial x_i}(a) = \lim_{h_i \to 0} \frac{f(a_1, a_2, \cdots, a_i + h_i, \cdots, a_n) - f(a_1, a_2, \cdots, a_i, \cdots, a_n)}{h_i} = f'_{x_i}(a). \quad (3.33)$$

In particular, if $f : \mathbb{R}^2 \longrightarrow \mathbb{R},\ a = (a_1, a_2)$,

$$\frac{\partial f}{\partial x}(a_1, a_2) = \lim_{h_1 \to 0} \frac{f(a_1 + h_1, a_2) - f(a_1, a_2)}{h_1} = f'_x(a_1, a_2), \quad (3.34)$$

and

$$\frac{\partial f}{\partial y}(a_1, a_2) = \lim_{h_2 \to 0} \frac{f(a_1, a_2 + h_2) - f(a_1, a_2)}{h_2} = f'_y(a_1, a_2). \quad (3.35)$$

Remark 3.4 ***It should be understood that one can only talk about partial derivatives at*** $a = (a_1, a_2)$ ***if the limits in*** (3.34) ***and*** (3.35) ***exist. When these limits exist, they are denoted by*** $\frac{\partial f}{\partial x}(a_1, a_2)$ ***and*** $\frac{\partial f}{\partial y}(a_1, a_2)$ ***respectively. This remark remains valid for functions defined on*** $\mathbb{R}^n$, ***where the limit in*** (3.33) ***is required to exist.***

When the partial derivatives exist, how can we compute them?

Example 3.18

$$f(x, y) = 2x^3y^2, \quad a = (-1, 2)$$

1. ***Consider*** y ***as a constant and differentiate with respect to*** x. ***Then,***

$$f'_x(x, y) = \frac{\partial f}{\partial x}(x, y) = 6y^2x^2.$$

2. ***Consider*** x ***as a constant and differentiate with respect to*** y. ***Then,***

$$f'_y(x, y) = \frac{\partial f}{\partial y}(x, y) = 4yx^3.$$

Thus,

$$\frac{\partial f}{\partial x}(-1, 2) = 6(-1)^2(2)^2 = 24,$$
$$\frac{\partial f}{\partial y}(-1, 2) = 4(2)(-1)^3 = -8.$$

Remark 3.5 ***The existence of partial derivatives at the point*** $a = (a_1, a_2)$ ***does not imply that*** f ***is continuous at this point.***

Example 3.19 ***Consider the function*** f ***defined by***

$$f(x,y)=\begin{cases}\dfrac{xy}{x^2+y^2}, & \textbf{\textit{if}}\ (x,y)\neq(0,0),\\ 0, & \textbf{\textit{if}}\ (x,y)=(0,0).\end{cases}$$

Study the existence of partial derivatives at the point $(0,0)$.

$\lim\limits_{h_1\to 0}\dfrac{f(h_1,0)-f(0,0)}{h_1}=\lim\limits_{h_1\to 0}\dfrac{0}{h_1}=0\Longrightarrow\dfrac{\partial f}{\partial x}(0,0)=0,$

$\lim\limits_{h_2\to 0}\dfrac{f(0,h_2)-f(0,0)}{h_2}=\lim\limits_{h_2\to 0}\dfrac{0}{h_2}=0\Longrightarrow\dfrac{\partial f}{\partial y}(0,0)=0.$

The partial derivatives at the point $(0,0)$ ***exist.***

3.4.3 Higher-Order Partial Derivatives

The definition is given for a function f of two variables, and it remains valid for functions of n $(n>2)$ variables.

Definition 3.13 ***If for a function*** $f(x,y)$ ***defined on*** $D\subset\mathbb{R}^2$, ***the partial derivatives*** $\dfrac{\partial f}{\partial x}(x,y)$ ***and*** $\dfrac{\partial f}{\partial y}(x,y)$ ***exist and are themselves functions of*** x ***and*** y, ***then*** $\dfrac{\partial f}{\partial x}$ ***and*** $\dfrac{\partial f}{\partial y}$ ***may have partial derivatives such that:***

$\dfrac{\partial^2 f}{\partial x^2}=\dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial x}\right)=f''_{xx}$ ***we differentiate twice with respect to*** x.

$\dfrac{\partial^2 f}{\partial x\partial y}=\dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right)=f''_{yx}$ ***we first differentiate with respect to*** y ***and then with respect to*** x.

$\dfrac{\partial^2 f}{\partial y\partial x}=\dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right)=f''_{xy}$ ***we first differentiate with respect to*** x ***and then with respect to*** y.

$\dfrac{\partial^2 f}{\partial y^2}=\dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial y}\right)=f''_{yy}$ ***we differentiate twice with respect to*** y.

Example 3.20 $f(x,y)=3x^4-2xy^3$

$$\frac{\partial f}{\partial x}=12x^3-2y^3,\quad\frac{\partial f}{\partial y}=-6xy^2.$$

$$\frac{\partial^2 f}{\partial x^2}=36x^2,\quad\frac{\partial^2 f}{\partial y^2}=-12xy.$$

$$\frac{\partial^2 f}{\partial y\partial x}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial}{\partial y}\left(12x^3-2y^3\right)=-6y^2.$$

$$\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial}{\partial x}\left(-6xy^2\right)=-6y^2.$$

In Example 3.20, we notice that $\dfrac{\partial^2 f}{\partial x\partial y}=\dfrac{\partial^2 f}{\partial y\partial x}$. Is this a coincidence, or is it always the case? The following theorem answers this question:

Theorem 3.1 ***(Schwarz's Theorem)***
Let $f : D \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ ***be a function continuous on*** D. ***If for all*** $i, j = 1, 2, \cdots, n$, ***the partial derivatives*** $\dfrac{\partial f}{\partial x_i}, \dfrac{\partial f}{\partial x_j}$ ***exist and are continuous, and the mixed partial derivatives*** $\dfrac{\partial}{\partial x_i}\left(\dfrac{\partial f}{\partial x_j}\right)$ ***exist and are also continuous, then for every*** $a = (a_1, a_2, \cdots, a_n) \in D$, ***and for*** $i, j = 1, 2, \cdots, n$,

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(a) = \frac{\partial^2 f}{\partial x_j \partial x_i}(adxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxc).$$

This theorem states that the order of differentiation with respect to the variables is irrelevant in the case where the partial derivatives exist and are continuous in the neighborhood of a point.

3.4.4 Derivatives of a Function Composed of Two Variables

Definition 3.14 ***Let*** $z = f(u, v)$ ***be a function of two variables*** u ***and*** v, ***where*** u ***and*** v ***themselves are functions of*** x ***and*** y, ***i.e.,***

$$u = u(x, y), \quad v = v(x, y).$$

Then z ***can be expressed explicitly as a function of*** x ***and*** y:

$$z(x, y) = f(u(x, y), v(x, y)).$$

Theorem 3.2 (Chain Rule for Two Variables) ***If*** f, u, ***and*** v ***are differentiable, the partial derivatives of*** $z(x, y)$ ***are given by:***

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x}, \quad \frac{\partial z}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y}.$$

Example 3.21 ***Let***

$$z(x, y) = \sin(u(x, y)\, v(x, y)), \quad u(x, y) = x^2 + y, \quad v(x, y) = xy.$$

Compute the partial derivatives $\dfrac{\partial z}{\partial x}$ ***and*** $\dfrac{\partial z}{\partial y}$.
Solution: Using the chain rule,

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = v(x, y)\cos(uv) \cdot 2x + u(x, y)\cos(uv) \cdot y,$$
$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = v(x, y)\cos(uv) \cdot 1 + u(x, y)\cos(uv) \cdot x.$$

3.4.5 Differential

We know that if a function $f : D \subset \mathbb{R} \longrightarrow \mathbb{R}$ is differentiable on D, then its derivative $f^{'}$ satisfies

$$\forall x \in D : \; f^{'}(x) = \frac{df}{dx} \tag{3.36}$$

and

$$\forall x \in D : \; df = f^{'}(x)\, dx \tag{3.37}$$

df is the differential of f. We generalize this result for functions of several variables.

Definition 3.15 ***Let $f : D \subset \mathbb{R}^n \longrightarrow \mathbb{R}$. We say that f is differentiable at $a \in D$ if there exists a linear map $L : \mathbb{R}^n \longrightarrow \mathbb{R}$, such that in the neighborhood of a we have***

$$f\,(a+h) = f(a) + Lh + o(\|h\|).$$

If such a map exists, it is called the differential of f at the point a and is denoted $df(a)$.

We will admit the following proposition:

Proposition 3.3 ***If a function $f : D \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ is continuous and has partial derivatives that are all continuous, then f is differentiable on D and we have:***

$$df = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \cdots + \frac{\partial f}{\partial x_n}dx_n.$$

1. Total Differentials

Definition 3.16 ***Let $z = f(x, y)$ be a function of two variables. The total differential of f at the point (x, y) is defined as:***

$$dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy,$$

where dx and dy are infinitesimal changes in x and y, respectively.

Remark 3.6 ***If $z(x, y) = cst$ then $dz = 0$.***

Example 3.22 ***Let***

$$z(x, y) = x^2 y + \sin(xy).$$

Then the total differential is:

$$\begin{aligned} dz &= \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy \\ &= (2xy + y\cos(xy))\, dx + \Big(x^2 + x\cos(xy)\Big)\, dy. \end{aligned}$$

Remark 3.7 ***The total differential*** dz ***gives an approximation of the change in*** z ***for small changes*** dx ***and*** dy***:***

$$\Delta z \approx dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy.$$

Example 3.23

$$f\left(x, y\right) = \sin\left(xy\right)$$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = y\cos(xy)dx + x\cos(xy)dy.$$

2. Exact Total Differentials

Definition 3.17 ***Let*** $M(x, y)$ ***and*** $N(x, y)$ ***be functions defined on a domain*** $D \subset \mathbb{R}^2$***. The differential expression***

$$\omega = M(x, y)\,dx + N(x, y)\,dy$$

is called a total differential of some function $f(x, y)$ ***if there exists a function*** f ***such that***

$$df(x, y) = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = M(x, y)\,dx + N(x, y)\,dy.$$

In this case, ω ***is said to be exact.***

Theorem 3.3 (Condition for Exactness) ***A differential form***

$$\omega = M(x, y)\,dx + N(x, y)\,dy$$

is exact in a simply connected domain D ***if*** M ***and*** N ***have continuous first partial derivatives and satisfy***

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Example 3.24 ***Consider the differential form***

$$\omega = (2xy + 3)\,dx + (x^2 + 4y)\,dy.$$

Check if ω ***is exact.***
We have

$$M(x, y) = 2xy + 3; \quad N(x, y) = x^2 + 4y.$$

Solution: Compute the partial derivatives:

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(2xy + 3) = 2x, \qquad \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(x^2 + 4y) = 2x.$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$***, the differential form is exact.***
Since

$$M(x, y) = \frac{\partial f}{\partial x}, \quad \textbf{\textit{and}}\ N(x, y) = \frac{\partial f}{\partial y},$$

to find the potential function $f(x,y)$, we first integrate M with respect to x:

$$f(x,y) = \int M\,dx = \int (2xy+3)\,dx = x^2y + 3x + g(y),$$

where $g(y)$ is a function of y.
Differentiate f with respect to y and set it equal to $N(x,y)$:

$$\frac{\partial f}{\partial y} = x^2 + g'(y) = N(x,y) = x^2 + 4y \implies g'(y) = 4y \implies g(y) = 2y^2.$$

Hence, the potential function is

$$f(x,y) = x^2y + 3x + 2y^2.$$

Particular Case: Finding the Potential Function $f(x,y)$
Consider the differential equation

$$M(x,y)\,dx + N(x,y)\,dy = 0,$$

where $M(x,y)$ and $N(x,y)$ are continuous functions with continuous first partial derivatives.

Definition 3.18 ***If the differential form $M(x,y)\,dx + N(x,y)\,dy$ is exact, there exists a function $f(x,y)$ such that***

$$df(x,y) = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = M(x,y)\,dx + N(x,y)\,dy.$$

Then the solution of the differential equation can be written as

$$f(x,y) = C,$$

where C is a constant.

Method 1 (Finding $f(x,y)$) ***To find the function $f(x,y)$:***

1. ***Integrate $M(x,y)$ with respect to x:***

$$f(x,y) = \int M(x,y)\,dx + g(y),$$

where $g(y)$ is an arbitrary function of y.

2. ***Differentiate $f(x,y)$ with respect to y:***

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\int M(x,y)\,dx\right) + g'(y),$$

and set it equal to $N(x,y)$:

$$\frac{\partial f}{\partial y} = N(x,y) \implies g'(y) = N(x,y) - \frac{\partial}{\partial y}\left(\int M(x,y)\,dx\right).$$

3. ***Integrate*** $g'(y)$ ***to find*** $g(y)$***.***

4. ***Substitute*** $g(y)$ ***back into*** $f(x, y)$ ***to obtain the potential function.***

Example 3.25 *Solve*

$$(2xy + 3)\,dx + (x^2 + 4y)\,dy = 0.$$

Solution:

1. ***Integrate*** $M(x, y) = 2xy + 3$ ***with respect to*** x***:***

$$f(x, y) = \int (2xy + 3)\,dx = x^2y + 3x + g(y).$$

2. ***Differentiate with respect to*** y ***and set equal to*** $N(x, y) = x^2 + 4y$***:***

$$\frac{\partial f}{\partial y} = x^2 + g'(y) = x^2 + 4y \implies g'(y) = 4y \implies g(y) = 2y^2.$$

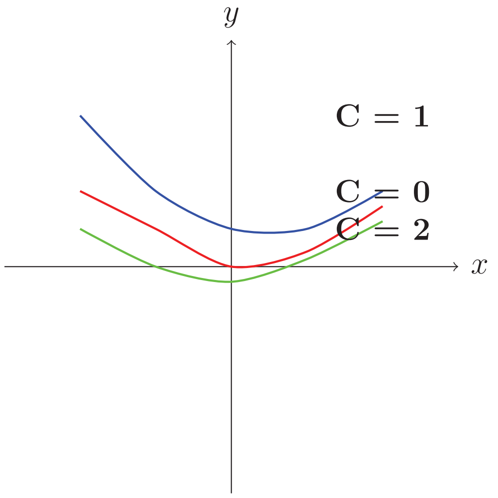
3. ***The potential function is***

$$f(x, y) = x^2y + 3x + 2y^2.$$

4. ***The general solution of the differential equation is***

$$f(x, y) = x^2y + 3x + 2y^2 = C.$$

Level Curves of $f(x, y) = x^2y + 3x + 2y^2$



Clarification

Point 1: What is $f(x, y)$?

In our example we have

$$f(x, y) = x^2y + 3x + 2y^2.$$

- This is a function of two variables, x and y.

- **Its value depends on the coordinates (x, y).**
- **For example:**
$$f(1,1) = 1^2 \cdot 1 + 3 \cdot 1 + 2 \cdot 1^2 = 6,$$
while
$$f(0,1) = 0 + 0 + 2 \cdot 1^2 = 2.$$
- **Therefore, $f(x, y)$ is not a constant; it varies with (x, y).**

Point 2: Where does the constant C come from?
From the exact differential equation

$$(2xy + 3)\, dx + (x^2 + 4y)\, dy = 0,$$

we know that
$$df = 0,$$
where
$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy.$$

- **The condition $df = 0$ means that the total change of f along a solution is zero.**
- **In other words, along a trajectory that satisfies the differential equation, the value of $f(x, y)$ remains the same.**
- **This is why the solution can be written as**
$$f(x, y) = C,$$
where C is a constant that depends on the chosen trajectory.

3. Solving a First-Order ODE Using the Exact Differential Method

Consider the first-order ODE:

$$xy' + y = x^2 \tag{3.38}$$

This is of the form

$$a(x)y' + b(x)y = c(x), \quad \textbf{with } a(x) = x,\ b(x) = 1,\ c(x) = x^2.$$

Step 1: Rewrite in differential form
Multiply both sides by dx:

$$x\, dy + y\, dx = x^2\, dx$$

Rewriting:

$$(y - x^2)\, dx + x\, dy = 0$$

where

$$M(x, y) = y - x^2, \quad N(x, y) = x$$

Step 2: Check exactness
The equation is exact if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Compute:

$$\frac{\partial M}{\partial y} = 1, \quad \frac{\partial N}{\partial x} = 1$$

Thus, the equation is exact.
Step 3: Find $f(x, y)$
We seek $f(x, y)$ such that

$$\frac{\partial f}{\partial x} = M(x, y) = y - x^2, \quad \frac{\partial f}{\partial y} = N(x, y) = x$$

Integrate $M(x, y)$ with respect to x:

$$f(x, y) = \int (y - x^2) dx = xy - \frac{x^3}{3} + g(y)$$

Step 4: Determine $g(y)$
Differentiate $f(x, y)$ with respect to y:

$$\frac{\partial f}{\partial y} = x + g'(y)$$

Compare with $N(x, y) = x$:

$$g'(y) = 0 \implies g(y) = C_0$$

Step 5: Solution
The general solution is:

$$\boxed{f(x, y) = xy - \frac{x^3}{3} = C}$$

where C is an arbitrary constant.

3.5 Elements of Partial Differential Equations (PDEs)

3.5.1 Generalities

1. Introduction
A partial differential equation (PDE) is an equation that relates an unknown function $u(x_1, x_2, \ldots, x_n)$ of several variables to its partial derivatives.
Unlike ordinary differential equations (ODEs), which involve functions of a single variable, PDEs model phenomena depending on multiple variables, often in space and time.
Examples:

- **Heat diffusion: $u(x,t)$ depends on space x and time t.**
- **Wave propagation: $u(x,t)$ represents displacement of a string or membrane.**
- **Electrostatics: potential $V(x,y,z)$ in a spatial domain.**

The general form of a PDE can be written as:

$$F\Big(x_1, \ldots, x_n, u, u_{x_1}, \ldots, u_{x_n}, u_{x_1x_1}, \ldots\Big) = 0$$

2. Order and Degree
Order: the highest order of partial derivative appearing in the equation.
Example:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \Rightarrow \text{second order}$$

Degree: the highest power to which a derivative is raised (if polynomial).
Example:

$$\left(\frac{\partial u}{\partial x}\right)^3 + u = 0 \quad \Rightarrow \text{degree 3}$$

3. Linearity
A PDE is linear if the unknown function u and its derivatives appear linearly: - No products of u and its derivatives - No powers higher than 1
Examples:

- **Linear: $u_t = u_{xx}$ (heat equation)**
- **Non-linear: $u_t = (u_x)^2$**

4. Classification of Second-Order PDEs
For a linear second-order PDE in two variables:

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G(x,y)$$

Discriminant: $D = B^2 - 4AC$

Type	Condition	Physical Example
Elliptic	$D < 0$	Laplace's equation (potential problems)
Parabolic	$D = 0$	Heat equation
Hyperbolic	$D > 0$	Wave equation

- Elliptic: smooth solutions, equilibrium problems.
- Parabolic: time-evolving solutions, diffusion problems.
- Hyperbolic: wave propagation with finite speed.

5. Initial and Boundary Conditions

To ensure a unique solution:

- Initial conditions: values of u and/or its derivatives at $t = 0$
  Example: $u(x, 0) = f(x)$
- Boundary conditions: values of u on the spatial domain boundary $\partial\Omega$
  - Dirichlet: $u = g(x)$
  - Neumann: $\frac{\partial u}{\partial n} = h(x)$

3.5.2 Methods for Solving PDEs

1. Separation of Variables

Assume the solution can be written as a product of functions, each depending on a single variable:

$$u(x, t) = X(x)\,T(t)$$

Example: Heat Equation

$$u_t = ku_{xx}, \quad 0 < x < L,\ t > 0$$

- Assume $u(x, t) = X(x)T(t)$.
- Substitute into the PDE: $X(x)T'(t) = kX''(x)T(t)$
- Separate variables: $\frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)} = -\lambda$
- Solve the resulting ODEs: $\begin{cases} X'' + \lambda X = 0 \\ T' + k\lambda T = 0 \end{cases}$
- Apply boundary conditions to determine possible λ and construct the general solution.

Remark:
The negative sign in $-\lambda$ is a practical convention. It ensures that the spatial function $X(x)$ satisfies the boundary conditions, such as $X(0) = X(L) = 0$.
Using $-\lambda$ gives the harmonic equation:

$$X'' + \lambda X = 0 \quad \Rightarrow \quad X(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$$

which allows for non-trivial solutions satisfying the boundary conditions.
If we had used $+\lambda$**, the solution would involve exponentials:**

$$X(x) = Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x}$$

which cannot satisfy $X(0) = X(L) = 0$ **unless** $X \equiv 0$**. Therefore, the negative sign simplifies finding physically meaningful solutions.**

2. Method of Characteristics

Used for first-order PDEs:

$$a(x, y)u_x + b(x, y)u_y = c(x, y, u)$$

- **Define characteristic curves** $(x(s), y(s))$ **along which the PDE reduces to an ODE.**
- **Characteristic equations:** $\frac{dx}{ds} = a(x, y), \frac{dy}{ds} = b(x, y), \frac{du}{ds} = c(x, y, u)$
- **Solve these ODEs to find** $u(x, y)$**.**

Example: $u_x + u_y = 0$
- Characteristics: $y - x =$ **constant - Solution:** $u(x, y) = f(y - x)$**,** f **determined by initial conditions.**

3. Transform Methods (Fourier and Laplace)

- Fourier transform: converts spatial PDE into temporal ODE, useful for infinite or periodic domains. - Laplace transform: converts temporal PDE into algebraic equation, convenient for initial conditions.
Example: Heat equation on $x \in [0, \infty)$ **with** $u(x, 0) = f(x)$**:**

- **Apply Laplace transform in** t**:** $U(x, s) = \mathcal{L}\{u(x, t)\}$
- **PDE becomes ODE in** x**:** $sU(x, s) - f(x) = kU_{xx}(x, s)$
- **Solve ODE, then apply inverse Laplace transform to obtain** $u(x, t)$

4. Numerical Methods

When analytical solution is not possible:

- **Finite Difference Method (FDM): approximate derivatives using discrete differences.**
- **Finite Element Method (FEM): approximate solution on a mesh and solve linear system.**
- **Finite Volume Method (FVM): conserve mass or energy in fluid mechanics.**