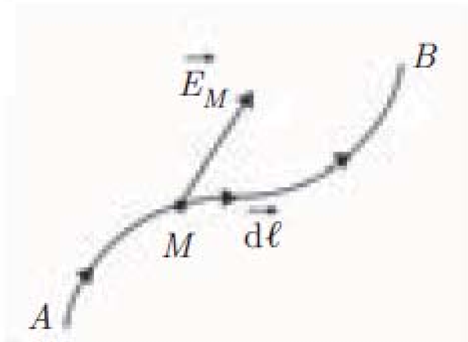
Chapter II : Electric Potential

2-1 Circulation of the electric field- work of the electrostatic force:

In mechanics, we have defined the elementary work dW of a force F along an infinitely small path MM ' by the scalar product:

$$dW = \vec{F}.\vec{dl} = F\,dl\cos\theta$$



The total work is given by following curvilinear integral:

$$W = \int_A^B \vec{F}.\,d\vec{l}$$

In electrostatic the force is electric between Q and q:

$$\vec{F} = K\frac{Q\,q}{r^2}\vec{u} = q\vec{E}(M)$$

the elementary work related to this force is

$$dW = \vec{F}.\overrightarrow{dr} = q\;\vec{E}.\overrightarrow{dr}$$

The total work :

$$W_{AB} = q\int_A^B \vec{E}.\overrightarrow{dr} = q\;\vec{E}.\overrightarrow{dr}$$

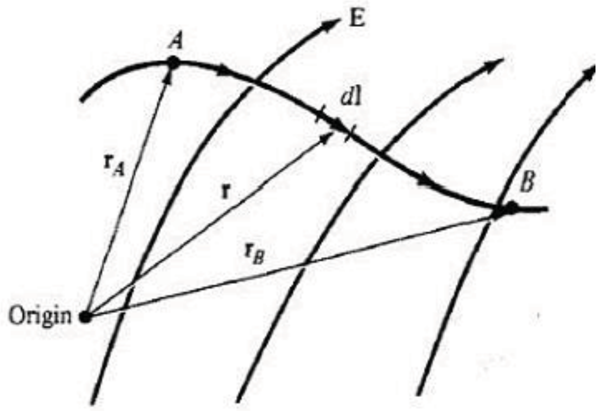
Replacing F, we get for the the elementary work

$$dW = \frac{KQq}{r^2}\;\vec{u}\;.\overrightarrow{dr} \qquad\qquad \vec{u} = \frac{\vec{r}}{r}$$

$$\begin{aligned} dW &= \frac{KQq}{r^3}\vec{r}\;.\overrightarrow{dr} \\ &= \frac{KQq}{r^3}r\;.dr \qquad (\vec{r}\;//\overrightarrow{dr}) \\ &= \frac{KQq}{r^2}\;.dr \end{aligned}$$

We also can write :

$$\begin{aligned} W_{AB} &= \int_A^B dW = KQq \int \frac{1}{r^2} dr \\ &= KQq\left(\frac{1}{r_A} - \frac{1}{r_B}\right) \\ &= q(V_A - V_B) \end{aligned}$$



$V_A - V_B$ is known as the difference between points A and B

V_A and V_B are the potentials at A and B.

If we choose as reference the potential at infinity is zero, the potential of a point charge Q is

$$V = \frac{KQ}{r}$$

The units of the potential is the Volt V

Remarks :

1. This circulation depend on the initial and the final point.
2. This circulation is conservative: it does not depend on the followed path.
3. The circulation of the electrostatic field on a closed curve (we return to A) is zero.
4. From the above relationship, along a field line, that is to say for $\vec{E} \cdot \vec{dl} > 0$ we have V(A)>V(B). The lines of electrostatic fields go in the direction of the decreasing potentials.

2.2 Case of n charges:

In the presence of several q_i charges, the electrostatic field is the sum of the electric potentials produced by each charge q_i:

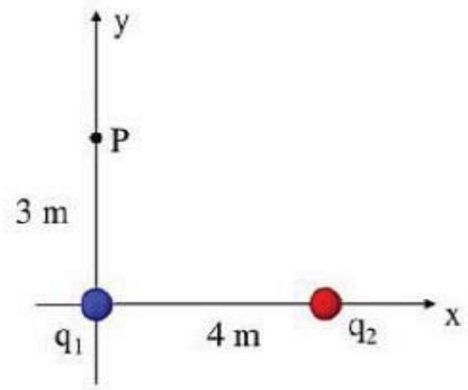
$$V(M) = \sum_{i=1}^{n} V_i(M)$$

.

$$V(M) = \sum_{i=1}^{n} \frac{1}{4\pi\varepsilon_0} \frac{q_i}{r_i} + V_0$$

Where r_i is the distance between the charge q_i and the point M.

Example: a 1 μC point charge is located at the origin and a -4 μC point charge 4 meters along the +x axis. Calculate the electric potential at a point P, 3 meters along the +y axis.



$$V_P = k\sum_i \frac{q_i}{r_i} = k\left(\frac{q_1}{r_1} + \frac{q_2}{r_2}\right)$$

$$= 9\times10^{9}\left(\frac{1\times10^{-6}}{3} + \frac{-4\times10^{-6}}{5}\right)$$

$$= -4.2\times10^{3}\ \text{V}$$

2-3 Case of a distribution of charges:

For linear λ, surface σ and volume ρ charge distributions, we obtain respectively:

$$V(M) = \frac{1}{4\pi\varepsilon_0}\int \frac{\lambda\, dl}{r} \qquad \lambda = dQ/dl$$

$$V(M) = \frac{1}{4\pi\varepsilon_0}\iint \frac{\sigma\, dS}{r} \qquad \sigma = dQ/dS$$

$$V(M) = \frac{1}{4\pi\varepsilon_0}\iiint \frac{\rho\, d\mathcal{V}}{r} \qquad \rho = dQ/d\mathcal{V}$$

2-4 Relation between field and potential:

Since $\vec{E}.\vec{dl} = -dV$, $\vec{dl} = dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k}$

and $dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz$

We get $E_x = -\frac{\partial V}{\partial x}$ $E_y = -\frac{\partial V}{\partial y}$ $E_z = -\frac{\partial V}{\partial z}$

As a result, the electric field intensity is the gradient of *V*

$$\vec{E} = -\overrightarrow{grad}\, V$$

The negative sign shows that the direction of E is opposite to the direction in which *V* increases.

2-4-1 Potential Energy:

We have seen that :

$$W = \int_A^B \vec{F}.d\vec{l} \;=\; q\int_A^B \vec{E}.d\vec{l} \;=\; q\,(V_A - V_B)$$

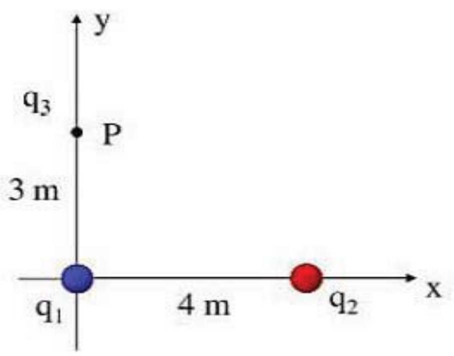
$$W_A^B = q\,(V_A - V_B)$$

The potential energy of a point charge placed in an external field is defined as the work of the electrostatic force acting on the charge, for a displacement of the latter from the point M, where it is located and where the potential is V_M, at a reference point R, where the charge is no longer subject to the action of the external field.
At this point, the potential is zero: $V_R = 0$, we have:

$$E_P\,(M) = \int_M^R \overrightarrow{F}.\overrightarrow{dl} \;=\; q\int_M^R \overrightarrow{E}.\overrightarrow{dl} \;=\; q(V_M - V_R)$$

The unit of E_P is the Joule $[E_P] = J$

Example: how much work is required to bring a +3 μC point charge from infinity to point P?



$$W_{external} = q_3\left(V_P - V_\infty\right), \quad V_\infty = 0$$

$$W_{external} = 3\times10^{-6}\left(-4.2\times10^{3}\right)$$

$$W_{external} = -1.26\times10^{-3}\,J$$

The work done by the external force was negative, so the work done by the electric field was positive. The electric field "pulled" q_3 in (keep in mind q_2 is 4 times as big as q_1).

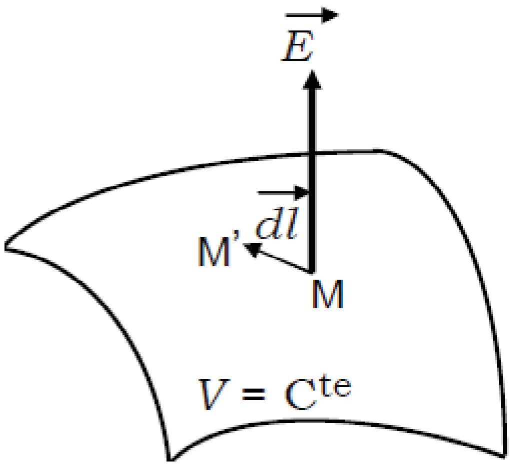
2-5 Topography of the field and electrical potential
Equipotential surface.

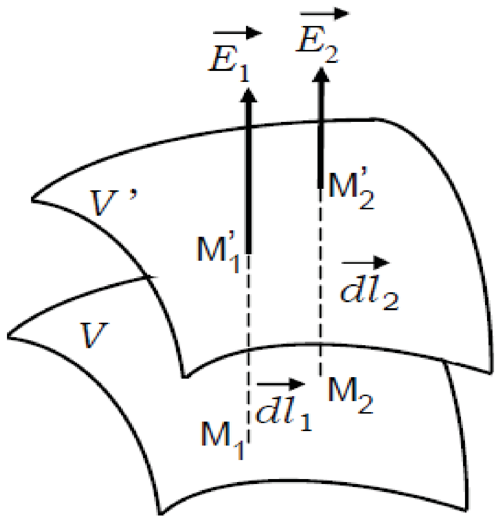
A virtual surface composed of points with the same electric potential is called an equipotential surface.
The work necessary to carry an electric charge, q, a small distance, dr, on an equipotential surface is zero from the equation of work.

Since this work is given by $q\,E\,dr$, we have

$$dV = -\vec{E}.\vec{dl} = -\vec{E}.\vec{MM'} = 0$$

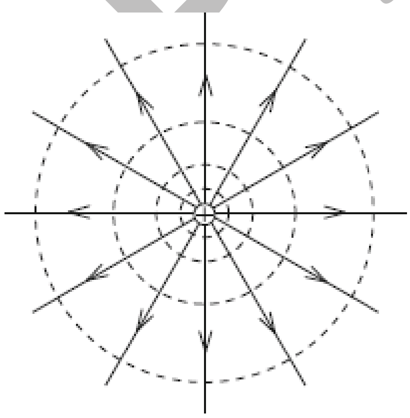
That is, E vector is normal to the equipotential surface. This can also be expressed by saying that the electric field lines are normal to the equipotential surface.



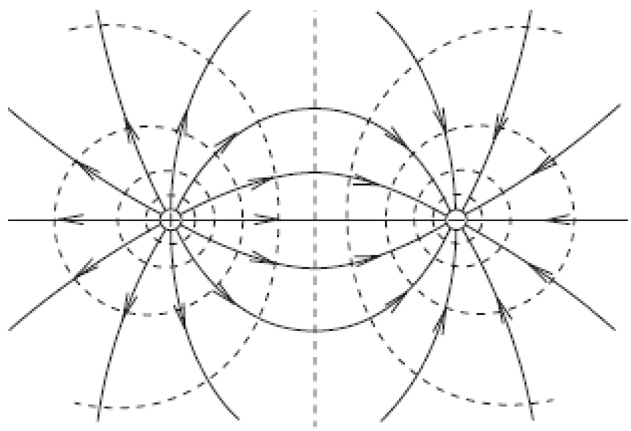


Example:

positive electric charge



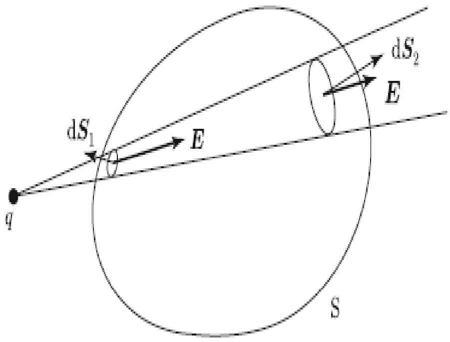
pair of positive and negative charges



2-6 Electrostatic flow-Gauss Law :

2-6-1 Flow of a vector through a surface:

At any point in space the field E is defined, in particular at all points of a surface S. Where S is A real or fictitious surface can be considered as consisting of a large number of elementary surfaces ds.



We call the flow of the electrostatic field E, produced by a charge, through any elementary surface dS oriented, is by definition:

$$d\Phi = \vec{E} \cdot \vec{dS}$$

Where

$$d\vec{S} = dS\ \vec{n}$$

$\vec{n}$ is a unit vector normal to the surface dS.

The total flow is given by: $\Phi = \oiint_S \vec{E} \cdot \vec{dS}$

2-6-2 Gauss Law:

The **total electric flow is the** Integral of the electric field (E) over a closed surface (S) and equal the total enclosed charge (Q) divided by the permittivity of free space (ε_0).

$$\Phi = \oiint_S \vec{E} \cdot \vec{dS} = \frac{Q_{\text{int}}}{\varepsilon_0}$$

-Physical Meaning: The total outward electric flow through any closed surface is proportional to the net charge enclosed by that surface.

-Gauss' theorem is used to calculate the electric field for charge distributions.

-General method:

-Finding the closed surface.

-Writing the definition of the flow.

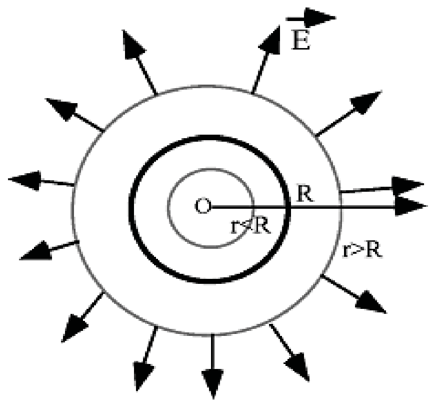
Applying the Gauss theory.

2-6-3 Example :

We consider a sphere with center O and radius R, charged with a surface distribution of charges σ. This distribution having a spherical symmetry

Calculate the resulting electrostatic field and potential in all space. We take V(∞)=0.

draw the graph E(r) and V(r).



Solution:

1-The field :

From the Gauss theory, we have :

$$\Phi = \oiint_S \vec{E} \cdot \vec{dS} = \oiint_S E(r)dS = E(r)\,4\pi r^2$$

$$= \frac{Q_{int}}{\varepsilon_0}$$

<u>**Case r<R**</u>

$$Q_{int} = 0 \Rightarrow \boldsymbol{E_1 = 0}$$

<u>**Case r>R**</u>

$$dq = \sigma ds \Rightarrow \; Q_{int} = \sigma s = \sigma 4\pi R^2$$

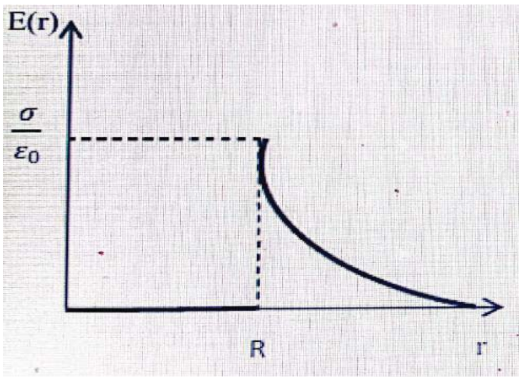
Therefor: $$E_2 4\pi r^2 = \frac{\sigma 4\pi R^2}{\varepsilon_0} \Rightarrow \boldsymbol{E_2 = \frac{\sigma R^2}{\varepsilon_0 r^2}}$$

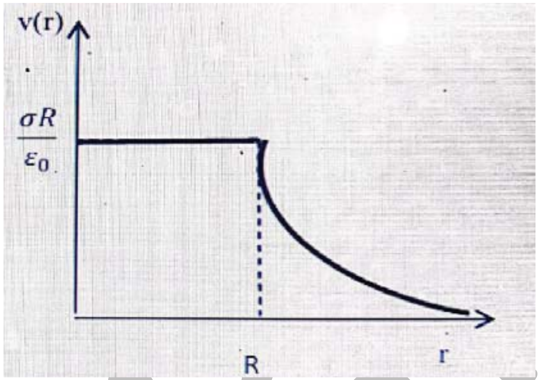
2-The potential:

<u>**Case r<R**</u> $$E_1 = 0 \Rightarrow v_1 = C_1$$

<u>**Case r>R**</u> $$E_2 = \frac{\sigma R^2}{\varepsilon_0 r^2} \Rightarrow \; v_2 = -\frac{\sigma R^2}{\varepsilon_0}\int \frac{dr}{r^2} = \frac{\sigma R^2}{\varepsilon_0 r} + C_2$$

Calculating of the constants:





Laplace and Poisson equations

Let S be a closed surface containing a volume density ρ, from the Gauss theory, we have :

$$\Phi = \iint_{(S)} \vec{E} \cdot \overrightarrow{\mathrm{d}S} = \frac{1}{\varepsilon_0} \iiint_{(\tau)} \rho \, \mathrm{d}\tau = \frac{Q_{\mathrm{int}}}{\varepsilon_0}$$

This expression constitutes the integral form of Gauss' theorem.
The divergence theorem (Green-Ostrogradsky) also allows us to write:

$$\Phi = \iint_S \vec{E} \cdot \overrightarrow{\mathrm{d}S} = \iiint_\tau \mathrm{div}\, \vec{E} \, \mathrm{d}\tau$$

From these relations, we deduce the following local form for Gauss' theorem:

$$\mathrm{div}\, \vec{E} = \frac{\rho}{\varepsilon_0}$$

Note: If there is no charge $\vec{\nabla} \cdot \vec{E} = 0.$

From the precedent local form of Gauss' theorem and from the relation between the field and the potential, $\vec{E} = -\vec{\nabla} V$, we obtain, in the presence of charges, the Poisson equation:

$$\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0 \ \Rightarrow \ -\vec{\nabla} \cdot \vec{\nabla} V = \rho/\epsilon_0 \ \Rightarrow \ \triangle V + \rho/\epsilon_0 = 0$$

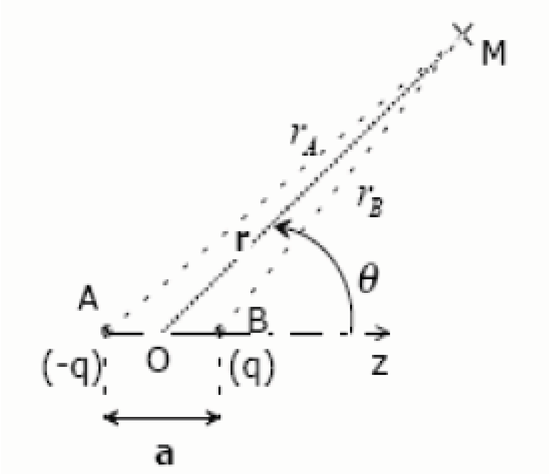
In the absence of charges, Laplace's equation:

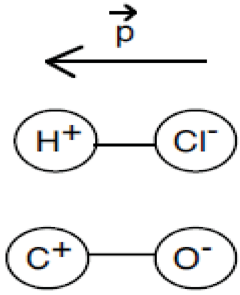
$$\triangle V = 0$$

2-8 Electric Dipole

An electrostatic dipole is a system of two point electric charges, +q and –q located at a distance d=2a very small compared to the distance r.

Molecules such as HCL,CO,H_2O,CO_2 constitute examples of electrostatic dipoles.





Definition: we call the electric dipole moment the quantity $\vec{p} = qd\,\vec{i} = 2aq\,\vec{i}$

The unit of the electric dipole moment is C.m

2.8.1 Potential of a dipole

We have

$$V(M) = V_{+q}(M) + V_{-q}(M)$$

$$= \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r_+} - \frac{1}{r_-}\right)$$

$$= \frac{q}{4\pi\varepsilon_0}\left(\frac{r_- - r_+}{r_+ \, r_-}\right)$$

With r_+ =MA and r_- =MB

As: $MA \simeq r + \frac{a}{2}\cos\theta$ $\quad MB \simeq r - \frac{a}{2}\cos\theta$

Consequently: $MB \cdot MA \simeq r^2$

The potentialis given by: $V(M) = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2}$

where $p = q.d$ is the magnitude of the electric dipole moment.

2.8.2 Field of a dipole

The relationship between field and potential is, $dV = -\vec{E}\,.\,d\vec{l}$

In polar coordinates :

$$dV = -\left(E_r\, dr + E_\theta\, r\, d\theta\right) = \left(\frac{\partial V}{\partial r}\right) dr + \left(\frac{\partial V}{\partial \theta}\right) d\theta$$

This gives

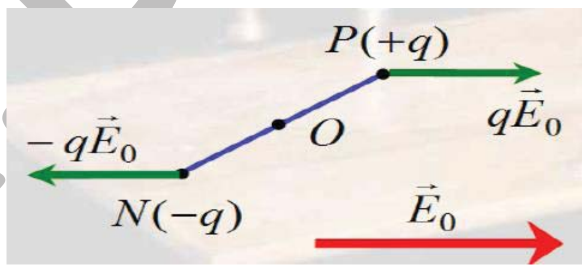
$$E_r = -\frac{\partial V}{\partial r} = \frac{2Kp\cos\theta}{r^3}$$

$$E_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta} = \frac{Kp\sin\theta}{r^3}$$

2.8.3. Interaction between an external field and the dipole

If we place a dipole, with electric moment p, in an external uniform field E_0, the charges, which constitute it, are subjected to equal forces and opposite.

The dipole is subjected to a moment couple with respect to o.



The charges, which constitute it, are subject to opposite and equal forces.

$$\vec{F}_1 = -\vec{F}_2 = q\vec{E}_0$$

The dipole then undergoes the action of a torque of magnitude:

$$\begin{aligned} L &= F.d.\sin\theta \\ &= q.E_0 d.\sin\theta \\ &= p.E_0.\sin\theta \end{aligned}$$

In vector form:

$$\vec{L} = \vec{p} \times \vec{E}_0$$

This torque rotates the dipole to align it parallel to the external field.

2-9 Interaction energy:

The potential energy of a dipole, placed in an E field, is calculated by adding the potential energies of each charge:

$$E_p = q \quad (V_+ - V_-) = q \ \Delta V$$

With :

$$E_p = -\vec{E}_0 . \vec{d}$$

For θ=0 The energy is minimum –Stable equilibrium position.

For θ=Π The energy is maximum – instable equilibrium position.