Chapter I: Basic definitions

As noted in the Introduction, **graph theory** is a branch of mathematics that deals with problems using specific types of diagrams called graphs. Graph theory has many applications across a wide range of fields, including operations research, physics, chemistry, computer science, and other scientific disciplines. In this chapter, we introduce some fundamental concepts of graph theory and provide a variety of examples. We also present several elementary results.

1. "Intuitive" definition of a graph:

Intuitively speaking, a graph is a diagram consisting of a finite number of points, called *vertices*, and a finite number of arrows, called *arcs*, that connect certain pairs of these points.

Example:

In the graph below:

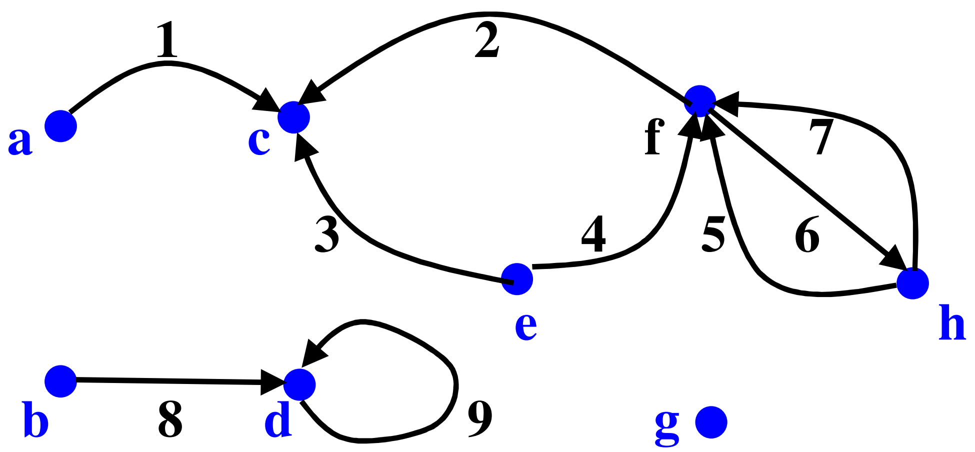


Figure 1: An example of a graph.

- **X = { a, b, c, d, e, f, g, h}** is the set of vertices, and
- **U={1, 2, 3, 4, 5, 6, 7, 8, 9}** is the set of arcs.

2. Mathematical definition of a graph:

Formally, a graph **G** is defined to be a pair **(X , U)**, where **X** is a finite *set* of nodes called *vertices,* and **U** is a set of pairs of vertices **(x , y)** of elements in **X** called *arcs*.

Remark:

The position of the vertices and the shape of the arcs in a graph drawing are not important; what matters is how the vertices are joined by arcs.

The two graphs shown in **Figures 1** and **2** have the same structure. We say that they are *isomorphic*, as defined below:

Two graphs $G_1=(X_1 , U_1)$ and $G_2=(X_2 , U_2)$ are said to be *isomorphic* if there exists a bijection **f** from X_1 to X_2 such that, for every pair **(x , y)** of elements in X_1, **(x , y)** is an arc in U_1 if and only, if **(f(x) , f(y))** is an arc in U_2.

For example: **(a , c)** is an arc of G_1, and **(f(a) , f(c))=(b , c)** is an arc of G_2. (b , d) is an arc of G_1, and **(f(a) , f(c))=(g , h)** is an arc of G_2, and so on.

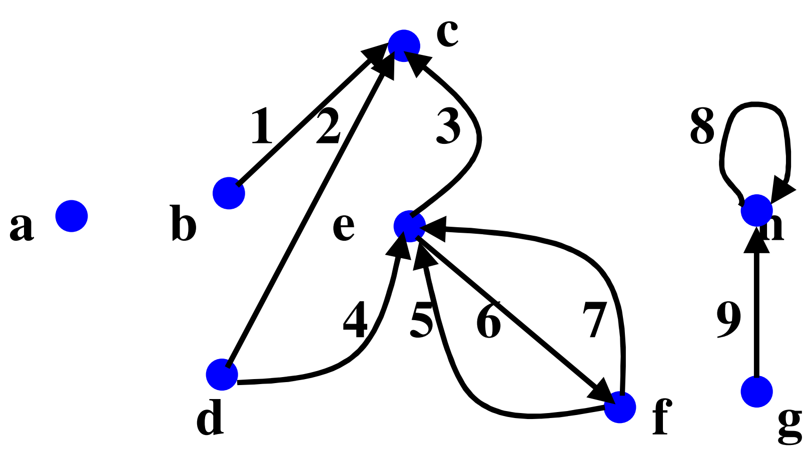


Figure 2: Graph G_2 isomorphic to the graph G_1 in the previous figure.

Notation:

In **Figure 2, arc 6** goes from vertex **e** to vertex **f**. It is said to be in the form **(e , f)**, and by convention, we write **6 = (e , f)**. Note that while the form

(e , f) is enough to uniquely identify arc **6**, the form **(f , e)** is not enough to uniquely identify arc **5**, because **7 = (f , e)** as well.

3. Order, orientation and multiplicity:

3.1. Order and Size of a Graph:

The *order* of a graph **G**, denoted by **|G|**, is the number of its vertices, and its *size*, denoted by **||G||**, is the number of its arcs.

Example:

The graph in the **Figure 1**, "an example of a graph", has an order of **8** and size of **9**, that is, **|G|** = 8 and **||G|| = 9.**

3.2. Orientation:

In the graph **G = (X , U)** shown in **Figure 1** or **2**, each arc **u = (x , y)** represents a directed line (an arrow) from vertex **x** to vertex **y**. A graph of this type is called a *directed graph.* If the connection between **x** and **y** is represented by a line without any direction it is called an *edge*. The graph is then called an *undirected graph.*

In what follows, the term *graph* will refer to a *directed graph*, while an *undirected graph* will be referred to as a multigraph.

Example:

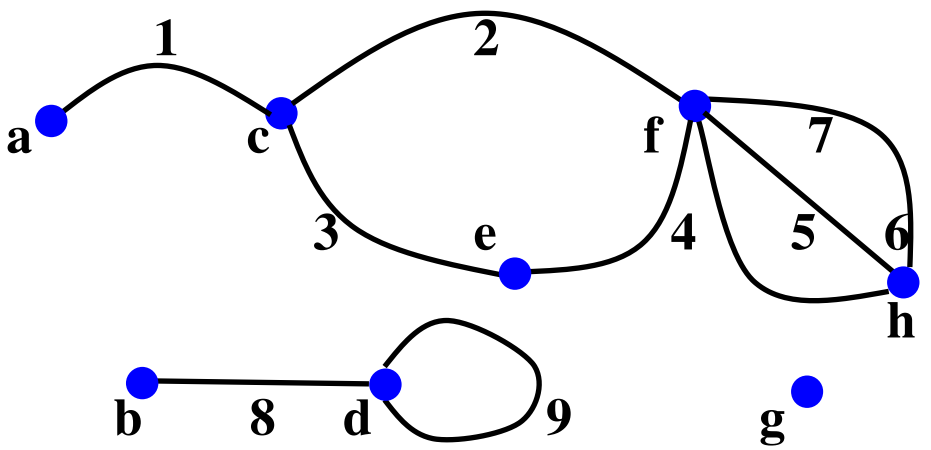


Figure 3: Multigraph associated with the graph in Figure 1.

Notation:

In a multigraph **G = (X , U)**, the notation **{x , y}**, where **x**$\in$**X** and **y**$\in$**X**, is typically used to represent an *edge*.

For example, in the multigraph shown in **Figure 3**, the edge **1** is denoted by **{a , c}**, that is, **1 = {a , c}**.

Remark:

As noted by C. Berge [Berge, 1973], it may appear convenient to distinguish between two separate theories: one for *directed* graphs and another for *undirected* graphs. However, this distinction is not strictly necessary. In reality, All graphs are directed, but sometimes the direction need not be specified. To apply a directed concept in a undirected graph we will actually consider the associated directed graph obtained by replacing each edge with two arcs in opposite directions. Conversely, an undirected concept can be applied to a directed graph by ignoring the orientation of the arcs.

3.3. Multiplicity:

Loop and Endpoints:

In a graph, an arc of the form **(x , x)** is called a *loop.* For an arc **(x , y)**, **x** and **y** are called the *endpoints* of the arc. Specifically, vertex **x** is its *initial endpoint,* and vertex **y** is its *terminal endpoint.*

Example:

In the graph below: arc **9 = (d , d)** is a loop. Vertex **b** is the initial endpoint of arc **8** and vertex **d** is its terminal endpoint.

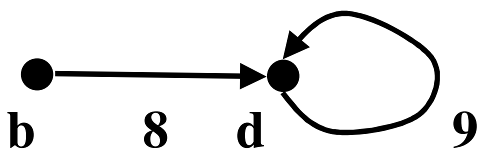


Figure 4: An example of a loop and endpoints.

Multiplicity of a pair x, y:

The multiplicity of a pair **x** , **y** is the number of arcs, having **x** as the initial endpoint and **y** as the final endpoint.

Notation:

The multiplicity of a pair **x** , **y** is denoted as $\mathbf{m_G^+(x,y)}$.

We define $\mathbf{m_G^-(x,y) = m_G^+(y,x)}$ and $\mathbf{m_G(x,y) = m_G^+(x,y) + m_G^-(x,y)}$.

If $\mathbf{x \neq y}$, then $\mathbf{m_G(x,y)}$ denotes the number of arcs with one endpoint at **x** and the other at **y**.

If $\mathbf{x = y}$, then $\mathbf{m_G(x,y)}$ is equal to twice the number of loops attached to vertex **x**.

Example:

In the graph shown in **Figure 1**: $\mathbf{m_G^+(h,f) = 2}$, $\mathbf{m_G^-(c,f) = 1}$, $\mathbf{m_G(d,d) = 2}$.

4. Relations between the elements of a graph:

4.1. Relations between vertices:

Neighboring vertices:

In a graph, vertex **x** is a *successor* of vertex **y** if there is an arc in the form **(y , x)**.

Vertex **x** is a *predecessor* of vertex **y** if there is an arc in the form **(x , y)**.
Vertex **x** is a *neighbor* of vertex **y** if **x** is either a successor or a predecessor of **y**.

The term *adjacent vertices* is also used to refer to neighboring vertices.

Notations:

The set of successors of vertex **x** is denoted as $\Gamma_G^+(\mathbf{x})$.

The set of predecessors of vertex x is denoted as $\Gamma_G^-(\mathbf{x})$.

The set of neighbors of vertex x is denoted as $\Gamma_G(\mathbf{x}) = \Gamma_G^+(\mathbf{x}) \cup \Gamma_G^-(\mathbf{x})$.

Example:

In the graph shown in **Figure 1**:

$\Gamma_G(f) = \Gamma_G^+(f) \cup \Gamma_G^-(f) = \{c, h\} \cup \{e, h\} = \{c, e, h\}$.

Isolated vertex:

It is possible that $\Gamma_G(x) = \varnothing$ (the empty set). If $\Gamma_G(x) = \varnothing$, **x** is called an *isolated vertex.*

Example:

In the graph in the **Figure 1**: **g** is an *isolated vertex* since $\Gamma_G(g) = \varnothing$.

Pendant vertex:

A *pendant vertex* is a vertex that has only one neighbor.

Example:

In the graph shown in **Figure 1**: **b** is a pendent vertex since $\Gamma_G(b) = \{d\}$.

Degree of a vertex:

In a multigraph, the *degree* of a vertex is the number of edges that have the vertex as an endpoint. If the vertex is connected to a loop, the loop is counted twice. A vertex has degree **0** if it is an isolate vertex, and degree **1** if it is a pendant vertex.

Notation:

The degree of a vertex **x** in the graph **G** is denoted by $d_G(x)$.

Example:

In the graph shown in **Figure 3**, an example of a multigraph, the degrees of the vertices are as follows:

x	a	b	c	d	e	f	g	h
$d_G(x)$	1	1	3	3	2	5	0	3

Vertex **g** is an isolated vertex, while vertices **a** and **b** are both pendant vertices.

In a graph, two types of degrees are associated with each vertex. Let **x** be a vertex in a graph **G**: the number of arcs of the form **(x , y)**, going from **x**, is denoted by $\mathbf{d}_{\mathbf{G}}^{+}(\mathbf{x})$ and is called the *outer demi-degree* of **x**. Similarly, the number of arcs of the form **(y , x)**, going to **x** is denoted by $\mathbf{d}_{\mathbf{G}}^{-}(\mathbf{x})$ and is called the *inner demi-degree* of **x**.

The *total degree* of vertex **x** is given by $\mathbf{d}_{\mathbf{G}}(\mathbf{x})=\mathbf{d}_{\mathbf{G}}^{+}(\mathbf{x})+\mathbf{d}_{\mathbf{G}}^{-}(\mathbf{x})$ where each loop is counted twice.

Example:

In the graph shown in **Figure** 1, the degrees of the vertices are as follows:

x	a	b	c	d	e	f	g	h
$\mathbf{d}_{\mathbf{G}}^{+}(\mathbf{x})$	1	1	0	1	2	2	0	2
$\mathbf{d}_{\mathbf{G}}^{-}(\mathbf{x})$	0	0	3	2	0	3	0	1
$\mathbf{d}_{\mathbf{G}}(\mathbf{x})$	1	1	3	3	2	5	0	3

Similarly, vertex **g** is an isolated vertex, while vertices **a** and **b** are both pendant vertices.

4.2. Relations between arcs and vertices:

Arc incident to a vertex:

In a graph, if a vertex **x** is the initial endpoint of an arc **u**, then **u** is said to be *incident out of* vertex **x**. Conversely, if **x** is the terminal endpoint of an arc **u**, then **u** is said to be *incident into* vertex **x**. In both cases, we say that the arc **u** is *incident to* vertex **x**. We also use the term *incoming arc* for incident inward and *outgoing arc* for incident outward. A *pendant arc* is incident to a *pendant vertex.*

Arc incident to a set of vertices:

In a graph **G = (X , U)**, let $\mathbf{A}\subset\mathbf{X}$. If the initial endpoint of an arc **u** belongs to **A** but its terminal endpoint does not, then **u** is said to be an arc *incident out of the set* **A**. Conversely, if the final endpoint of an arc **u** belongs to **A** but its initial endpoint does not, then **u** is said to be an arc *incident into the set* **A**.

Notation:

If **u** is incident out of **A**, we write: $\mathbf{u} \in \omega_{\mathbf{G}}^{+}(\mathbf{A})$.

If **u** is incident into **A**, we write: $\mathbf{u} \in \omega_{\mathbf{G}}^{-}(\mathbf{A})$.

The set of arcs incident to the set **A** is denoted as $\omega_{\mathbf{G}}(\mathbf{A}) = \omega_{\mathbf{G}}^{+}(\mathbf{A}) \cup \omega_{\mathbf{G}}^{-}(\mathbf{A})$.

Example:

In the graph shown in **Figure 1**, if **A={c,e,f}** then $\omega_{\mathbf{G}}(\mathbf{A}) = \omega_{\mathbf{G}}^{+}(\mathbf{A}) \cup \omega_{\mathbf{G}}^{-}(\mathbf{A}) = \{6\} \cup \{1,5,7\} = \{1,5,6,7\}$.

Adjacent arcs, adjacent edges:

Two arcs (or two edges) are said to be *adjacent* if they have at least one common endpoint.

Example:

In the graph of **Figure 4**, arc **8** and arc **9** are adjacent because **d** is a common vertex.

4.3. Graph qualifiers:

p-Graph:

If, in a graph **G**, the number of arcs from a vertex **x** to another vertex **y** does not exceed a number **p**, the graph is called a **p**-graph, and **p** is referred to as the *multiplicity* of the graph.

Example:

The graph in **Figure 1**, is a **2**-graph.

Simple Graph:

A multigraph is *simple* if it contains no loops and no more than one edge between any pair of distinct vertices.

Example:

The following graph is simple.

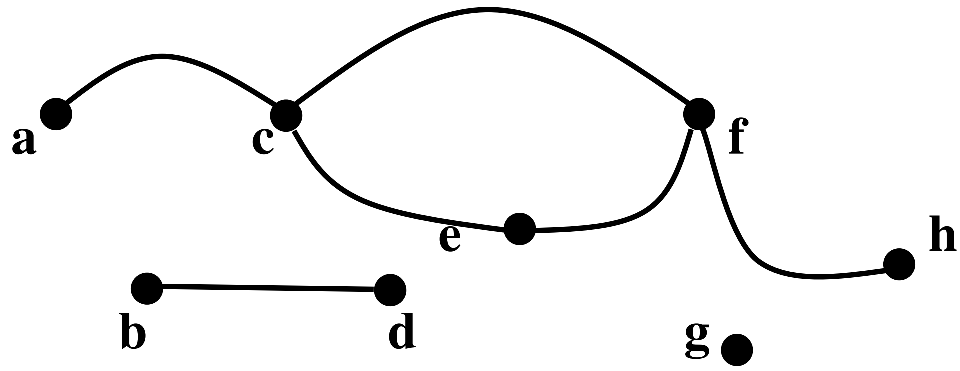


Figure 5: An example of a simple graph.

Regular graph:

If in a graph or in a multigraph each vertex has the same degree, this graph is said to be *regular***.**

Example:

The graph in **Figure 1**: is not regular.

Symmetric or antisymmetric graph:

A graph is *symmetric* if for every pair of vertices **x** and **y**, there are as many arcs of the form **(x , y)** as there are arcs of the form **(y , x)**.

A directed graph is *antisymmetric* if for every arc of the form **(x , y)**, there is no other arc of the form **(x , y)** or of the form **(y , x)**.

Example:

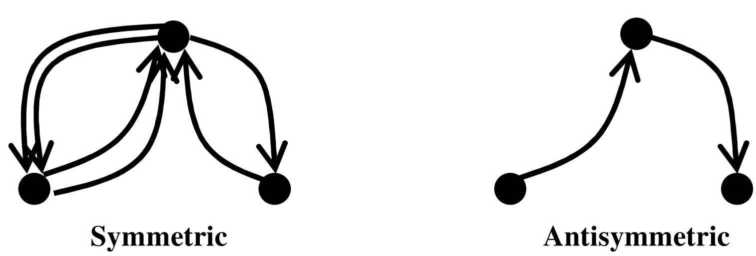


Figure 6: Symmetric and antisymmetric graphs.

<u>Complete Graph, Clique</u>:

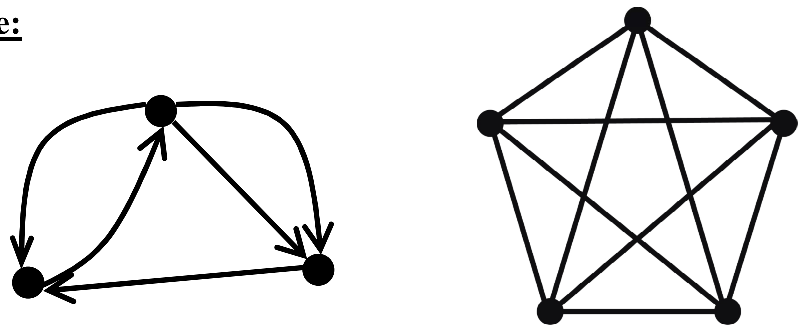
A graph is said to be *complete* if, for every pair of distinct vertices **x** and **y**, there is at least one arc of the form **(x , y)** or **(y , x)**. Similarly, a multigraph is said to be *complete* if every pair of distinct vertices is connected by an edge.

A *simple complete* graph of order **n** is called an **n**-clique.

<u>Notation:</u>

An **n**-clique is denoted as $\mathbf{K_n}$.

<u>Example:</u>



Complete graph **Simple and Complete multigraph**

Figure 7: Complete graph and the **5**-clique $\mathbf{K}_5$.

<u>Bipartite graph, Complete bipartite graph:</u>

A graph or a multigraph is *bipartite* if its vertex set can be partitioned into two classes $\mathbf{X}_1$ and $\mathbf{X}_2$ such that two vertices in the same class are never neighbors.

<u>Notation:</u>

A bipartite graph can be denoted as $\mathbf{G} = (\mathbf{X}_1, \mathbf{X}_2, \mathbf{U})$. A simple complete bipartite graph with classes $|\mathbf{X}_1| = \mathbf{p}$ and $|\mathbf{X}_2| = \mathbf{q}$ is denoted as $\mathbf{K_{p,q}}$.

Example:

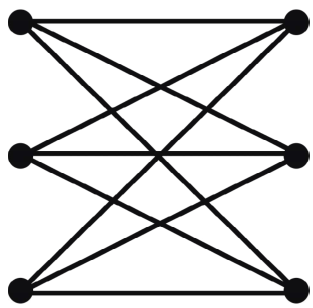


Figure 8: Simple complete bipartite graph $\mathbf{K}_{3,3}$.

Stable set:

In a graph or a multigraph, a set **S** of vertices is called a *stable set* if no arc or edge joins two distinct vertices in **S**.

Example:

In the graph below, the subset {a , d , e} is stable.

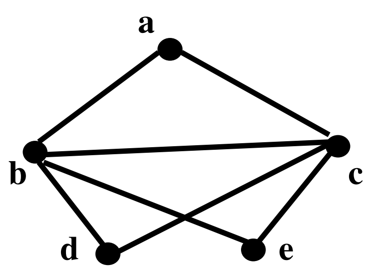


Figure 9: A set stable.

Complement graph:

Given a simple graph or multigraph **G = (X, U)** of order **n**, its *complement* graph **G'=(X , U')** is of the same order **n** and is defined as follows:
An arc (edge) in **U** does not belong to **U'**, and vice versa.
Note that the union $\mathbf{G} \cup \mathbf{G}'$ forms a clique $\mathbf{K}_{\mathbf{n}}$.

Example:

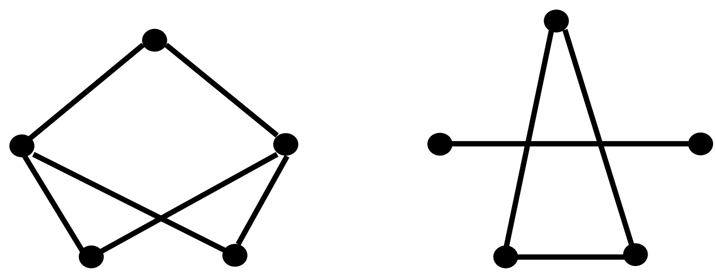


Figure 10: Graph and its complement.

Subgraph of G generated by A⊂X:

Let **G = (X , U**) be a graph or a multigraph, and let **A⊂X**.

The *subgraph* of **G** generated by **A** is the graph G_A, whose vertices are the elements of **A** and whose arcs (edges) are the arcs (edges) of **G** that have both endpoints in **A**.

Partial graph of G generated by V⊂ U:

Let **G = (X , U**) be a graph or a multigraph, and let **V⊂U**.

The *partial graph* generated by **V** is the graph **G' = (X , V)** with the same set **X** of vertices as **G**, and whose arcs (edges) are the arcs (edges) of **V**. Removing from **G** the arcs (edges) in **U\V**.

Partial subgraph of G:

Let **G = (X , U**) be a graph or a multigraph.

A *partial subgraph* of **G** is a subgraph of a partial graph of **G** or a partial graph of a subgraph of **G**.

Example:

The graph in **Figure** 4, where **A={b , d},** is a subgraph of the graph in the **Figure 1**.

The graph in **Figure 11**, where **V={8}**, is a partial graph of the graph in **Figure** 4.

Therefore, the graph in **Figure 11** given below is a partial subgraph of the graph in **Figure 1**.

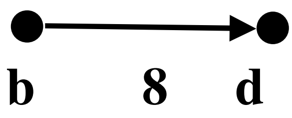


Figure 11: A partial graph of the graph in Figure 4 and a partial subgraph of the graph in Figure 1.

5. Matrices Associated with a Graph:

5.1. Vertex-arc incidence matrix:

Let **G** be a loopless graph of order **n** and size **m**.

The *vertex-arc incidence* matrix of **G** is a matrix $\mathbf{A} = (\mathbf{a_{ij}})$ of size $\mathbf{n} \times \mathbf{m}$, where each row **i** corresponds to the vertex $\mathbf{x_i}$ of **G**, and each column **j** corresponds to the arc $\mathbf{u_j}$ of **G**. The matrix **A** is defined as follows:

- $\mathbf{a_{ij}} = +1$ if vertex $\mathbf{x_i}$ is the initial endpoint of arc $\mathbf{u_j}$,
- $\mathbf{a_{ij}} = -1$ if vertex $\mathbf{x_i}$ is the terminal endpoint of arc $\mathbf{u_j}$,
- $\mathbf{a_{ij}} = 0$ in all other cases.

Note that the number of entries equal to +1 (respectively, -1) in row **i** gives the outer demi-degree $\mathbf{d_G^+}(\mathbf{x_i})$ (respectively, the inner demi-degree $\mathbf{d_G^-}(\mathbf{x_i})$) of the corresponding vertex $\mathbf{x_i}$.

Example:

The vertex-arc incidence matrix of the graph in **Figure 1**, with the arc **9** (the loop) removed is the following:

A	1	2	3	4	5	6	7	8
a	+1	0	0	0	0	0	0	0
b	0	0	0	0	0	0	0	+1
c	-1	-1	-1	0	0	0	0	0
d	0	0	0	0	0	0	0	-1
e	0	0	+1	+1	0	0	0	0
f	0	+1	0	-1	-1	+1	-1	0
g	0	0	0	0	0	0	0	0
h	0	0	0	0	+1	-1	+1	0

We can observe, for example, that: $\mathbf{d}_G^+(\mathbf{f}) = 2$ and $\mathbf{d}_G^-(\mathbf{f}) = 3$.

5.2. <u>Adjacency matrix or vertex-vertex incidence matrix:</u>

The *adjacency matrix* (also called the *vertex-vertex incidence matrix*) of a graph **G** of order **n** is a square matrix $\mathbf{A} = (\mathbf{a}_{ij})$ of size $\mathbf{n} \times \mathbf{n}$, where each row and each column corresponds to a vertex of **G**. The element $\mathbf{a}_{ij}$ represents the number of arcs from vertex $\mathbf{x}_i$ to vertex $\mathbf{x}_j$.

Unlike the vertex-arcs incidence matrix, loops can be represented using this matrix.

In this matrix, the sum of the entries in row **i** gives the outer demi-degree $\mathbf{d}_G^+(\mathbf{x}_i)$ of the corresponding vertex $\mathbf{x}_i$, while the sum of the entries in column **j** gives the inner demi-degree $\mathbf{d}_G^-(\mathbf{x}_j)$ of the corresponding vertex $\mathbf{x}_j$.

<u>Example:</u>

The adjacency matrix associated with the graph shown in **Figure** 1 is the following:

A	a	b	c	d	e	f	g	h
a	0	0	1	0	0	0	0	0
b	0	0	0	1	0	0	0	0
c	0	0	0	0	0	0	0	0
d	0	0	0	1	0	0	0	0
e	0	0	1	0	0	1	0	0
f	0	0	1	0	0	0	0	1
g	0	0	0	0	0	0	0	0
h	0	0	0	0	0	2	0	0

We can observe, for example, that: $\mathbf{d}_{\mathbf{G}}^{+}(\mathbf{d})=1$ and $\mathbf{d}_{\mathbf{G}}^{-}(\mathbf{d})=2$.

5.3. Condensed form of sparse matrices:

It is immediately noticeable that both incidence matrices are *sparse matrices*; many terms are zero. Therefore, it is possible to write each of the matrices in *compressed form*, that is, to "locate" the non-zero terms by their position in the matrix or by using the IFV matrix (Initial Vertex Final Vertex) if the arcs are numbered.

Example:

The compressed form of the vertex-vertex incidence matrix of the graph in **Figure 1** is as follows:

x	y	$\mathbf{m}_{\mathbf{G}}^{+}(\mathbf{x}, \mathbf{y})$
a	**c**	**1**
b	**d**	**1**
d	**d**	**1**
e	**c**	**1**
e	**f**	**1**
f	**c**	**1**
f	**h**	**1**
h	**f**	**2**

Since the arcs in **Figure 1** are numbered from 1 to 9, we can write the IFV matrix:

Arc	IV	TV
1	a	c
2	f	c
3	e	c
4	e	f
5	h	f
6	f	h
7	h	f
8	b	d
9	d	d

6. Vocabulary related to connected graph:

6.1. Chain, path, length:

A *chain* is a sequence $\mu = (\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_\mathbf{q})$ of adjacent arcs, i.e., each arc in the sequence has one endpoint in common with its predecessor and the other endpoint in common with its successor. The number **q > 0** of arcs in the sequence is the *length* of the chain.

In 1-graph, a chain is completely determined by the sequence of vertices it passes through. If the chain passes through vertices $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{\mathbf{q}+1}$, we may write it as: $\mu = (\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_{\mathbf{q}+1})$. Here, vertex $\mathbf{x}_1$ is called the *initial endpoint* and vertex $\mathbf{x}_{\mathbf{q}+1}$ is the *terminal endpoint* of chain μ.

A chain that does not visit any vertex more than once is called *elementary*.

A chain that does not use the same edge twice is called *simple.*

A *path* of *length* $\mathbf{q} > \mathbf{0}$ is a chain $\mu = (\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_\mathbf{q})$ in which the terminal vertex of arc $\mathbf{u}_\mathbf{i}$ is the initial vertex of arc $\mathbf{u}_{\mathbf{i}+1}$ for all $\mathbf{i} < \mathbf{q}$.

Example:

In the graph below:

The sequence **(1,4,5,6)** is a simple and elementary chain of length **3**.

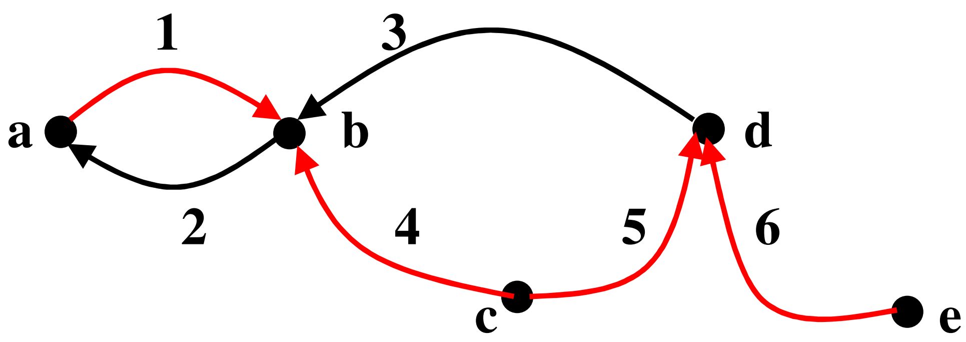


Figure 12: An example of a chain.

In the following graph: the sequence **(5,3,2,1)** is a simple path of length **3**, but it is not elementary, since it may revisit the vertex **b**.

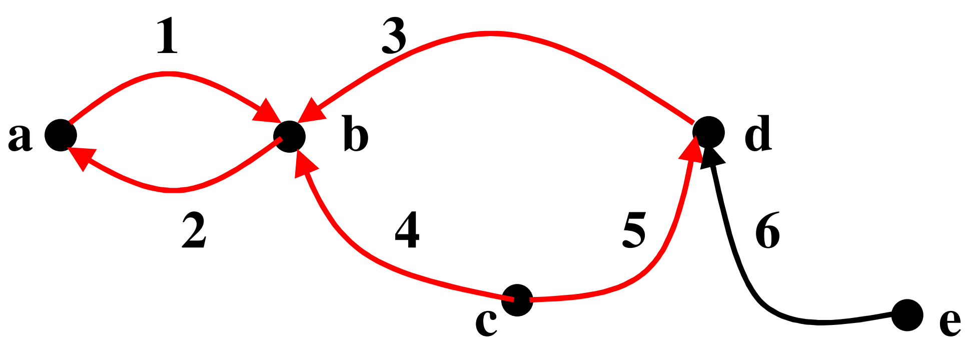


Figure 13: An example of a path.

6.2. <u>Connected graph:</u>

A graph is said to be *connected* if, for every pair of distinct vertices **x** and **y**, there exists a chain that links **x** to **y**.

A *connected component* of a graph is a maximal subset of vertices such that every pair of distinct vertices in the subset is connected by a chain, and no vertex outside the subset is connected by a chain to any vertex within it.

A graph is said to be *strongly connected* if, for every pair of distinct vertices **x** and **y**, there exists a path from **x** to **y** and another path from **y** to **x**.

A *strongly connected component* of a graph is a maximal subset of vertices in which every pair of distinct vertices **x** and **y** of the subset satisfies the condition that there is a path from **x** to **y** and a path from **y** to **x**. Moreover, for any vertex **z** outside this subset, there is no path from **z** to any vertex **x** in the subset, or no path from **x** to **z**.

<u>Example:</u>

The graph below is not connected and has three connected components: {**a**, **c**, **e**, **f**, **h**}, {**b**, **d**} and {**g**}.

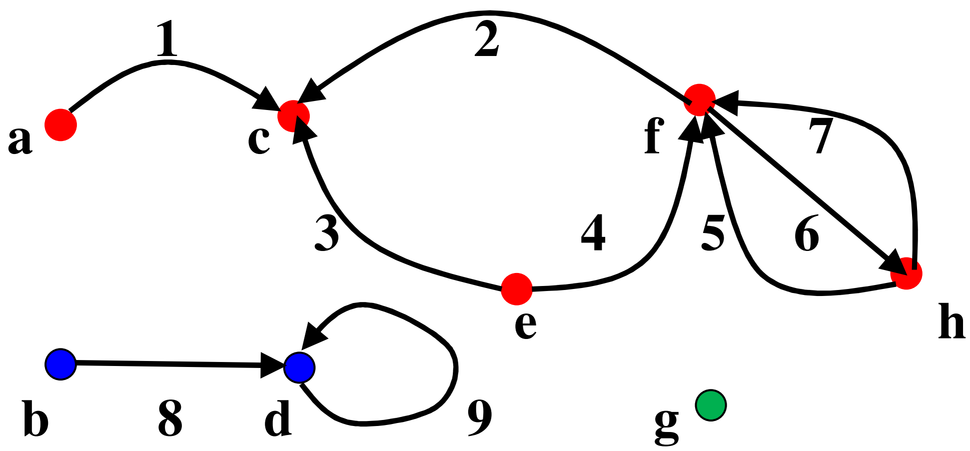


Figure 14: Connected components.

The graph below is strongly connected.

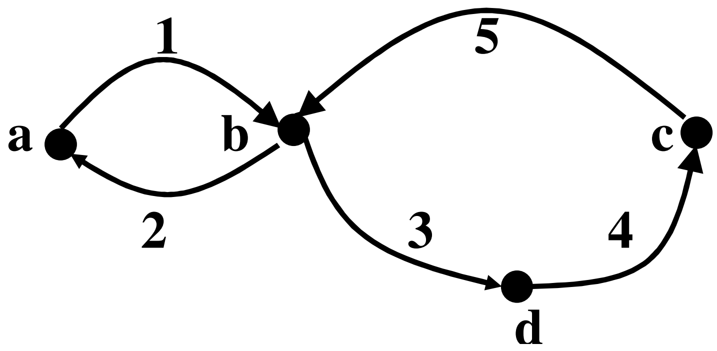


Figure 15: Strongly connected graph.

6.3. <u>Cycle and circuit:</u>

A *cycle* is a simple chain whose two endpoints coincide. A loop is a cycle of length 1.

A *circuit* is a cycle $\mu = (\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_q)$ such that for all **i < q** the terminal vertex of arc $\mathbf{u}_i$ is the initial vertex of arc $\mathbf{u}_{i+1}$.

<u>Notation</u>:

We denote by μ^+ the set of arcs of the cycle oriented in the traversal direction and by μ^- the set of the other arcs of the cycle.

If the arcs are numbered **1**, **2**, **…**, **m**, every cycle can be associated with a vector $\vec{\mu} = (\mu_1, \mu_2, ..., \mu_m)$ in $\mathbb{R}^m$, with:

$$\mu_i = \begin{cases} +1 \text{ if } i \in \mu^+ \\ -1 \text{ if } i \in \mu^- \\ 0 \text{ if } i \notin \mu^+ \cup \mu^- \end{cases}.$$

<u>Example:</u>

In the graph below: **(3,4,5)** is a cycle of length **3**. The associated vector is $\vec{\mu} = (0,0,1,-1,1,0)$.

(1,2) is a circuit of length **2**. The associated vector is $\vec{\mu}=(1,1,0,0,0,0)$.

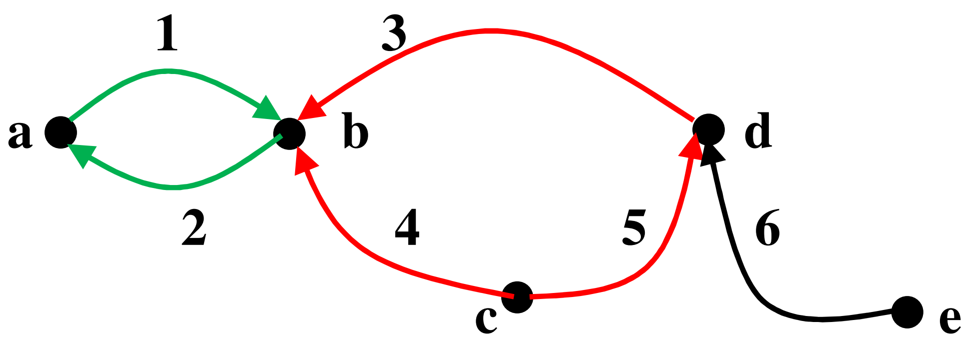


Figure 16: An example of a cycle and a circuit

6.4. Cocycle and Cocircuit:

Let **G = (X , U)** a graph and $A \subset X$. The set of arcs incident to the set **A**, be denoted as $\omega_G(A)=\omega_G^+(A)\cup\omega_G^-(A)$ (see § 4.2, incident arc).

A *cocycle* is a non-empty set of arcs of the form $\omega_G(A)$, partitioned into two classes $\omega_G^+(A)$ and $\omega_G^-(A)$.

A **cocircuit** is a cocycle, either $\omega_G(A)=\omega_G^+(A)$ or $\omega_G(A)=\omega_G^-(A)$, in which all arcs are oriented in the same direction, i.e. into set **A,** or out of set **A.**

Notation:

If the arcs are numbered **1**, **2**, **…**, **m**, every cocycle can be associated with a vector $\vec{\omega}=(\omega_1,\omega_2,\dots,\omega_m)$ in $\mathbb{R}^m$, with:

$$\omega_i=\begin{cases}+1 \text{ if } i\in\omega_G^+(A)\\ -1 \text{ if } i\in\omega_G^-(A)\\ 0 \text{ if } i\notin\omega_G(A)\end{cases}$$

Example:

In the following graph :$\{2\}\cup\{1,6\}$ is a cocycle with $A=\{b,c,d\}$.

The associated vector is $\vec{\omega}=(-1,1,0,0,0,-1)$.

$\{6\}$ is a cocircuit with $\mathbf{A}=\{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}\}$. The associated vector is $\vec{\boldsymbol{\omega}}=(0,0,0,0,0,-1)$.

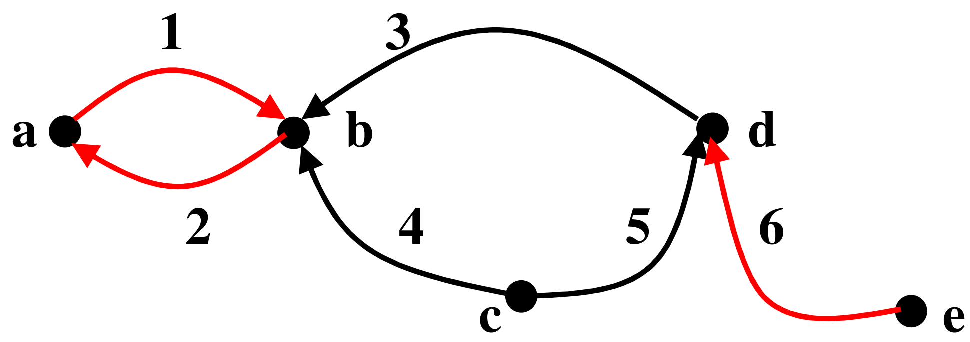


Figure 17: An example of a cycle and a circuit

An *elementary cocycle* is composed of the set of arcs joining two connected subgraphs $\mathbf{A}_1$ and $\mathbf{A}_2$, such that:

$$\begin{cases} \mathbf{A}_1 \neq \varnothing,\ \mathbf{A}_2 \neq \varnothing \\ \mathbf{A}_1 \cap \mathbf{A}_2 = \varnothing \\ \mathbf{A}_1 \cup \mathbf{A}_2 = \mathbf{C} \end{cases}.$$

where **C** is a connected component.

<u>**Example:**</u>

In the graph below: **{3,5}** is an elementary cocycle with $\mathbf{A}_1=\{\mathbf{a},\mathbf{b},\mathbf{c}\}$ and $\mathbf{A}_2=\{\mathbf{d},\mathbf{e}\}$.

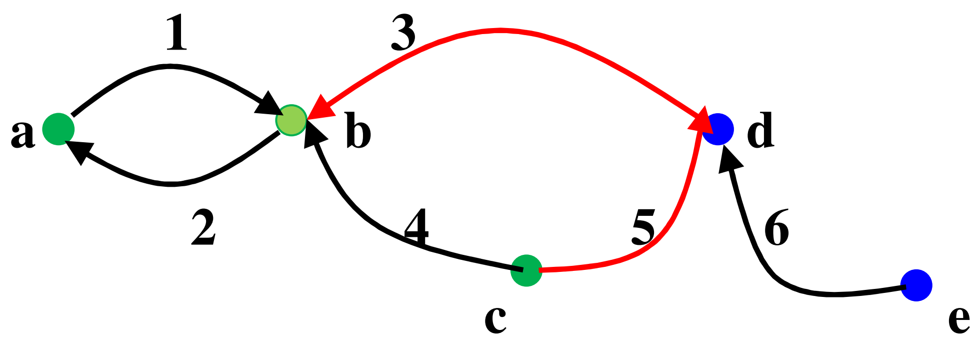


Figure 18: An example of an elementary cocycle.