

Tutorial Session 2 (Solution)

Exercise 1:

1. Each arc of the graph gives the in-degree once and the out-degree once.

Therefore, $\sum_{x \in X} \mathbf{d}_G^-(\mathbf{x}) = \sum_{x \in X} \mathbf{d}_G^+(\mathbf{x})$.

2. When summing the degrees of the vertices, an arc is counted twice, once for each endpoint. Therefore, $\sum_{x \in X} \mathbf{d}_G(\mathbf{x}) = 2|\mathbf{U}|$.

3. This follows immediately from (2).

4. We have: $\sum_{x \in X} \mathbf{d}_G(\mathbf{x}) = \sum_{\substack{x \in X \\ \mathbf{d}_G(\mathbf{x}) \bmod 2 = 0}} \mathbf{d}_G(\mathbf{x}) + \sum_{\substack{x \in X \\ \mathbf{d}_G(\mathbf{x}) \bmod 2 = 1}} \mathbf{d}_G(\mathbf{x})$.

Since the first sum is even, the second sum must also be even.

Since each term of the second sum is odd, this sum must contain an even number of terms.

Exercise 2:

The essential idea is to represent the problem with a graph $G = (X, U)$: each club is represented by a vertex, and an edge connects two clubs that play a match against each other. After testing all possible scenarios, we conclude that it is not possible for all teams to play the same number of matches. But what happens if the size of the data is large? With 15 clubs, for example, it is quite difficult to start working on this problem. A good idea is to use the mathematical properties of graphs that we have studied:

If the problem has a solution, then we have: $\forall x \in X : \mathbf{d}_G(\mathbf{x}) = 3$, since each team must play 3 matches. If we count the total number of participations, we get: $\sum_{x \in X} \mathbf{d}_G(\mathbf{x}) = 3 \times 7 = 21$, which is absurd (the sum of the degrees must be an even number). Organizing such a tournament is not possible. More generally, we see that, for the same reason, if there is an odd number of teams, it is not possible for them all to play an odd number of matches.

Exercise 3:

If we think of representing each segment by a vertex, and connecting two vertices if the corresponding segments intersect, we see, exactly as in the previous exercise, that the proposed exercise is impossible.

Exercise 4:

The graph G is simple, so each vertex in the graph is adjacent to at most 3 other vertices. In other words, $\forall \mathbf{x} \in \mathbf{X} : \mathbf{d}_G(\mathbf{x}) \leq 3$. The sum of the degrees of the graph satisfies the inequality $\sum_{\mathbf{x} \in \mathbf{X}} \mathbf{d}_G(\mathbf{x}) = 3 \times 4 = 12 = 2 \times 6$. Consequently,

$\|\mathbf{G}\| \leq 6$, this is absurd. Therefore, it is not possible to construct a simple graph with 4 vertices and 7 edges.

Exercise 5:

The situation is impossible, at least if we assume that friendship is reciprocal. Indeed, by drawing the friendship graph, we see that the assumption implies there are 11 vertices of odd degree, which cannot happen.