Chapter 2

Improper Integrals

Introduction

Improper integrals naturally appear in physics, for example in gravitational or electrostatic potentials, quantum mechanics, or Green's functions. An improper integral is an integral where either:

- **one of the bounds is infinite, or**
- **the integrand becomes infinite at some point of the interval.**

2.1 Improper Integrals of Functions Defined on an Unbounded Interval (First Kind)

Definition 2.1 ***Let f be a continuous function. An integral of the form***

$$\int_a^{+\infty} f(x)\,dx, \quad \int_{-\infty}^{b} f(x)\,dx, \quad \textbf{\textit{or}} \quad \int_{-\infty}^{+\infty} f(x)\,dx$$

is called an improper integral of the first kind, because the interval of integration is unbounded. These are defined through limits:

$$\int_a^{+\infty} f(x)\,dx := \lim_{t\to+\infty}\int_a^t f(x)\,dx,$$

$$\int_{-\infty}^{b} f(x)\,dx := \lim_{t\to-\infty}\int_t^b f(x)\,dx,$$

$$\int_{-\infty}^{+\infty} f(x)\,dx := \lim_{A\to-\infty,\,B\to+\infty}\int_A^B f(x)\,dx,$$

provided these limits exist and are finite. If the limit does not exist or is infinite, the integral is said to diverge.

2.1.1 Convergence criteria

1. p-integral test

Theorem 2.1 ***Let p be an arbitrary power, that can be any real number.***

$$\int_1^{+\infty} \frac{1}{x^p}\,dx \quad \textbf{\textit{converges if }} p > 1, \quad \textbf{\textit{diverges if }} p \le 1.$$

Examples 2.1 ***1. Convergent integral:***

$$\int_1^{+\infty} \frac{1}{x^2}\,dx = \lim_{t\to+\infty}\int_1^t \frac{1}{x^2}\,dx = \lim_{t\to+\infty}\left[\frac{1}{x}\right]_1^t = \lim_{t\to+\infty}\left(\frac{1}{t}-1\right) = 1.$$

We can easily see that this integral converge by p-integral test for $p = 2$.

2. Divergent integral:

$$\int_1^{+\infty} \frac{1}{x}\,dx = \lim_{t\to+\infty}\ln(t) = +\infty.$$

We can easily see that this integral diverge by p-integral test for $p = 1$.

3. Integral over the whole real line:

$$\int_{-\infty}^{+\infty} e^{-x^2}\,dx = \sqrt{\pi}.$$

We want to compute:

$$I = \int_{-\infty}^{+\infty} e^{-x^2}\,dx.$$

Step 1. Square the integral. We consider

$$I^2 = \left(\int_{-\infty}^{+\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{+\infty} e^{-y^2}\,dy\right).$$

This can be written as a double integral over the plane:

$$I^2 = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-(x^2+y^2)}\,dx\,dy.$$

Step 2. Switch to polar coordinates. Recall that in polar coordinates:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad dx\,dy = r\,dr\,d\theta.$$

Thus,

$$I^2 = \int_0^{2\pi}\int_0^{+\infty} e^{-r^2}\,r\,dr\,d\theta.$$

Step 3. Compute the radial integral. First compute the inner integral:

$$\int_0^{+\infty} e^{-r^2}\,r\,dr.$$

Use the substitution $u = r^2$, so $du = 2r\,dr$ and hence $r\,dr = \frac{1}{2}\,du$. Therefore,

$$\int_0^{+\infty} e^{-r^2}\,r\,dr = \frac{1}{2}\int_0^{+\infty} e^{-u}\,du = \frac{1}{2}.$$

In fact,

$$\int_0^{+\infty} e^{-r^2}\,r\,dr = \frac{1}{2}\int_0^{+\infty} e^{-u}\,du = \frac{1}{2}\lim_{t\to+\infty}\int_0^t e^{-u}\,du = \frac{1}{2}\lim_{t\to+\infty}\left[-e^{-u}\right]_0^t = \frac{1}{2}\lim_{t\to+\infty}\left(1-e^{-t}\right) = \frac{1}{2}\cdot 1$$

Step 4. Compute the angular integral. Now,

$$I^2 = \int_0^{2\pi} \left(\frac{1}{2}\right) d\theta = \pi.$$

Step 5. Conclude. Since $I > 0$, we have

$$I = \sqrt{\pi}.$$

Final Answer

$$\int_{-\infty}^{+\infty} e^{-x^2}\, dx = \sqrt{\pi}.$$

2. Comparison Test (Useful Criterion)

Let f and g be continuous functions on $[a, +\infty)$ such that:
$0 \le f(x) \le g(x)$ for all $x \ge a$.

- **If $\int_a^{+\infty} g(x)\, dx$ converges, then $\int_a^{+\infty} f(x)\, dx$ also converges.**
- **If $\int_a^{+\infty} f(x)\, dx$ diverges, then $\int_a^{+\infty} g(x)\, dx$ also diverges.**

Example 2.1 ***Consider the improper integral***

$$\int_1^{+\infty} \frac{1}{x^2+1}\, dx.$$

For all $x \ge 1$, we have

$$0 \le \frac{1}{x^2+1} \le \frac{1}{x^2}.$$

Now, we know that

$$\int_1^{+\infty} \frac{1}{x^2}\, dx$$

is a convergent improper integral (it is a p-integral with $p = 2 > 1$). Therefore, by the Comparison Test,

$$\int_1^{+\infty} \frac{1}{x^2+1}\, dx$$

also converges.

Example 2.2 ***Consider the improper integral***

$$\int_1^{+\infty} \frac{1}{\sqrt{x}}\, dx.$$

For all $x \ge 1$, we have

$$\frac{1}{\sqrt{x}} \ge \frac{1}{x}.$$

Now, we know that

$$\int_1^{+\infty} \frac{1}{x}\, dx$$

is a divergent improper integral (it is a p-integral with $p = 1$). Therefore, by the Comparison Test,

$$\int_1^{+\infty} \frac{1}{\sqrt{x}}\, dx$$

also diverges.

3. Limit Comparison Test

Let f and g be continuous and positive on $[a,\infty)$. Assume

$$L := \lim_{x\to\infty} \frac{f(x)}{g(x)} \in [0,+\infty].$$

1. **If $0 < L < \infty$, then $\int_a^\infty f(x)\,dx$ converges iff $\int_a^\infty g(x)\,dx$ converges.**

   Remarks 2.1

   $$\int_a^\infty f(x)\,dx \ \textit{\textbf{converges}} \quad \Longleftrightarrow \quad \int_a^\infty g(x)\,dx \ \textit{\textbf{converges}}.$$

   Implications

   - **If $\displaystyle\int_a^\infty g(x)\,dx$ converges, then $\displaystyle\int_a^\infty f(x)\,dx$ converges.**
   - **If $\displaystyle\int_a^\infty f(x)\,dx$ converges, then $\displaystyle\int_a^\infty g(x)\,dx$ converges.**

   Contrapositives (for divergence)

   - **If $\displaystyle\int_a^\infty f(x)\,dx$ diverges, then $\displaystyle\int_a^\infty g(x)\,dx$ diverges.**
   - **If $\displaystyle\int_a^\infty g(x)\,dx$ diverges, then $\displaystyle\int_a^\infty f(x)\,dx$ diverges.**

   (a) **When $0 < L < \infty$, the two integrals always share the same behavior: they both converge or they both diverge.**

   (b) **The explicit writing of implications and their contrapositives avoids ambiguity when dealing with divergence.**

2. **If $L = 0$ and $\int_a^\infty g(x)\,dx$ converges, then $\int_a^\infty f(x)\,dx$ converges.**

3. **If $L = +\infty$ and $\int_a^\infty g(x)\,dx$ diverges, then $\int_a^\infty f(x)\,dx$ diverges.**

Example 2.3 ***Study*** $\displaystyle\int_1^\infty \frac{1}{x^2+3x+1}\,dx.$

Take $f(x) = \dfrac{1}{x^2+3x+1}$ ***and*** $g(x) = \dfrac{1}{x^2}$. ***Then***

$$\frac{f(x)}{g(x)} = \frac{x^2}{x^2+3x+1} = \frac{1}{1+\frac{3}{x}+\frac{1}{x^2}} \xrightarrow[x\to\infty]{} 1.$$

Thus $L = 1 \in (0,\infty)$. ***Since*** $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ ***converges (p-integral with*** $p = 2 > 1$***), the Limit Comparison Test implies that***

$$\int_1^\infty \frac{1}{x^2+3x+1}\,dx$$

also converges.

Example 2.4 ***Study*** $\int_1^{\infty} \frac{1}{\sqrt{x^2+1}}\,dx$.

Take $f(x) = \frac{1}{\sqrt{x^2+1}}$ ***and*** $g(x) = \frac{1}{x}$***. Then***

$$\frac{f(x)}{g(x)} = \frac{x}{\sqrt{x^2+1}} = \frac{1}{\sqrt{1+\frac{1}{x^2}}} \xrightarrow[x\to\infty]{} 1.$$

Thus $L = 1$***. Since*** $\int_1^{\infty} \frac{1}{x}\,dx$ ***diverges (harmonic integral), the Limit Comparison Test implies that***

$$\int_1^{\infty} \frac{1}{\sqrt{x^2+1}}\,dx$$

also diverges.

4. Absolutely Convergent Integrals

Definition 2.2 ***Let*** $f : [a, +\infty[\to \mathbb{R}$ ***be a measurable function. The improper integral***

$$\int_a^{\infty} f(x)\,dx$$

is said to be absolutely convergent if

$$\int_a^{\infty} |f(x)|\,dx$$

is convergent.

Theorem 2.2 ***If the integral of*** $|f(x)|$ ***is convergent, then the integral of*** $f(x)$ ***is also convergent:***

$$\int_a^{\infty} |f(x)|\,dx \ \textbf{\textit{convergent}} \quad \Longrightarrow \quad \int_a^{\infty} f(x)\,dx \ \textbf{\textit{convergent}}.$$

Example 2.5 ***Consider the integral***

$$\int_1^{\infty} \frac{\cos(x)}{x^2}\,dx.$$

We observe that

$$\left|\frac{\cos(x)}{x^2}\right| \leq \frac{1}{x^2}.$$

Now, the integral

$$\int_1^{\infty} \frac{1}{x^2}\,dx$$

is convergent.
Therefore,

$$\int_1^{\infty} \left|\frac{\cos(x)}{x^2}\right| dx$$

is convergent, and consequently

$$\int_1^{\infty} \frac{\cos(x)}{x^2}\,dx$$

is absolutely convergent.

2.2 Integrals of Functions Defined on a Bounded Interval, Infinite at One Endpoint (Second Kind)

Definition 2.3 ***Improper Integrals of the Second Kind (Integration of an Unbounded Integrand)***
Let f be a function defined on an interval $(a, b]$, $[a, b)$, or (a, b), but not necessarily at one or both endpoints because it becomes infinite there. An integral of the form

$$\int_a^b f(x)\,dx$$

is called an improper integral of the second kind if f is unbounded at a, b, or both.
Formally:

- ***If f is unbounded at a:***

$$\int_a^b f(x)\,dx := \lim_{\varepsilon\to 0^+} \int_{a+\varepsilon}^b f(x)\,dx$$

- ***If f is unbounded at b:***

$$\int_a^b f(x)\,dx := \lim_{\varepsilon\to 0^+} \int_a^{b-\varepsilon} f(x)\,dx$$

- ***If f is unbounded at both a and b, split the interval at some $c \in (a, b)$:***

$$\int_a^b f(x)\,dx := \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

  where each integral is defined as above.

Remark 2.1

- ***f is not defined at the singular point(s) because it diverges.***
- ***The convergence depends on the behavior of f near the singularity. For example, if $f(x) \sim (x - a)^{-\alpha}$ near a, then:***
  - ***Convergent if $\alpha < 1$***
  - ***Divergent if $\alpha \geq 1$***
- ***Similarly for singularities near b.***

Method of Calculation

1. **Replace the singular endpoint by a variable $\varepsilon > 0$.**
2. **Compute the integral on the modified interval as a normal (proper) integral.**
3. **Take the limit as $\varepsilon \to 0^+$.**

Example 2.6 ***Convergent integral***

$$\int_0^1 \frac{dx}{\sqrt{x}}$$

Solution:

$$\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{\varepsilon\to 0^+}\int_\varepsilon^1 x^{-1/2}dx = \lim_{\varepsilon\to 0^+}\left[2\sqrt{x}\right]_\varepsilon^1 = \lim_{\varepsilon\to 0^+}(2-2\sqrt{\varepsilon}) = 2$$

Example 2.7 ***Divergent integral***

$$\int_0^1 \frac{dx}{x}$$

Solution:

$$\int_0^1 \frac{dx}{x} = \lim_{\varepsilon\to 0^+}\int_\varepsilon^1 \frac{dx}{x} = \lim_{\varepsilon\to 0^+}(-\ln\varepsilon) = +\infty$$

Example 2.8 ***Singular at both endpoints***

$$\int_0^1 \frac{dx}{\sqrt{x(1-x)}}$$

Solution: Split at $c = \frac{1}{2}$***:***

$$\int_0^{1/2} \frac{dx}{\sqrt{x(1-x)}} + \int_{1/2}^1 \frac{dx}{\sqrt{x(1-x)}}$$

Both integrals converge $\Rightarrow$ ***total integral convergent.***

2.3 Integrals of Functions with Discontinuities on the Interval of Integration (*Third kind of improper integrals*)

We consider integrals of the form

$$I = \int_a^b f(x)\,dx,$$

where the function f is ***discontinuous*** at one or more points inside the interval $[a, b]$. Such discontinuities prevent us from computing the integral in the usual Riemann sense. Therefore, these are treated as ***improper integrals.*** This case is often referred to as the ***third kind of improper integrals.*** If f has a discontinuity at some point $c \in (a, b)$, the integral is defined by splitting at c:

$$\int_a^b f(x)\,dx = \lim_{\substack{\alpha\to c^-\\ \beta\to c^+}}\left(\int_a^\alpha f(x)\,dx + \int_\beta^b f(x)\,dx\right).$$

- If this limit exists and is finite, the integral is said to converge.

- **If the limit does not exist or is infinite, the integral is said to diverge.**

Exercise 2.1 ***Study the improper integral***

$$\int_{-1}^{1} \frac{dx}{x}.$$

Solution 2.1 ***The integrand*** $f(x) = 1/x$ ***is not defined at*** $x = 0$***, which lies inside the integration interval*** $[-1, 1]$***. Thus we must treat the integral as an improper integral by splitting at the singular point*** 0***.***

1. Split into one-sided integrals.
By definition, the improper integral (ordinary definition) is

$$\int_{-1}^{1} \frac{dx}{x} = \lim_{\alpha \to 0^-} \int_{-1}^{\alpha} \frac{dx}{x} + \lim_{\beta \to 0^+} \int_{\beta}^{1} \frac{dx}{x},$$

provided both one-sided limits exist and are finite.

2. Compute the one-sided integrals.
For any $a < b$ ***not containing*** 0 ***we have***

$$\int_a^b \frac{dx}{x} = \ln|x|\Big|_a^b = \ln|b| - \ln|a|.$$

Apply this to the two pieces:

$$\int_{-1}^{\alpha} \frac{dx}{x} = \ln|\alpha| - \ln 1 = \ln|\alpha|, \qquad (\alpha < 0),$$

and

$$\int_{\beta}^{1} \frac{dx}{x} = \ln 1 - \ln\beta = -\ln\beta, \qquad (\beta > 0).$$

3. Take the limits.
As $\alpha \to 0^-$***,*** $|\alpha| \to 0^+$ ***so*** $\ln|\alpha| \to -\infty$***. As*** $\beta \to 0^+$***,*** $-\ln\beta \to +\infty$***.***
Therefore the left one-sided limit equals $-\infty$ ***and the right one-sided limit equals*** $+\infty$***. Because the two one-sided limits are not finite, the ordinary improper integral does not converge.***

> ***The improper integral*** $\int_{-1}^{1} \frac{dx}{x}$ ***diverges.***

4. Cauchy principal value (optional but important).
Although the ordinary improper integral diverges, one can consider the Cauchy principal value ***(symmetrically approaching the singularity):***

$$\text{p. v.} \int_{-1}^{1} \frac{dx}{x} := \lim_{\varepsilon \to 0^+} \left(\int_{-1}^{-\varepsilon} \frac{dx}{x} + \int_{\varepsilon}^{1} \frac{dx}{x} \right).$$

Compute the combined integral for fixed ε***:***

$$\int_{-1}^{-\varepsilon} \frac{dx}{x} + \int_{\varepsilon}^{1} \frac{dx}{x} = \Big(\ln|-\varepsilon| - \ln 1\Big) + \Big(\ln 1 - \ln\varepsilon\Big) = \ln\varepsilon - \ln\varepsilon = 0.$$

Since the value is 0 ***for every*** $\varepsilon > 0$***, the limit exists and***

$$\text{p. v.} \int_{-1}^{1} \frac{dx}{x} = 0.$$

Remark. ***The principal value is a different notion from the ordinary improper integral:***

- ***The ordinary improper integral requires both one-sided limits to be finite, here they are infinite, so the integral diverges.***
- ***The principal value uses a symmetric limiting process; it can exist even when the ordinary improper integral does not. The principal value is often useful in applications (Fourier transforms, distributions), but it does not restore ordinary convergence.***

5. A symmetry dependence.
If the two limits approaching 0 ***are taken at different rates (non-symmetric), the combined limit*** $\lim_{\alpha \to 0^-, \beta \to 0^+} \left(\int_{-1}^{\alpha} 1/x \, dx + \int_{\beta}^{1} 1/x \, dx \right)$ ***need not exist (and typically will diverge). The principal value chooses the symmetric path*** $\alpha = -\varepsilon, \ \beta = \varepsilon$***, which yields the finite value*** 0.

Conclusion. The improper integral $\int_{-1}^{1} \frac{dx}{x}$ ***diverges, since the one-sided limits at*** 0 ***are infinite. However, its Cauchy principal value exists and equals*** 0.

Example 2.9 ***Decide convergence (improper integrals with an interior singularity) and compute the value when it converges.***

1. $J_1 = \displaystyle\int_{-1}^{1} \frac{dx}{|x|^{1/2}}.$
2. $J_2 = \displaystyle\int_{-1}^{1} \frac{dx}{x^2}.$

Solution 2.2 ***(A)*** $J_1 = \displaystyle\int_{-1}^{1} \frac{dx}{|x|^{1/2}}.$
Step 1 - split at the singularity.

$$J_1 = \int_{-1}^{0} \frac{dx}{|x|^{1/2}} + \int_{0}^{1} \frac{dx}{|x|^{1/2}}.$$

Since $|x|^{-1/2} = (-x)^{-1/2}$ ***on*** $[-1, 0)$ ***and*** $= x^{-1/2}$ ***on*** $(0, 1]$***, both one-sided integrals are identical.***
Step 2 - test near 0. ***Compare with the model*** x^{-p} ***with*** $p = \frac{1}{2}$***. We know*** $\int_0^{\varepsilon} x^{-1/2} \, dx = 2\sqrt{\varepsilon} < \infty$***. Hence each one-sided integral is finite.***
Step 3 - compute. ***Use symmetry:***

$$J_1 = 2 \int_{0}^{1} x^{-1/2} \, dx = 2 \Big[2\sqrt{x} \Big]_0^1 = 2 \cdot 2 = 4.$$

J_1 ***converges and*** $J_1 = 4.$

(B) $J_2 = \displaystyle\int_{-1}^{1} \frac{dx}{x^2}.$

Step 1 - split at the singularity.

$$J_2 = \lim_{\alpha \to 0^-} \int_{-1}^{\alpha} \frac{dx}{x^2} + \lim_{\beta \to 0^+} \int_{\beta}^{1} \frac{dx}{x^2},$$

provided both one-sided limits are finite.

Step 2 - compute one-sided integrals. ***A primitive of*** x^{-2} ***is*** $-x^{-1}$***. For*** $0 < \beta < 1$***,***

$$\int_{\beta}^{1} \frac{dx}{x^2} = \left[-\frac{1}{x}\right]_{\beta}^{1} = -1 + \frac{1}{\beta} = \frac{1-\beta}{\beta}.$$

As $\beta \to 0^+$***,*** $\frac{1}{\beta} \to +\infty$***, so the right-hand one-sided integral diverges to*** $+\infty$***.***
Similarly the left-hand piece diverges to $+\infty$***.***
Conclusion. ***For the integral***

$$J_2 = \int_{-1}^{1} \frac{dx}{x^2},$$

we split at the singular point $x = 0$***.***
- On the interval $(0, 1]$***:***

$$\int_{\beta}^{1} \frac{dx}{x^2} = \frac{1-\beta}{\beta} \qquad \xrightarrow[\beta \to 0^+]{} +\infty.$$

Hence, the right-hand side diverges to $+\infty$***.***
- On the interval $[-1, 0)$***:***

$$\int_{-1}^{\alpha} \frac{dx}{x^2} = \frac{1+\alpha}{|\alpha|} \qquad \xrightarrow[\alpha \to 0^-]{} +\infty.$$

Thus, the left-hand side also diverges to $+\infty$***.***
Since both one-sided integrals are infinite, the improper integral cannot converge in the usual sense.
Moreover, the **Cauchy principal value** ***of an improper integral with a singularity inside the interval is defined by taking symmetric limits around the singular point. In our case:***

$$\textit{\textbf{p.v.}} \int_{-1}^{1} \frac{dx}{x^2} = \lim_{\varepsilon \to 0^+} \left(\int_{-1}^{-\varepsilon} \frac{dx}{x^2} + \int_{\varepsilon}^{1} \frac{dx}{x^2} \right).$$

Step 1 - compute the left-hand integral.

$$\int_{-1}^{-\varepsilon} \frac{dx}{x^2} = \left[-\frac{1}{x}\right]_{-1}^{-\varepsilon} = \left(-\frac{1}{-\varepsilon}\right) - \left(-\frac{1}{-1}\right) = \frac{1}{\varepsilon} - 1.$$

As $\varepsilon \to 0^+$***, we have*** $\frac{1}{\varepsilon} \to +\infty$***. Thus, the left-hand part diverges to*** $+\infty$***.***

Step 2 - compute the right-hand integral.

$$\int_{\varepsilon}^{1} \frac{dx}{x^2} = \left[-\frac{1}{x}\right]_{\varepsilon}^{1} = (-1) - \left(-\frac{1}{\varepsilon}\right) = \frac{1}{\varepsilon} - 1.$$

Again, as $\varepsilon \to 0^+$, $\frac{1}{\varepsilon} \to +\infty$. So the right-hand part also diverges to $+\infty$.
Step 3 - combine the two parts.

$$\int_{-1}^{-\varepsilon} \frac{dx}{x^2} + \int_{\varepsilon}^{1} \frac{dx}{x^2} = \left(\frac{1}{\varepsilon} - 1\right) + \left(\frac{1}{\varepsilon} - 1\right) = \frac{2}{\varepsilon} - 2.$$

Taking the limit as $\varepsilon \to 0^+$:

$$\lim_{\varepsilon \to 0^+} \left(\frac{2}{\varepsilon} - 2\right) = +\infty.$$

In this case, both one-sided integrals diverge to $+\infty$, so there is no possibility of cancellation (unlike the case of $\int_{-1}^{1} \frac{dx}{x}$). Therefore, the Cauchy principal value does not exist either.

$$\boxed{J_2 \textit{ diverges and has no finite principal value.}}$$

2.4 Solved Exercises

Improper Integrals

Exercise 2.2 ***Compute***

$$I = \int_0^{\infty} e^{-x}\, dx.$$

Solution 2.3 ***We write the improper integral as a limit:***

$$I = \lim_{R \to \infty} \int_0^{R} e^{-x}\, dx = \lim_{R \to \infty} \Big[-e^{-x}\Big]_0^{R} = \lim_{R \to \infty} \left(1 - e^{-R}\right) = 1.$$

Therefore $I = 1$.

Exercise 2.3 ***Compute***

$$I = \int_1^{\infty} \frac{\ln x}{x^2}\, dx.$$

Solution 2.4 ***Use integration by parts on the finite integral and then pass to the limit. For $R > 1$,***

$$\int_1^{R} \frac{\ln x}{x^2}\, dx$$

let $u = \ln x$ and $dv = x^{-2}dx$. Then $du = \frac{1}{x}dx$ and $v = -\frac{1}{x}$. Thus

$$\int_1^{R} \frac{\ln x}{x^2}\, dx = \left[-\frac{\ln x}{x}\right]_1^{R} + \int_1^{R} \frac{1}{x} \cdot \frac{1}{x}\, dx = -\frac{\ln R}{R} + 0 + \int_1^{R} \frac{1}{x^2}\, dx.$$

Compute the remaining integral:

$$\int_1^R \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^R = 1 - \frac{1}{R}.$$

So

$$\int_1^R \frac{\ln x}{x^2}\,dx = -\frac{\ln R}{R} + 1 - \frac{1}{R}.$$

Let $R \to \infty$***. Since*** $\frac{\ln R}{R} \to 0$ ***and*** $\frac{1}{R} \to 0$***, we get***

$$\int_1^\infty \frac{\ln x}{x^2}\,dx = 1.$$

Exercise 2.4 ***Compute***

$$I = \int_0^1 \frac{1}{\sqrt{x}}\,dx.$$

Solution 2.5 ***This is an improper integral at*** 0***. Write it as a limit:***

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{\varepsilon\to 0^+} \int_\varepsilon^1 x^{-1/2}\,dx = \lim_{\varepsilon\to 0^+} \left[2x^{1/2}\right]_\varepsilon^1 = \lim_{\varepsilon\to 0^+} \left(2 - 2\sqrt{\varepsilon}\right) = 2.$$

Thus the integral converges and equals 2***.***

Exercise 2.5 ***Study the convergence of the improper integral:***

$$J = \int_0^1 \frac{dx}{x^p}, \qquad p \in \mathbb{R}.$$

Solution 2.6 ***The integrand*** $\dfrac{1}{x^p}$ ***is unbounded at*** $x = 0$***. We write***

$$J = \lim_{\varepsilon\to 0^+} \int_\varepsilon^1 \frac{dx}{x^p}.$$

Case 1: $p \neq 1$***. We compute the primitive:***

$$\int \frac{dx}{x^p} = \frac{x^{1-p}}{1-p}, \quad (p \neq 1).$$

Thus,

$$J = \lim_{\varepsilon\to 0^+} \left[\frac{x^{1-p}}{1-p}\right]_\varepsilon^1 = \frac{1}{1-p}\left(1 - \varepsilon^{1-p}\right).$$

- If $p < 1$***, then*** $1 - p > 0$ ***and*** $\varepsilon^{1-p} \to 0$***, so*** $J = \dfrac{1}{1-p}$ ***(finite).***
- If $p > 1$***, then*** $1 - p < 0$ ***and*** $\varepsilon^{1-p} \to +\infty$***, so*** $J = +\infty$ ***(divergent).***
Case 2: $p = 1$***.***

$$\int_\varepsilon^1 \frac{dx}{x} = \ln(1) - \ln(\varepsilon) = -\ln(\varepsilon).$$

As $\varepsilon \to 0^+$***,*** $-\ln(\varepsilon) \to +\infty$***, hence the integral diverges.***

The integral converges if and only if $p < 1$***, with*** $J = \dfrac{1}{1-p}$***.***

Exercise 2.6 ***Determine whether the following integral converges:***

$$K = \int_0^1 \ln(x)\, dx.$$

Solution 2.7 ***The function*** $\ln(x)$ ***is unbounded at*** $x = 0$***. We write***

$$K = \lim_{\varepsilon \to 0^+} \int_\varepsilon^1 \ln(x)\, dx.$$

Integration by parts: set

$$u = \ln(x), \quad dv = dx, \quad du = \frac{dx}{x}, \quad v = x.$$

Then,

$$\int \ln(x)\, dx = x\ln(x) - \int x \cdot \frac{dx}{x} = x\ln(x) - x.$$

Thus,

$$K = \lim_{\varepsilon \to 0^+} \Big[x\ln(x) - x \Big]_\varepsilon^1.$$

At $x = 1$***, we get*** $1 \cdot \ln(1) - 1 = -1$***. At*** $x = \varepsilon$***, we get*** $\varepsilon\ln(\varepsilon) - \varepsilon$***.***
As $\varepsilon \to 0^+$***: -*** $\varepsilon\ln(\varepsilon) \to 0$***, -*** $\varepsilon \to 0$***.***
So the lower limit tends to 0***. Hence,***

$$K = -1 - 0 = -1.$$

The integral converges and its value is -1.

Chapter 3

Differential Equations

3.1 Ordinary Differential Equations

3.1.1 Generalities

Definition 3.1 ***A differential equation is an equation in which the unknown is a function and where some derivatives of the unknown function appear.***

Example 3.1 ***Let*** u ***be a function, the following equations are differential equations.***

1. $u^{'} = 2u$
2. $u^{''} - 3u^{'} + 1 = 0$
3. $u^{(3)} = u$

Definition 3.2 ***Let*** $u = u(x)$ ***be an unknown function of the variable*** x***. An equation of the form***

$$F\Big(x, u, u', u'', \ldots, u^{(n)}\Big) = 0, \tag{3.1}$$

with $n \in \mathbb{N}$, ***is called a differential equation of order*** n. ***Here*** $u', u", \ldots, u^{(n)}$ ***denote the derivatives of*** u ***of orders*** $1, 2, \ldots, n$***, respectively.***

Remarks 3.1

1. ***The equation*** (3.1) ***involves*** $(n+2)$ ***variables.***
2. ***The unknown function may also be denoted by*** $y, t, \ldots$
3. ***In a differential equation, when we write*** $u, u', u'', \ldots, u^{(n)}$***, it is understood that we mean*** $u(x), u'(x), u''(x), \ldots, u^{(n)}(x)$.

Definition 3.3 ***A solution of equation*** (3.1) ***on the interval*** I ***is a function that is*** n ***times differentiable on*** I ***and satisfies*** (3.1)***.***

Example 3.2 ***It can be easily verified that the function*** $u(x) = ce^{4x},\ c \in \mathbb{R}$, ***is a solution of the differential equation*** $u' = 4u$. ***Indeed, it is clear that if*** $u(x) = ce^{4x}$ ***then*** $u'(x) = 4ce^{4x} = 4u(x)$.

Definition 3.4 ***A differential equation of the type*** $u'f(u) = g(x)$ ***is called a*** **separable variables equation*.***

Example 3.3 ***The equation*** $u' = \dfrac{e^{-u}}{x^2}$ ***can be rewritten as*** $u'e^u = \dfrac{1}{x^2}$. ***We can easily find the solutions of this equation. By integrating both sides, we obtain***

$$e^u = -\frac{1}{x} + k, \qquad k \in \mathbb{R}.$$

This yields

$$u(x) = \ln\left| -\frac{1}{x} + k\right|, \qquad k \in \mathbb{R}.$$

Definition 3.5 ***The*** **order** ***of a differential equation is the order of the highest derivative appearing in the equation.***

Example 3.4

1. $2xu' + u = 0$ ***is a first-order differential equation.***
2. $u'' + u' - 3u = \ln(x)$ ***is a second-order differential equation.***

Definition 3.6

1. ***A differential equation of order*** n ***is said to be*** **linear** ***if it has the form***

$$f_0(x)u + f_1(x)u' + f_2(x)u'' + \cdots + f_n(x)u^{(n)} = f(x), \qquad (3.2)$$

where $f_0, f_1, f_2, \ldots, f_n, f$ ***are real continuous functions on an interval*** $I \subset \mathbb{R}$.

2. ***If*** $f(x) = 0$ ***for all*** $x \in I$, ***then equation*** (3.2) ***is called*** **homogeneous*, and it has the form***

$$f_0(x)u + f_1(x)u' + f_2(x)u'' + \cdots + f_n(x)u^{(n)} = 0.$$

3. ***Equation*** (3.2) ***is said to have*** **constant coefficients** ***if the functions*** $f_0, f_1, f_2, \ldots, f_n$ ***are constants on*** I. ***In other words, equation*** (3.2) ***can be written as***

$$a_0u + a_1u' + a_2u'' + \cdots + a_nu^{(n)} = f(x),$$

where $a_0, a_1, a_2, \ldots, a_n$ ***are real constants.***

Remark 3.1 ***In a linear differential equation, none of the terms*** $u, u', u'', \ldots, u^{(n)}$ ***are raised to a power.***

Example 3.5

1. $e^xu + x^{\frac{1}{2}}u'' = x^2 + 1$ ***is a linear differential equation, and*** $e^xu + x^{\frac{1}{2}}u'' = 0$ ***is the associated homogeneous equation.***
2. $2u' - 3u'' + \frac{1}{5}u^{(3)} = x$ ***is a linear differential equation with constant coefficients.***
3. ***The equation*** $(u')^2 + u'' + 3u = 0$ ***is*** **not** ***a linear differential equation.***

Proposition 3.1 ***If*** u_1, u_2 ***are two solutions of the linear homogeneous equation***

$$f_0(x)u + f_1(x)u' + f_2(x)u'' + \cdots + f_n(x)u^{(n)} = 0, \tag{3.3}$$

then $\alpha u_1 + \beta u_2$ ***is also a solution of*** (3.3)***, for any constants*** $\alpha, \beta \in \mathbb{R}$***.***

Consider the linear differential equation (3.2) and its associated homogeneous equation (3.3). The following proposition allows us to find the general solution of equation (3.2).

Proposition 3.2 ***If*** u_0 ***is a particular solution of*** (3.2) ***and*** u_1 ***is a solution of the homogeneous equation*** (3.3)***, then***

$$u = u_1 + u_0$$

is a general solution of (3.2)***.***

Remark 3.2 ***Recall that a particular solution of equation*** (3.2) ***is a function that is*** n ***times differentiable and satisfies*** (3.2)***.***

3.2 First-Order Differential Equation

3.2.1 First-Order Linear Differential Equation Without a Nonhomogeneous Term

Consider the equation

$$a_1(x)u' + a_0(x)u = 0, \quad \textbf{with} \quad a_1(x) \neq 0, \tag{3.4}$$

which is equivalent to the separable equation

$$u' + a(x)u = 0, \quad \textbf{where} \quad a(x) = \frac{a_0(x)}{a_1(x)}. \tag{3.5}$$

To solve equation (3.5), we follow the steps:

$$\begin{aligned}
u' + a(x)u = 0 &\iff u' = -a(x)u,\\
&\iff \frac{du}{dx} = -a(x)u,\\
&\iff \frac{du}{u} = -a(x)\,dx,\\
&\iff \int \frac{du}{u} = -\int a(x)\,dx,\\
&\iff \ln|u| = -A(x) + k,\\
&\iff |u| = e^{-A(x)+k},\\
&\iff u = Ke^{-A(x)},
\end{aligned}$$

where $A(x)$ is an antiderivative of $a(x)$ and $K = \pm e^k,\ k \in \mathbb{R}$.

Example 3.6 ***The solution of the equation***

$$u' - \sqrt{x}u = 0, \quad x > 0,$$

is given by

$$u(x) = Ke^{-A(x)},$$

with $K = \pm e^k$ ***and***

$$-A(x) = \int \sqrt{x}\, dx = \frac{2}{3}\sqrt{x^3}.$$

3.2.2 First-Order Linear Differential Equation with a Nonhomogeneous Term

Consider the equation

$$a_1(x)u' + a_0(x)u = f_1(x), \quad \text{with} \quad a_1(x) \neq 0, \tag{3.6}$$

which is equivalent to

$$u' + a(x)u = f(x), \quad \text{where} \quad a(x) = \frac{a_0(x)}{a_1(x)} \quad \text{and} \quad f(x) = \frac{f_1(x)}{a_1(x)}. \tag{3.7}$$

According to Proposition 3.2, the solution of equation (3.7) is of the form

$$u(x) = u_0(x) + u_1(x),$$

where $u_1(x) = Ke^{-A(x)}$ is the solution of the homogeneous equation associated with (3.7), and $u_0(x)$ is a particular solution of (3.7).

Example 3.7 ***Consider the equation***

$$u' - \sqrt{x}u = 1 - x\sqrt{x}, \quad x > 0. \tag{3.8}$$

From the previous example, $u_1(x) = Ke^{\frac{2}{3}\sqrt{x^3}}$ ***is a solution of the homogeneous equation associated with*** (3.8)***. On the other hand, it can be easily verified that*** $u_0(x) = x$ ***is a particular solution of*** (3.8)***.***
Therefore, the general solution of equation (3.8) ***is***

$$u(x) = x + Ke^{\frac{2}{3}\sqrt{x^3}} = x + Ke^{\frac{2}{3}x\sqrt{x}}, \quad K \in \mathbb{R}.$$

The question now is: how can we find a particular solution?

3.2.3 Finding a Particular Solution: The Method of Variation of the Constant

We know that the solution of the homogeneous equation associated with (3.7) is of the form

$$u_1(x) = Ke^{-A(x)}, \quad K \in \mathbb{R}.$$

The *method of variation of the constant* consists in looking for a particular solution of (3.7) **of the form**

$$u_0(x) = K(x)e^{-A(x)},$$

where $K(x)$ **is a function of the variable** x **instead of a constant.**
Saying that $u_0(x) = K(x)e^{-A(x)}$ **is a solution of** (3.7) **means that**

$$u_0'(x) + a(x)u_0(x) = f(x), \quad \textbf{with} \quad A'(x) = a(x). \tag{3.9}$$

$$\begin{aligned} u_0^{'}(x) + a(x)u_0(x) = f(x) &\iff \left(K(x)e^{-A(x)}\right)^{'} + a(x)K(x)e^{-A(x)} = f(x) \\ &\iff K^{'}(x)e^{-A(x)} - K(x)A^{'}(x)e^{-A(x)} + a(x)K(x)e^{-A(x)} = f(x) \\ &\iff K^{'}(x)e^{-A(x)} - K(x)a(x)e^{-A(x)} + a(x)K(x)e^{-A(x)} = f(x) \\ &\iff K^{'}(x)e^{-A(x)} = f(x) \\ &\iff K^{'}(x) = f(x)e^{A(x)} \\ &\iff K(x) = \int f(x)e^{A(x)}\,dx. \end{aligned}$$

Thus, a particular solution of (3.7) **can be written in the form**

$$u_0(x) = \left(\int f(x)e^{A(x)}\,dx\right)e^{-A(x)}.$$

Exercise 3.1 ***Find the solutions of the equation***

$$u' - 2u = e^{3x+1}. \tag{3.10}$$

Proof. Finding the solution of the homogeneous equation:
The solution of the homogeneous equation

$$u' - 2u = 0$$

associated with equation (3.10) **is given by**

$$u_1(x) = Ke^{2x}, \quad K \in \mathbb{R}.$$

$$\begin{aligned} u^{'} = 2u &\iff \frac{du}{dx} = 2u \\ &\iff \frac{du}{u} = 2dx \\ &\iff \ln|u| = 2x + k, \quad k \in \mathbb{R} \\ &\iff u = Ke^{2x}, \ K = \pm e^{k}. \end{aligned}$$

Then

$$u_1(x) = Ke^{2x}$$

Finding the particular solution:

We look for a function $K(x)$ such that a particular solution of equation (3.10) is of the form

$$u_0(x) = K(x)e^{2x}.$$

$u_0(x) = K(x)e^{2x}.$

$$\begin{aligned}
u_0^{'}(x) - 2u_0(x) = e^{3x+1} &\iff \left(K(x)e^{2x}\right)^{'} - 2K(x)e^{2x} = e^{3x+1} \\
&\iff K^{'}(x)e^{2x} + 2K(x)e^{2x} - 2K(x)e^{2x} = e^{3x+1} \\
&\iff K^{'}(x)e^{2x} = e^{3x+1} \\
&\iff K^{'}(x) = e^{3x+1}e^{-2x} \\
&\iff K(x) = \int e^{x+1}\, dx \\
&\Longrightarrow K(x) = e\int e^{x}\, dx \\
&\Longrightarrow K(x) = e^{x+1}.
\end{aligned}$$

The solution of equation (3.10) is of the form

$$u(x) = e^{x+1}e^{2x} + Ke^{2x} = \left(e^{x+1} + K\right)e^{2x}, \quad K \in \mathbb{R}.$$

■

3.2.4 First-Order Linear Differential Equation with Constant Coefficients

Consider equations of the form

$$a_1u' + a_0u = f_1(x). \tag{3.11}$$

This is a special case of equations (3.6). Equation (3.11) is solved in the same way as (3.6).

3.2.5 Bernoulli Differential Equation

Definition 3.7 ***Any equation of the form***

$$u' + a(x)u + b(x)u^n = 0 \tag{3.12}$$

is called a Bernoulli equation.

Solving the Bernoulli Equation

1. **If $n = 0$, equation (3.12) becomes of the form (3.7).**
2. **If $n = 1$, equation (3.12) becomes of the form (3.5).**

3. **If $n \neq 0$ and $n \neq 1$, we try to transform equation (3.12) into a first-order linear differential equation. To do this, we follow the following method: we divide by u^n and equation (3.12) becomes**

$$u^{-n}u^{'} + a(x)u^{1-n} + b(x) = 0 \tag{3.13}$$

We set $y = u^{1-n}$, so that

$$\frac{1}{1-n}y' = u^{-n}u'.$$

Therefore, equation (3.13) becomes

$$\frac{1}{1-n}y' + a(x)y + b(x) = 0. \tag{3.14}$$

Equation (3.14) is of the form (3.6).

Example 3.8 ***Solve the following equation:***

$$u' + e^x u + e^x u^3 = 0. \tag{3.15}$$

Proof. **Dividing by u^3 we obtain**

$$u^{-3}u' + e^x u^{-2} = -e^x. \tag{3.16}$$

By making the change of variable $y = u^{-2}$ and differentiating both sides, we obtain

$$y' = -2u'u^{-3}.$$

In other words,

$$u'u^{-3} = -\frac{1}{2}y.$$

This change of variable allows us to write equation (3.16) in the form

$$-\frac{1}{2}y' + e^x y = -e^x, \tag{3.17}$$

which is a first-order linear equation with a nonhomogeneous term whose solution follows the previous steps. ■

3.2.6 Homogeneous Differential Equation

Let H be a numerical function defined and continuous on a domain $D \subset \mathbb{R}$.

Definition 3.8 ***A differential equation is called*** **homogeneous** ***if it is of the form***

$$F(x, u, u') = 0$$

and remains unchanged when x is replaced by αx and u by αu, while leaving u' unchanged. These equations are of the form

$$u' = H\left(\frac{u}{x}\right). \tag{3.18}$$

The solution of equation (3.18) generally reduces to solving a simple equation using the change of variable $t = \frac{u}{x}$, with $u = tx$ and $u' = t'x + t$. The solutions are in the form (x, u).

Example 3.9 ***Solve the equation***

$$2xuu' = u^2 - x^2.$$

Proof. When x is replaced by αx and u by αu, leaving u' unchanged, we obtain

$$2\alpha^2 xuu' = \alpha^2(u^2 - x^2),$$

which is exactly

$$2xuu' = u^2 - x^2.$$

Thus, the equation is homogeneous.
We use the change of variable $t = \frac{u}{x}$, with $u = tx$ and $u' = t'x + t$. The given equation becomes

$$\begin{aligned}
2xuu^{'} = u^2 - x^2 \iff\ & 2uu^{'} = \frac{u^2}{x} - x \\
\iff\ & 2tx\left(t^{'}x + t\right) = \left(\frac{u}{x}\right)u - x \\
\iff\ & 2x^2tt^{'} + 2t^2x = t^2x - x \\
\iff\ & 2x^2tt^{'} = -\left(t^2 + 1\right) \\
\iff\ & \frac{2t}{t^2+1}\,dt = -\frac{1}{x}\,dx \\
\iff\ & \ln\left(t^2+1\right) = -\ln|x| + k, \quad k \in \mathbb{R} \\
\iff\ & \ln\left(t^2+1\right)|x| = k \\
\iff\ & x = \frac{K}{t^2+1},\ u = \frac{Kt}{t^2+1},\ K = \pm e^k.
\end{aligned}$$

■

3.3 Second-Order Differential Equations

We consider equations of the form

$$F(x, u, u', u'') = 0.$$

To solve these equations, we distinguish several cases.