Tutorial Session 3 (Solution)

Exercise 1:

We know that: $\forall \mathbf{x} \in \mathbf{X}:\ \delta(\mathbf{G}) \leq \mathbf{d_G}(\mathbf{x}) \leq \Delta(\mathbf{G})$.

By summing these inequalities over all vertices **x**, we obtain: $\sum_{\mathbf{x}\in\mathbf{X}} \delta(\mathbf{G}) \leq \sum_{\mathbf{x}\in\mathbf{X}} \mathbf{d_G}(\mathbf{x}) \leq \sum_{\mathbf{x}\in\mathbf{X}} \Delta(\mathbf{G})$.

Since $\delta(\mathbf{G})$ and $\Delta(\mathbf{G})$ are independent of **x**, and $\sum_{\mathbf{x}\in\mathbf{X}} \mathbf{d_G}(\mathbf{x}) = 2|\mathbf{U}|$, we get:

$\delta(\mathbf{G}) \times |\mathbf{X}| \leq 2|\mathbf{U}| \leq \Delta(\mathbf{G}) \times |\mathbf{X}|$.

Dividing by $|\mathbf{X}|$, we finally obtain: $\delta(\mathbf{G}) \leq \frac{2|\mathbf{U}|}{|\mathbf{X}|} \leq \Delta(\mathbf{G})$.

Exercise 2:

We have $\sum_{\mathbf{x}\in\mathbf{X}} \mathbf{d_G}(\mathbf{x}) = 2|\mathbf{U}|$. Moreover, **G** is **r**-regular, that is:

$\forall \mathbf{x} \in \mathbf{X} : \mathbf{d_G}(\mathbf{x}) = \mathbf{r}$.

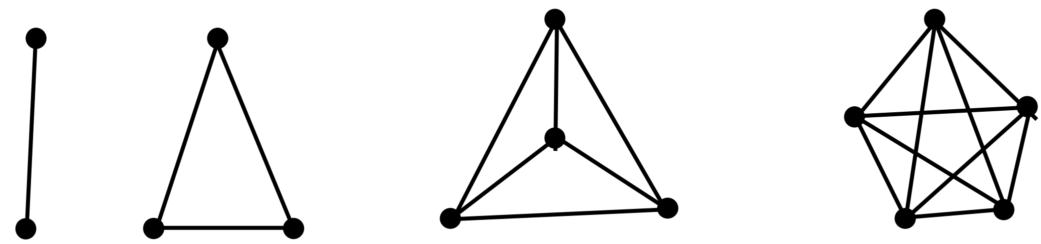
By substitution, we obtain: $\sum_{\mathbf{x}\in\mathbf{X}} \mathbf{r} = 2|\mathbf{U}|$.

Hence, $\mathbf{r}|\mathbf{X}| = 2|\mathbf{U}|$.

Therefore, $|\mathbf{U}| = \frac{\mathbf{r}|\mathbf{X}|}{2}$.

Exercise 3:

The simple and complete graphs are shown below. Since the graph $\mathbf{K_n}$ contains all possible edges, it has as many edges as there are ways to choose 2 elements from **n**, that is: $\mathbf{C_n^2} = \frac{\mathbf{n}(\mathbf{n}-1)}{2}$.



Exercise 4:

A little reflection shows that if a graph has all its vertices of degree 2, then, starting from any vertex, one returns to it in a uniquely determined way after traversing a polygon. One might think that this completely determines the graph as soon as its order is known, but this is not the case. Indeed, it is not specified that the graph is connected; therefore, a graph of order 6 may consist of a hexagon or of two triangles.

However, in a simple graph where every vertex has degree 2, each connected component contains at least three vertices (the given vertex and the two distinct vertices adjacent to it). Hence, for orders 3, 4, and 5, there can be only one connected component, and the graph is uniquely determined. Starting from order 6, there are as many possible graphs as there are ways to write the order as a sum of integers greater than or equal to 3.

Exercise 5:

Consider a graph of order **2p** in which every vertex has a degree greater than or equal to **p**. Suppose the graph is not connected. Then there exist two distinct vertices **a** and **b** such that no chain connects them. In particular, they are not adjacent. There are at least **p** vertices adjacent to **a**, and **p** vertices adjacent to **b**. These vertices must all be distinct; otherwise, there would exist a chain of length 2 joining **a** and **b**. They are also distinct from **a** and **b** themselves. But then there would be **p + p + 2 = 2p + 2** distinct vertices, which is impossible.

Exercise 6:

In addition to being simple and bipartite, suppose that the graph is also complete. If we can prove that the inequality $|\mathbf{U}| \leq \frac{|\mathbf{X}|^2}{4}$ holds for a complete graph, then it will also hold for all other graphs. This is because a complete graph has the maximum possible number of edges. In bipartite graph the vertex set is divided into two disjoint classes X_1 and X_2 with $|\mathbf{X}_1| = \mathbf{p}$ and $|\mathbf{X}_2| = \mathbf{q}$. Because it is simple and complete, then $|\mathbf{U}| = \mathbf{pq}$. Consequently, the inequality

$|\mathbf{U}| \le \frac{|\mathbf{X}|^2}{4}$ is equivalent to $\mathbf{pq} \le \frac{(\mathbf{p}+\mathbf{q})^2}{4}$ which can be rewritten as $(\mathbf{p}-\mathbf{q})^2 \ge 0$.

Since $(\mathbf{p}-\mathbf{q})^2 \ge 0$ is true, the inequality $|\mathbf{U}| \le \frac{|\mathbf{X}|^2}{4}$ also holds.

From Exercise 1, we have already shown that $\delta(\mathbf{G}) \le \frac{2|\mathbf{U}|}{|\mathbf{X}|}$. If we choose **x** as a vertex with minimum degree in the graph, then $\mathbf{d}_{\mathbf{G}}(\mathbf{x}) = \delta(\mathbf{G}) \le \frac{2|\mathbf{U}|}{|\mathbf{X}|}$. Using the result from the first question of this exercise, we obtain $\mathbf{d}_{\mathbf{G}}(\mathbf{x}) = \delta(\mathbf{G}) \le \frac{2|\mathbf{U}|}{|\mathbf{X}|} \le \frac{2\frac{|\mathbf{X}|^2}{4}}{|\mathbf{X}|} = \frac{|\mathbf{X}|}{2}$.

Exercise 7:

In a simple graph with **n** vertices, the maximum number of edges is when the graph is complete, i.e., a clique. In this case number of edges is $\frac{\mathbf{n}(\mathbf{n}-1)}{2}$. We want this to be at least 1000**.** So we must have $\frac{\mathbf{n}(\mathbf{n}-1)}{2} \ge 1000$. We can test integer values:

- For **n = 45**, we have only 990 edges (< 1000).
- For **n = 46**, we have **1035 edges** (≥ 1000).

We conclude that the smallest number of vertices required is **n = 46**.