Khemis Miliana University – Djilali BOUNAAMA
Faculty of Science and Technology
Department of Physics

Electromagnetism

L2 Fundamental Physics

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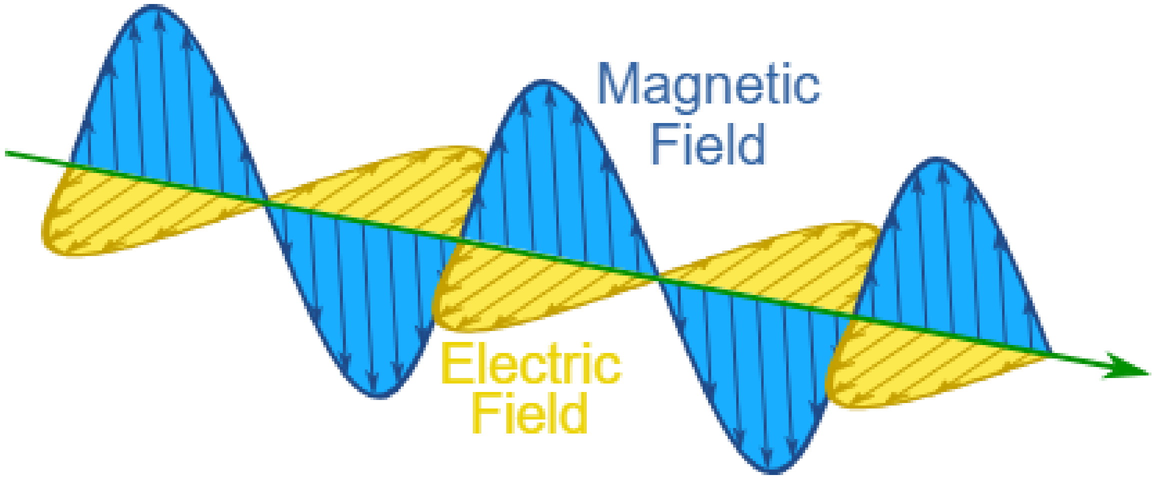
Chapter 03

Propagation of EM waves



III. Propagation of E.M Waves

1. Differential equation of the wave

We call the 2nd order differential equation of the following form (1D space):

$$\frac{\partial^2}{\partial x^2}F(x,t) - \alpha\frac{\partial^2}{\partial t^2}F(x,t) = 0$$

the wave's equation, and the function $F(x,t)$ verifying this equation (solution of the equation) is called the wave function.

The unit homogeneity implies that:

$$\alpha[s^2.m^{-2}] \rightarrow \frac{1}{\alpha}[m^2.s^{-2}] \equiv \left(v\left[\frac{m}{s}\right]\right)^2$$

This allows to rewrite the wave's equation:

$$\frac{\partial^2}{\partial x^2}F(x,t) - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}F(x,t) = 0$$

One can deduce that v represent the propagation velocity of the wave.

In 3D space, the wave's equation can be generalized:

$$\Delta F(x,y,z,t) - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}F(x,y,z,t) = 0$$

With: $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

In this case, the propagation velocity could be given as 3D vector:

$$\vec{v} = v_x\vec{i} + v_y\vec{j} + v_z\vec{k}$$

III. Propagation of E.M Waves

2. Solution of the wave's equation

Let's focus on the 1D space equation:

$$\frac{\partial^2}{\partial x^2}F(x,t) - \alpha\frac{\partial^2}{\partial t^2}F(x,t) = 0$$

We can recognize the difference of two squares identity: $(a^2 - b^2) = (a-b).(a+b)$ *in the differential operator:*

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right)F(x,t)$$
$$= \left(\frac{\partial}{\partial x} - \frac{1}{v}\frac{\partial}{\partial t}\right).\left(\frac{\partial}{\partial x} + \frac{1}{v}\frac{\partial}{\partial t}\right)F(x) = 0$$

Consequently a variable change could be performed here.

The following variable change is considered:

$$\begin{cases} X(x,t) = x - vt \\ Y(x,t) = x + vt \end{cases}$$

It will be easy to prove that:

$$\begin{cases} \dfrac{\partial}{\partial x} - \dfrac{1}{v}\dfrac{\partial}{\partial t} = 2\dfrac{\partial}{\partial X} \\ \dfrac{\partial}{\partial x} + \dfrac{1}{v}\dfrac{\partial}{\partial t} = 2\dfrac{\partial}{\partial Y} \end{cases}$$

Which leads to the new form of the differential equation:

$$\frac{\partial}{\partial X}\left(\frac{\partial}{\partial Y}\right)F(X,Y) = 0$$

Supporting a solution of the type:

$$F(X,Y) = A(X) + B(Y)$$

III. Propagation of E.M Waves

2. Solution of the wave's equation

The wave's function could be written with original variables x and t:

$$F(x,t) = A(x - vt) + B(x + vt)$$

Indicating that both solutions A and B represent propagation in both directions $+v$ and $-v$.

Besides that, A and B functions should be periodic functions to satisfy the 2nd order differential equation of the wave:

$$A(x,t) = a_1.sin(x - vt) + a_2.cos(x - vt)$$

$$B(x,t) = b_1.sin(x + vt) + b_2.cos(x + vt)$$

The coefficients a_i and b_i could be determined by initial and boundary conditions.

In the simplest case of 1D space, the propagating wave in the $+x$ direction, then only the function $A(x - vt)$ is considered $(b_1 = b_2 = 0)$.

Besides that, if we consider at $t = 0$ and $x = 0$ position we have: $A(0,0) = 0$, We can deduce easily that the solution is of the form $(a_2 = 0)$:

$$A(x,t) = a_1.sin(x - vt)$$

This corresponds to a sinusoidal function with an amplitude a_1.

III. Propagation of E.M Waves

3. Derivation of E.M wave's equations

Let's consider the general set of Maxwell's equations for a given medium characterized with an electric permittivity ε and magnetic permeability μ:

$$\begin{cases} \vec{\nabla}.\vec{E} = \dfrac{\rho}{\varepsilon} & (I) \\ \vec{\nabla} \wedge \vec{E} = -\dfrac{\partial \vec{B}}{\partial t} & (II) \\ \vec{\nabla}.\vec{B} = 0 & (III) \\ \vec{\nabla} \wedge \vec{B} = \mu \vec{J} + \mu\varepsilon \dfrac{\partial \vec{E}}{\partial t} & (IV) \end{cases}$$

By applying the following rule on (II) and (IV):

$$\vec{\nabla} \wedge (\vec{\nabla} \wedge \vec{A}) = \vec{\nabla}(\vec{\nabla}.\vec{A}) - \Delta\vec{A}$$

From (II) we get:

$$\vec{\nabla} \wedge (\vec{\nabla} \wedge \vec{E}) = \vec{\nabla}(\vec{\nabla}.\vec{E}) - \Delta\vec{E}$$

$$\vec{\nabla} \wedge \left(-\frac{\partial \vec{B}}{\partial t}\right) = \vec{\nabla}\left(\frac{\rho}{\varepsilon}\right) - \Delta\vec{E}$$

$$-\frac{\partial}{\partial t}(\vec{\nabla} \wedge \vec{B}) = \vec{\nabla}\left(\frac{\rho}{\varepsilon}\right) - \Delta\vec{E}$$

$$-\frac{\partial}{\partial t}\left(\mu\vec{J} + \mu\varepsilon\frac{\partial \vec{E}}{\partial t}\right) = \vec{\nabla}\left(\frac{\rho}{\varepsilon}\right) - \Delta\vec{E}$$

$$\Delta\vec{E} - \mu\varepsilon\frac{\partial^2 \vec{E}}{\partial t^2} - \mu\frac{\partial}{\partial t}\vec{J} = \vec{\nabla}\left(\frac{\rho}{\varepsilon}\right)$$

Since $\vec{J} = \sigma\vec{E}$:

$$\Delta\vec{E} - \mu\varepsilon\frac{\partial^2 \vec{E}}{\partial t^2} - \mu\sigma\frac{\partial}{\partial t}\vec{E} = \frac{1}{\varepsilon}\vec{\nabla}\rho \qquad (Eq.3.1)$$

III. Propagation of E.M Waves

3. Derivation of E.M wave's equations

From (IV) one can also derive the following equation in the same way:

$$\Delta\vec{B} - \mu\varepsilon\frac{\partial^2\vec{B}}{\partial t^2} - \mu\sigma\frac{\partial\vec{B}}{\partial t} = 0 \qquad (Eq.3.2)$$

Finally, we will get the following system of 2nd degree differential equations:

$$\begin{cases} \underbrace{\Delta\vec{E}}_{Spat.var.} - \underbrace{\mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2}}_{Propagation} - \underbrace{\mu\sigma\frac{\partial}{\partial t}\vec{E}}_{Dispersion} = \underbrace{\frac{1}{\varepsilon}\vec{\nabla}\rho}_{Source} \\ \underbrace{\Delta\vec{B}}_{Spat.var.} - \underbrace{\mu\varepsilon\frac{\partial^2\vec{B}}{\partial t^2}}_{Propagation} - \underbrace{\mu\sigma\frac{\partial\vec{B}}{\partial t}}_{Dispersion} = 0 \end{cases}$$

In the case of void (air) medium with no charge ($\rho = 0, \varepsilon = \varepsilon_0, \mu = \mu_0$):

$$\begin{cases} \Delta\vec{E} - \mu_0\varepsilon_0\frac{\partial^2\vec{E}}{\partial t^2} - \mu_0\sigma\frac{\partial\vec{E}}{\partial t} = 0 \qquad (Eq.3.3) \\ \Delta\vec{B} - \mu_0\varepsilon_0\frac{\partial^2\vec{B}}{\partial t^2} - \mu_0\sigma\frac{\partial\vec{B}}{\partial t} = 0 \qquad (Eq.3.4) \end{cases}$$

Which shows that we get a 2nd degree differential equations without constant terms (homogeneous equations).

It should be noticed that 1st degree terms :

$\mu_0\sigma\frac{\partial\vec{E}}{\partial t}$ and $\mu_0\sigma\frac{\partial\vec{B}}{\partial t}$ came from the presence of non-null current.

III. Propagation of E.M Waves

3. Derivation of E.M wave's equations

In the same medium, both equations (3.3) and (3.4) will be reduced in case of absence of currents $(j = 0)$:

$$\begin{cases} \Delta\vec{E} - \mu_0\varepsilon_0 \dfrac{\partial^2\vec{E}}{\partial t^2} = 0 & (Eq.3.5) \\ \Delta\vec{B} - \mu_0\varepsilon_0 \dfrac{\partial^2\vec{B}}{\partial t^2} = 0 & (Eq.3.6) \end{cases}$$

These equations are identical to general wave's equation (3D), and by identification we can find that propagation velocity:

$$v = \frac{1}{\sqrt{\mu\varepsilon}}$$

Application (5min):

Calculate $v_0 = \frac{1}{\sqrt{\mu_0\varepsilon_0}}$ in the void (Air), and comment your finding.

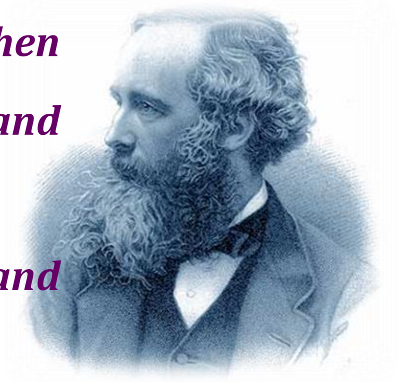
$$\varepsilon = \varepsilon_0 = 8.85 \times 10^{-12}[C^2.N^{-1}.m^{-2}]$$

$$\mu = \mu_0 = 4\pi \times 10^{-7}[N.A^{-2}]$$

We find:

$$v_0 = \frac{1}{\sqrt{\mu_0\varepsilon_0}} = 2.99 \times 10^8 \left[\frac{m}{s}\right]$$

The first measurements of light speed by Bradley in 1729 ($3.01 \times 10^8[m/s]$), then Fizeau in 1849 ($3.15 \times 10^8[m/s]$), and Foucault in 1862 ($2.98 \times 10^8[m/s]$). Maxwell's treatise in Electricity and Magnetism was published in 1873!!!



III. Propagation of E.M Waves

3. Derivation of E.M wave's equations

The most important results of Maxwell's work was the linking between light and Electromagnetic fields:

"Light is electromagnetic wave propagating in the void with a speed c $\cong 3 \times 10^{8} [m/s]$*"*

The differential equations (3.5) and (3.6) will support a periodic functions as solutions of the following form:

$$\vec{E}(\vec{r},t) = \vec{E}_0 . \cos\left(\vec{r}.\vec{k} - \omega t\right)$$

$$\vec{B}(\vec{r},t) = \vec{B}_0 . \cos\left(\vec{r}.\vec{k} - \omega t\right)$$

Where $\vec{k}$ *is the wave vector to be determined.*

III. Propagation of E.M Waves

1. The general solution in free space

Let's go back to the first system of 2nd differential equations including 1st order time term ($\rho = 0$):

$$\begin{cases} \Delta\vec{E} - \mu_0\varepsilon_0\dfrac{\partial^2\vec{E}}{\partial t^2} - \mu_0\sigma\dfrac{\partial\vec{E}}{\partial t} = 0 & (Eq.3.3) \\ \Delta\vec{B} - \mu_0\varepsilon_0\dfrac{\partial^2\vec{B}}{\partial t^2} - \mu_0\sigma\dfrac{\partial\vec{B}}{\partial t} = 0 & (Eq.3.4) \end{cases}$$

10min Test: *We propose the following form as general solutions of (3.3) & (3.4). Replace them and deduce the new differential equations of space phasors $\tilde{e}(\vec{r})$ & $\tilde{b}(\vec{r})$:*

$$\tilde{E}(\vec{r},t) = \tilde{e}(\vec{r}).e^{i\omega t} = \tilde{e}(x,y,z).e^{i\omega t}$$

$$\tilde{B}(\vec{r},t) = \tilde{b}(\vec{r}).e^{i\omega t} = \tilde{b}(x,y,z).e^{i\omega t}$$

With: $e^{i\omega t} = \cos\omega t + i.\sin\omega t, i^2 = -1$

We need just to replace both solutions in equations (3.3) and (3.4):

$$\Delta\left(\tilde{e}(\vec{r}).e^{i\omega t}\right) - \mu\varepsilon\frac{\partial^2\left(\tilde{e}(\vec{r}).e^{i\omega t}\right)}{\partial t^2} - \mu\sigma\frac{\partial}{\partial t}\left(\tilde{e}(\vec{r}).e^{i\omega t}\right) = 0$$

$$\Delta\left(\tilde{b}(\vec{r}).e^{i\omega t}\right) - \mu\varepsilon\frac{\partial^2\left(\tilde{b}(\vec{r}).e^{i\omega t}\right)}{\partial t^2} - \mu\sigma\frac{\partial}{\partial t}\left(\tilde{b}(\vec{r}).e^{i\omega t}\right) = 0$$

This will give us the new space-differential equations:

$$e^{i\omega t}\Delta\tilde{e}(\vec{r}) + \mu\varepsilon\omega^2\tilde{e}(\vec{r})e^{i\omega t} - i\omega\mu\sigma\tilde{e}(\vec{r})e^{i\omega t} = 0$$

$$e^{i\omega t}\Delta\tilde{b}(\vec{r}) + \mu\varepsilon\omega^2\tilde{b}(\vec{r})e^{i\omega t} - i\omega\mu\sigma\tilde{b}(\vec{r})e^{i\omega t} = 0$$

To be reduced to (phasor's equations):

$$\Delta\tilde{e}(\vec{r}) + \left[\mu\varepsilon\omega^2 - i\omega\mu\sigma\right]\tilde{e}(\vec{r}) = 0$$

$$\Delta\tilde{b}(\vec{r}) + \left[\mu\varepsilon\omega^2 - i\omega\mu\sigma\right]\tilde{b}(\vec{r}) = 0$$

III. Propagation of E.M Waves

1. The general solution in free space

By introducing complex permittivity:

$$\varepsilon_c = \varepsilon - i\frac{\sigma}{\omega} = \varepsilon' - i\varepsilon'', \varepsilon' = \varepsilon, \varepsilon'' = \frac{\sigma}{\omega}$$

We got: $k^2 = \mu\varepsilon_c\omega^2 = \mu\omega^2\left[\varepsilon - i\frac{\sigma}{\omega\varepsilon}\right] = -\gamma^2 = (i\gamma)^2$

Finally, the 2^{nd} order space differential equations known as Helmholtz equation of E.M wave could be written :

$$\begin{cases} \Delta\tilde{e}(\vec{r}) + k^2\tilde{e}(\vec{r}) = 0 & (Eq.3.7) \\ \Delta\tilde{b}(\vec{r}) + k^2\tilde{b}(\vec{r}) = 0 & (Eq.3.8) \end{cases}$$

Consequently, solutions are of the form:

$$\tilde{e}(\vec{r}) = \vec{E}_0 e^{\pm ik(\vec{r}.\vec{u})} = \vec{E}_0 e^{\pm i(\vec{r}.\vec{k})}$$
$$\tilde{b}(\vec{r}) = \vec{B}_0 e^{\pm ik(\vec{r}.\vec{u})} = \vec{B}_0 e^{\pm i(\vec{r}.\vec{k})}$$

E_0 and B_0: maximal amplitudes.

Where: $k = \omega\sqrt{\mu\varepsilon}\sqrt{1 - i\,\sigma/_{\omega\varepsilon}} = \alpha + i\beta$

is known as "Wave number".

And the parameter γ is called "propagation constant"

In the specific case of lossless medium:

$$\sigma = 0 \ \rightarrow \varepsilon'' = 0$$

The wave number is purely real and the propagation is done without loss of the strength of E.M wave, and we have:

$$k = \omega\sqrt{\mu\varepsilon} = \frac{\omega}{v} = \frac{2\pi}{vT} = \frac{2\pi}{\lambda}\left[\frac{rad}{m}\right]$$

III. Propagation of E.M Waves

1. The general solution in free space

Replacing now $\vec{e}(\vec{r})$ and $\vec{b}(\vec{r})$ in the general expression:

$$\widetilde{\vec{E}}(\vec{r},t) = \tilde{\vec{e}}(\vec{r}).e^{i\omega t} = \vec{E}_0 e^{i(\omega t \pm \vec{r}.\vec{k})}$$

$$\widetilde{\vec{B}}(\vec{r},t) = \tilde{\vec{b}}(\vec{r}).e^{i\omega t} = \vec{B}_0 e^{i(\omega t \pm \vec{r}.\vec{k})}$$

Since $\vec{E}_0$ and $\vec{B}_0$ are amplitudes at initial conditions they could be written:

$$\vec{E}_0 = \vec{E}(0,0) = E_0\vec{u}_E = |E_0|e^{i\varphi_0}\vec{u}_E$$

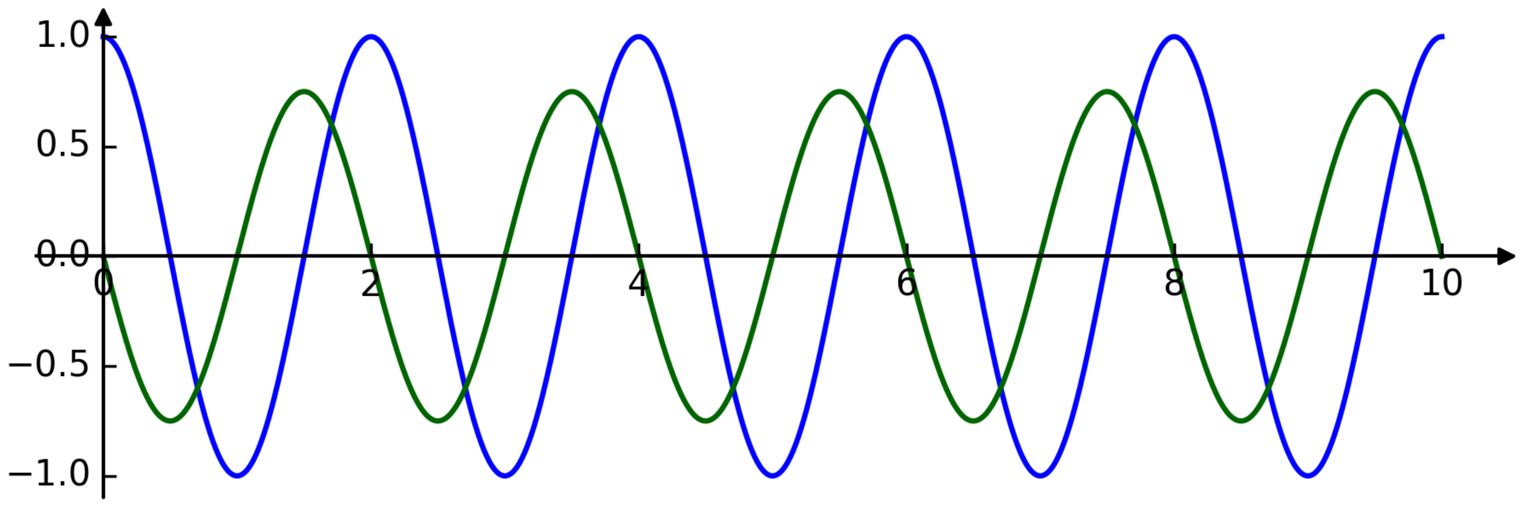
$$\vec{B}_0 = \vec{B}(0,0) = B_0\vec{u}_B = |B_0|e^{i\varphi_0}\vec{u}_B$$

φ_0: initial phase of the wave

For instance, if we consider two waves represented by their electric fields, taken as the real part of complex phasors:

$$\vec{E}_1(\vec{r},t) = |E_{10}|.\Re e\left[e^{i(\omega t \pm \vec{r}.\vec{k})}\right]\vec{u};\ \varphi_1 = 0$$

$$\vec{E}_2(\vec{r},t) = |E_{20}|\Re e\left[e^{i(\omega t \pm \vec{r}.\vec{k} + \pi/2)}\right]\vec{u};\ \varphi_2 = \pi/2$$



III. Propagation of E.M Waves

2. Phasors Maxwell's equations

One of the important results of the previous solutions given in complex notation, is the new form of Maxwell equations. Indeed, let's take the following expressions of E.M fields:

$$\tilde{E}(\vec{r},t) = \tilde{e}(\vec{r}).e^{i\omega t}$$

$$\tilde{H}(\vec{r},t) = \tilde{h}(\vec{r}).e^{i\omega t}$$

When replaced in the Maxwell questions, taking in consideration that (similarly for $\tilde{H}(\vec{r},t)$) :

$$\frac{\partial \tilde{E}(\vec{r},t)}{\partial t} = \frac{\partial\left[\tilde{e}(\vec{r}).e^{i\omega t}\right]}{\partial t} = \tilde{e}(\vec{r})\frac{\partial e^{i\omega t}}{\partial t} = i\omega\tilde{e}(\vec{r}).e^{i\omega t}$$

$$\begin{cases} \vec{\nabla}.\tilde{e} = \dfrac{\tilde{\rho}}{\varepsilon} & (I) \\ \vec{\nabla}\wedge\tilde{e} = -i\omega\mu\tilde{h} & (II) \\ \vec{\nabla}.\tilde{h} = 0 & (III) \\ \vec{\nabla}\wedge\tilde{h} = \tilde{j} + i\omega\varepsilon\tilde{e} & (IV) \end{cases}$$

Which could be rewritten by taking $\tilde{j} = \sigma\tilde{e}$, we get in free space ($\rho = 0 \rightarrow \tilde{\rho} = 0$):

$$\begin{cases} \vec{\nabla}.\tilde{e} = 0 & (I) \\ \vec{\nabla}\wedge\tilde{e} = -i\omega\mu\tilde{h} & (II) \\ \vec{\nabla}.\tilde{h} = 0 & (III) \\ \vec{\nabla}\wedge\tilde{h} = i\omega\varepsilon_c\tilde{e} & (IV) \end{cases}$$

With: $\varepsilon_c = \varepsilon - i\frac{\sigma}{\omega}$ as introduced above.

III. Propagation of E.M Waves

3. Spherical and Planar waves

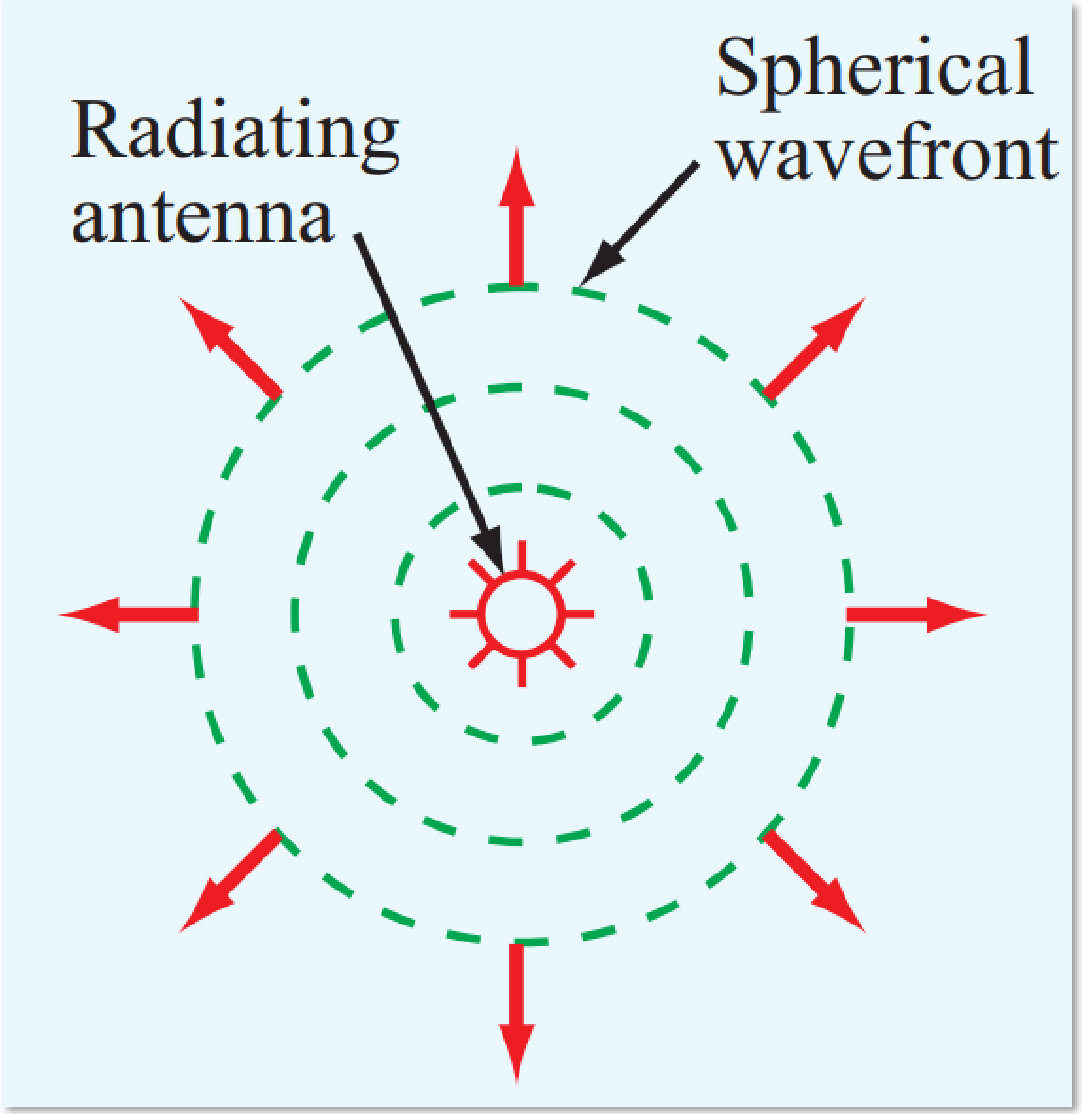
According to previous results, both electric and magnetic fields verifying differential equations are of the form:

$$\begin{cases} \tilde{E}(\vec{r},t) = \vec{E}_0 e^{\pm i(\vec{r}.\vec{k})} e^{i\omega t} = \vec{E}_0 e^{i(\omega t \pm \vec{r}.\vec{k})} & (Eq.3.9) \\ \tilde{B}(\vec{r},t) = \vec{B}_0 e^{\pm i(\vec{r}.\vec{k})} e^{i\omega t} = \vec{B}_0 e^{i(\omega t \pm \vec{r}.\vec{k})} & (Eq.3.10) \end{cases}$$

Along positive direction, physical solutions are:

$$\begin{cases} \vec{E}(\vec{r},t) = \vec{E}_0.\Re e\left[e^{i(\omega t - \vec{r}.\vec{k})}\right] & (Eq.3.11) \\ \vec{B}(\vec{r},t) = \vec{B}_0.\Re e\left[e^{i(\omega t - \vec{r}.\vec{k})}\right] & (Eq.3.12) \end{cases}$$

Such wave is propagating in all directions with the same intensities, therefore it constitutes a spherical wave.

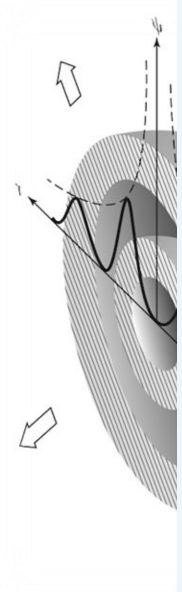


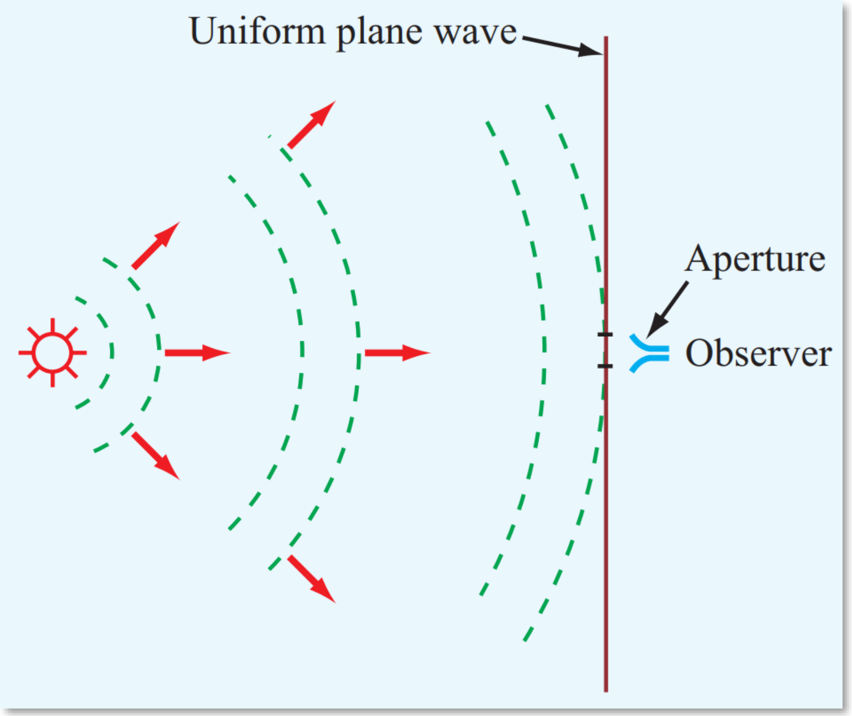
III. Propagation of E.M Waves

3. Spherical and plane waves

A wave produced by a localized source, such as an antenna, expands outwardly in the form of a spherical wave. Even though an antenna may radiate more energy along some directions than along others, the spherical wave travels at the same speed in all directions.

To an observer very far away from the source, however, the wavefront of the spherical wave appears approximately planar, as if it were part of a uniform plane wave with identical properties at all points in the plane tangent to the wavefront. Plane waves are easily described using a Cartesian coordinate system, which is mathematically easier to work with than the spherical coordinate system needed to describe spherical waves.





4. Uniform plane waves

It the case of plane waves, it is possible to choose an arbitrary cartesian direction to point the propagation direction along one of the XYZ axes. For instance if we take the +z-direction, so one can write the wave number vector: $\vec{k} = k\vec{u}_z$

And the scalar product will reduce the spatial term to: $\vec{r}.\vec{k} = (x\vec{u}_x + y\vec{u}_y + z\vec{u}_z).k\vec{u}_z = kz$

Thus, the expression of electric field will be:

$$\tilde{E}(\vec{r},t) = \tilde{e}(\vec{r}).e^{i\omega t} = |E_0|e^{i(\omega t - kz + \varphi_0)}\vec{u}_E$$

$\vec{u}_E = a\vec{u}_x + b\vec{u}_y + c\vec{u}_z$; *a, b, c are cosine directors*

When replaced in the first Maxwell equation a free space as propagation medium $(\rho = 0)$:

$$\vec{\nabla}.\vec{E} = \vec{\nabla}.\left(|E_0|e^{i(\omega t - \vec{r}.\vec{k} + \varphi_0)}\vec{u}_E\right) = 0$$

$$\leftrightarrow \underbrace{\partial_x e^{-i(kz)}}_{=0}(\vec{u}_x.\vec{u}_E) + \underbrace{\partial_y e^{-i(kz)}}_{=0}(\vec{u}_y.\vec{u}_E)$$

$$+ \underbrace{\partial_z e^{-i(kz)}}_{=-ike^{-i(kz)}\neq 0}(\vec{u}_z.\vec{u}_E) = 0 \rightarrow \vec{u}_z.\vec{u}_E = 0$$

Which means that $c = 0$:

$$\vec{u}_E = a\vec{u}_x + b\vec{u}_y$$

III. Propagation of E.M Waves

4. Uniform plane waves

The previous result, will allow us to write the electric field with its XY components:

$$\widetilde{E}(\vec{r},t) = |E_0|e^{i(\omega t - kz + \varphi_0)}\left(a\vec{u}_x + b\vec{u}_y\right)$$

Now let's use the second Maxwell equation: $\vec{\nabla} \wedge \vec{E} = -\frac{\partial \vec{B}}{\partial t} \leftrightarrow$

$$\vec{\nabla} \wedge \left(|E_0|e^{i(\omega t - kz + \varphi_0)}\left(a\vec{u}_x + b\vec{u}_y\right)\right) = -\frac{\partial\left(|B_0|e^{i(\omega t - kz + \varphi_0)}\vec{u}_B\right)}{\partial t} = -i\omega|B_0|e^{i(\omega t - kz + \varphi_0)}\vec{u}_B$$

Performing the curl on the left hand and simplifying similar terms will produce:

$$-ikE\left(-b\vec{u}_x + a\vec{u}_y\right) = -i\omega B\vec{u}_B \rightarrow \vec{u}_B = \frac{kE}{B}\left(-b\vec{u}_x + a\vec{u}_y\right)$$

Consequently, it will be easy to verify that $\vec{u}_B \perp \vec{u}_E$*, which implies that* $\vec{E}(\vec{r},t)$ *and* $\vec{B}(\vec{r},t)$ *are orthogonal.*

III. Propagation of E.M Waves

4. Uniform plane waves

Therefore, the plane electromagnetic wave propagating in the +z-direction, could be represented by both electric and magnetic fields lying on XY plane, with a practical choice ($a = 1, b = 0$):

$$\tilde{\vec{E}}(\vec{r},t) = |E_0|e^{i(\omega t - kz + \varphi_0)}\vec{u}_x$$

$$\tilde{\vec{B}}(\vec{r},t) = |B_0|e^{i(\omega t - kz + \varphi_0)}\vec{u}_y$$

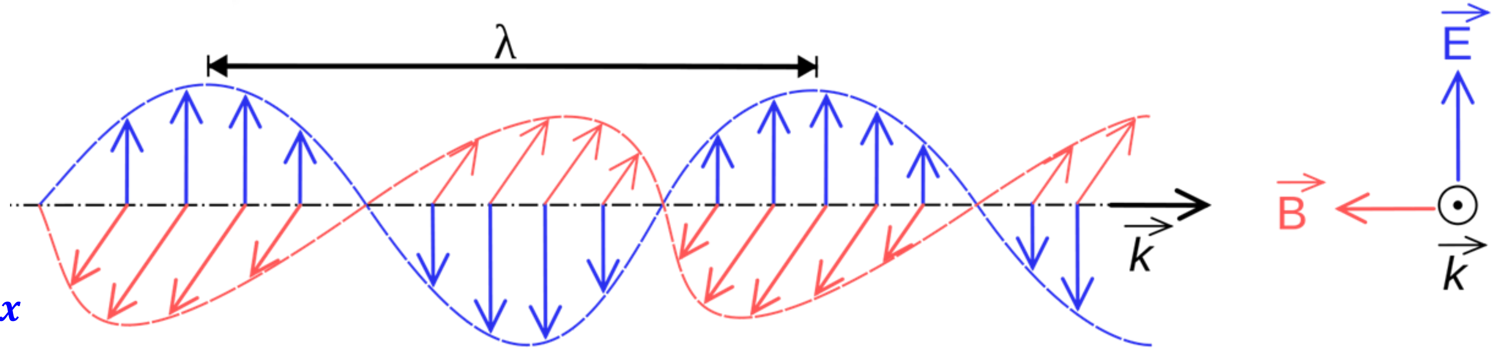
Taking the real part of each phasor:

$$\vec{E}(\vec{r},t) = E_0 cos(\omega t - kz + \varphi_0)\vec{u}_x$$

$$\vec{B}(\vec{r},t) = B_0 cos(\omega t - kz + \varphi_0)\vec{u}_y$$

Thus, the plane E.M wave propagating in a given direction, is represented by two orthogonal E.M fields lying on the perpendicular plan of the propagation direction given by the wave vector $\vec{k}$.

The vectors $\vec{E}, \vec{B}$ and $\vec{k}$ form a direct trihedral.



III. Propagation of E.M Waves

5. Relation between E and H: intrinsic impedance

By considering now that both E.M fields are lying on XY-plane and oriented along $\vec{u}_x$ and $\vec{u}_y$, respectively, the use of the second Maxwell equation will provide the following relation between $\vec{E}$ and $\vec{H}$ (or between $\vec{E}$ and $\vec{B}$), called the "intrinsic impedance" of the given medium of propagation:

$$-ikE = -i\omega\mu H \rightarrow \frac{E[V/m]}{H[A/m]} = \frac{\mu\omega}{k} = \eta[\Omega] = \frac{\mu\omega}{\omega\sqrt{\mu(\varepsilon' - i\varepsilon'')}} = |\eta|e^{i\theta} \leftrightarrow H = \frac{E}{\eta} = \frac{E}{|\eta|}e^{-i\theta}$$

<u>10min Test:</u> In the case of free space, where: $\mu = \mu_0, \varepsilon = \varepsilon_0, \varepsilon'' = 0$, Calculate η_0.

$\mu_0 = 4\pi \times 10^{-7} S.I \,;\, \varepsilon_0 = 8.85 \times 10^{-12} S.I$

III. Propagation of E.M Waves

5. Relation between E and H: intrinsic impedance

By considering now that both E.M fields are lying on XY-plane and oriented along $\vec{u}_x$ and $\vec{u}_y$, respectively, the use of the second Maxwell equation will provide the following relation between $\vec{E}$ and $\vec{H}$ (or between $\vec{E}$ and $\vec{B}$), called the "intrinsic impedance" of the given medium of propagation:

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10min Test: *In the case of free space, where: $\mu = \mu_0, \varepsilon = \varepsilon_0, \varepsilon'' = 0$, Calculate η_0.*

$\mu_0 = 4\pi \times 10^{-7} S.I\ ;\ \varepsilon_0 = 8.85 \times 10^{-12} S.I$

The intrinsic impedance of free space:

$$k = \omega\sqrt{\mu\varepsilon} \rightarrow \eta_0 = \frac{\mu_0\omega}{\omega\sqrt{\mu_0\varepsilon_0}} = \sqrt{\frac{\mu_0}{\varepsilon_0}}$$

$$= \sqrt{\frac{4\pi \times 10^{-7}}{8.85 \times 10^{-12}}} \cong 377[\Omega] \equiv 120\pi[\Omega]$$

III. Propagation of E.M Waves

6. Wave propagation in dielectrics

We now extend our analytical treatment of the uniform plane wave to propagation in a dielectric of permittivity ε and permeability μ. The medium is assumed to be homogeneous (having constant μ and ε with position) and isotropic (in which μ and ε are invariant with field orientation). The expression of the wave number obtained before is:

$$k = \omega\sqrt{\mu\varepsilon}\sqrt{1 - i\,{\varepsilon''}/{\varepsilon}} = \alpha + i\beta$$

With: $\varepsilon'' = \frac{\sigma}{\omega}$

Resolving this equation, one can find α and β:

$$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}}\left(\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}+1\right)^{1/2}$$

$$\beta = \omega\sqrt{\frac{\mu\varepsilon}{2}}\left(\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}-1\right)^{1/2}$$

Back to the general form of E.M fields and replacing with complex form of wave number:

$$\tilde{E}(\vec{r},t) = |E_0|e^{i(\omega t-(\alpha+i\beta)kz+\varphi_0)}\vec{u}_x$$

$$\tilde{B}(\vec{r},t) = |B_0|e^{i(\omega t-(\alpha+i\beta)z+\varphi_0)}\vec{u}_y$$

III. Propagation of E.M Waves

6. Wave propagation in dielectrics

The E.M phasors could be separated into complex and real exponential functions:

$$\tilde{E}(\vec{r},t) = |E_0|e^{-\beta z}e^{i(\omega t-\alpha z+\varphi_0)}\vec{u}_x$$

$$\tilde{B}(\vec{r},t) = |B_0|e^{-\beta z}e^{i(\omega t-\alpha z+\varphi_0)}\vec{u}_y$$

Now, taking the real parts, physical E.M fields are written as:

$$\vec{E}(\vec{r},t) = |E_0|e^{-\beta z}cos(\omega t-\alpha z+\varphi_0)\vec{u}_x$$

$$\vec{B}(\vec{r},t) = |B_0|e^{-\beta z}cos(\omega t-\alpha z+\varphi_0)\vec{u}_y$$

This indicates the phase wave velocity:

$$v_p = \frac{\omega}{\alpha} \to \lambda = \frac{2\pi}{\alpha}$$

For a good dielectric medium ($\sigma \ll$), one can assume with good approximation that: $\frac{\varepsilon''}{\varepsilon'} \ll 1$, which implies using limited development:

$$\alpha \cong \omega\sqrt{\mu\varepsilon};\ \beta = \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}$$

$$v_p \cong \frac{\omega}{\omega\sqrt{\mu\varepsilon}} \equiv \frac{1}{\sqrt{\mu\varepsilon}} \to \lambda = \frac{2\pi}{\omega\sqrt{\mu\varepsilon}}$$

The intrinsic impedance could be obtained :

$$\eta = \frac{\mu\omega}{k} = \frac{\mu\omega}{\omega\sqrt{\mu(\varepsilon'-i\varepsilon'')}} = \sqrt{\frac{\mu}{\varepsilon'}}\frac{1}{\sqrt{1-i\,\varepsilon''/\varepsilon'}}$$

With approach of small numbers expansion:

$$\eta \cong \sqrt{\frac{\mu}{\varepsilon'}}\left(1+i\frac{\varepsilon''}{2\varepsilon'}\right) \equiv \sqrt{\frac{\mu}{\varepsilon'}}\left(1+i\frac{\sigma}{2\omega\varepsilon}\right)$$

III. Propagation of E.M Waves

7. Wave propagation in conductors

In this case $\sigma \gg$, we can assume with a good approximation that : $\frac{\varepsilon''}{\varepsilon'} = \frac{\sigma}{\varepsilon\omega} \gg 1$, the wave number :

$$k = \sqrt{\mu\varepsilon\omega^2 - i\omega\mu\sigma} = \omega\sqrt{\mu\varepsilon'}\sqrt{1 - i\frac{\varepsilon''}{\varepsilon'}}$$

Could be rewritten with good approximation:

$$k \cong \omega\sqrt{\mu\varepsilon}\sqrt{-i\frac{\sigma}{\varepsilon\omega}} \equiv \sqrt{\mu\sigma\omega}\sqrt{-i} = \sqrt{\frac{\mu\sigma\omega}{2}}(1 - i)$$

By identification: $k = \alpha + i\beta$:

$$\alpha = -\beta = \sqrt{\frac{\mu\sigma\omega}{2}}$$

Similarly, taking the real parts, physical E.M fields are written as:

$$\vec{E}(\vec{r}, t) = |E_0|e^{-\beta z}\cos(\omega t - \alpha z + \varphi_0)\vec{u}_x$$

$$\vec{B}(\vec{r}, t) = |B_0|e^{-\beta z}\cos(\omega t - \alpha z + \varphi_0)\vec{u}_y$$

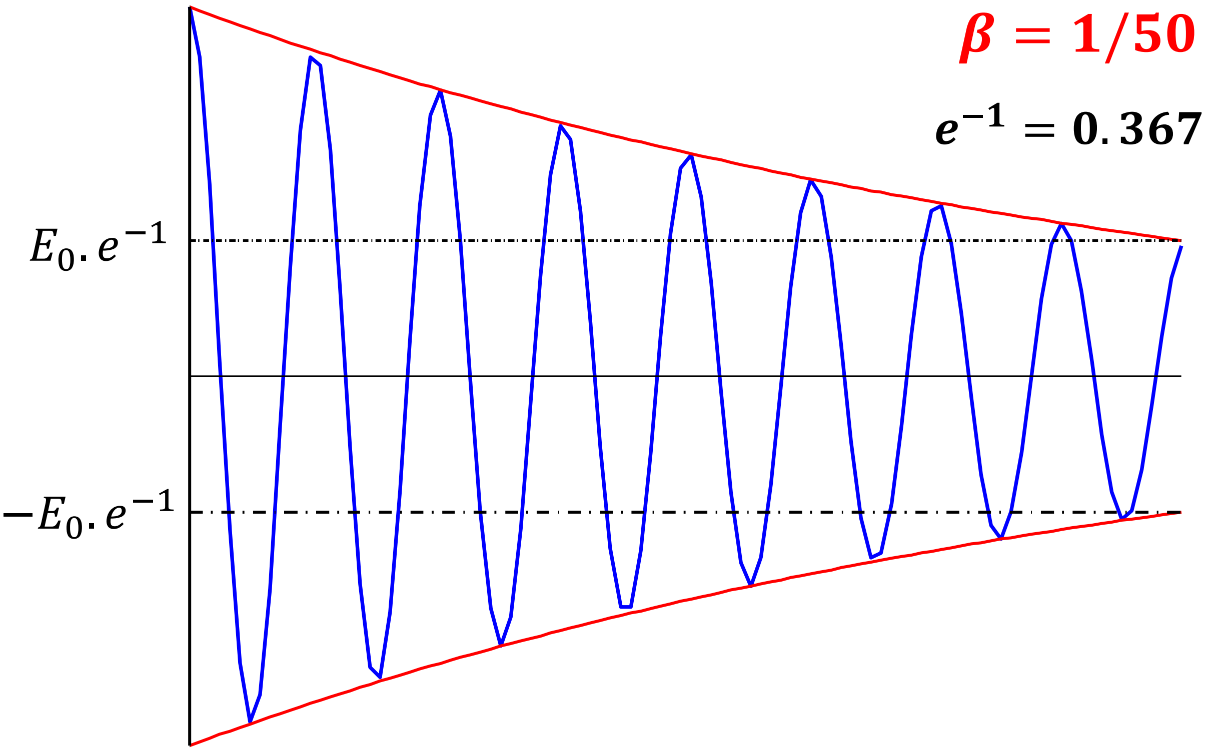
The intrinsic impedance in the case could be obtained in similar way:

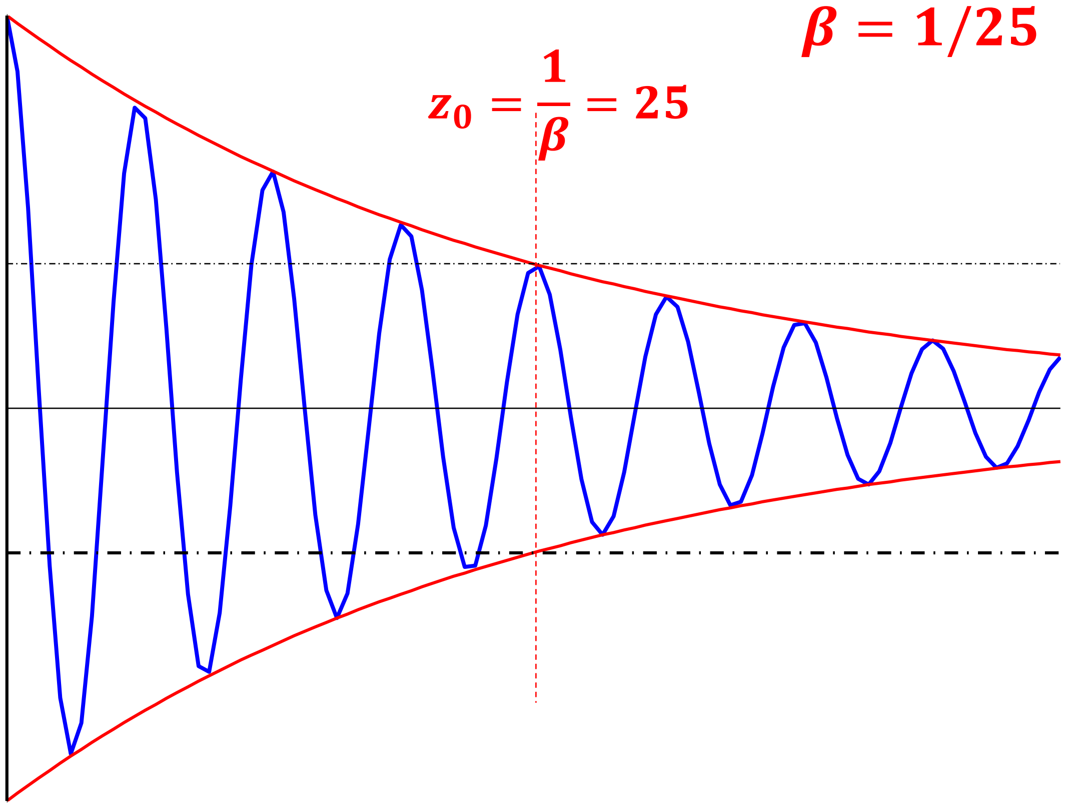
$$\eta = \frac{\mu\omega}{k} = \frac{\mu\omega}{\alpha + i\beta} = \sqrt{\frac{\mu\omega}{\sigma}}\frac{\sqrt{2}}{(1 - i)} = \sqrt{\frac{\mu\omega}{2\sigma}}(1 + i)$$

III. Propagation of E.M Waves

8. Wave attenuation and skin depth

$$\vec{E}(\vec{r},t) = |E_0|e^{-\beta z}\cos(\omega t - \alpha z + \varphi_0)\vec{u}_x$$





III. Propagation of E.M Waves

8. Wave attenuation and skin depth

It is clear that the physical signification of β is the attenuation of E.M fields strength, by a coefficient of $^1/_e = 0.135$ each specific distance:

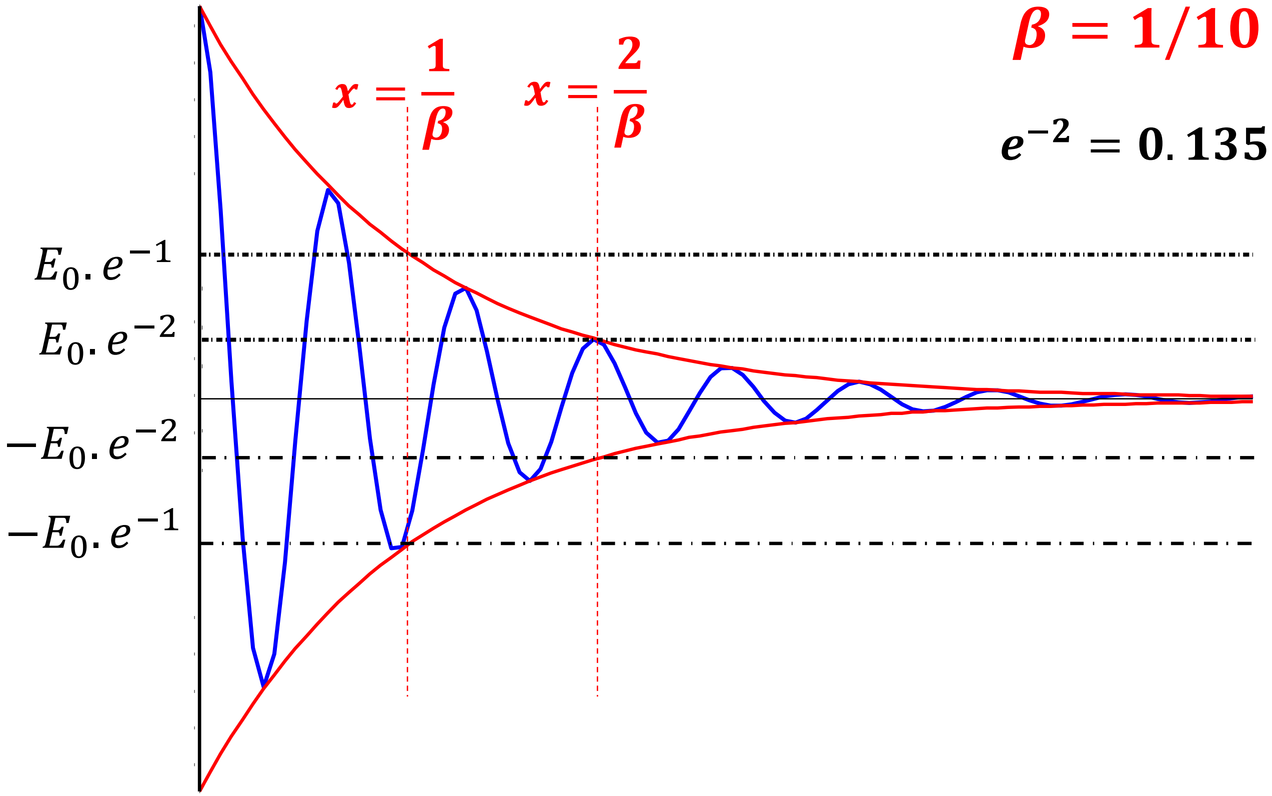
$$\delta = \frac{1}{\beta}$$

Called "Skin depth" or "penetration depth".

In the specific case of good conductors:

$$\delta[m] = \frac{1}{\beta} = \sqrt{\frac{2}{\mu\sigma\omega}} = \frac{1}{\sqrt{\pi\mu\sigma f}}$$

To measure the attenuation, the argument βz of attenuated exponential $e^{-\beta z}$ is called "Neper" and β is measured by $[Np/m]$.



III. Propagation of E.M Waves

8. Poynting's theorem and wave power

In order to find the power flow associated with an electromagnetic wave, it is necessary to develop a power theorem for the electromagnetic field known as the Poynting theorem. It was originally postulated in 1884 by an English physicist, John H. Poynting.

The development begins with the fourth Maxwell's equation, in which we assume that the medium may be conductive:

$$\vec{\nabla} \wedge \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

Next, we take the scalar product of both sides with $\vec{E}$:

$$\vec{E}.\left(\vec{\nabla} \wedge \vec{H}\right) = \vec{E}.\vec{J} + \vec{E}.\frac{\partial \vec{D}}{\partial t}$$

Using the following vectors identity (Chap01):

$$\vec{\nabla}.\left(\vec{E} \wedge \vec{H}\right) = \vec{H}.\vec{\nabla} \wedge \vec{E} - \vec{E}.\vec{\nabla} \wedge \vec{H}$$

Using the latter equation in the left side of IV Maxwell's equation:

$$\vec{H}.\left(\vec{\nabla} \wedge \vec{E}\right) - \vec{\nabla}.\left(\vec{E} \wedge \vec{H}\right) = \vec{E}.\vec{J} + \vec{E}.\frac{\partial \vec{D}}{\partial t}$$

$$\vec{H}.\left(-\frac{\partial \vec{B}}{\partial t}\right) - \vec{\nabla}.\left(\vec{E} \wedge \vec{H}\right) = \vec{E}.\vec{J} + \vec{E}.\frac{\partial \vec{D}}{\partial t}$$

III. Propagation of E.M Waves

8. Poynting's theorem and wave power

Make few adjustments about derivatives, since we know that: $\vec{D} = \varepsilon\vec{E};\ \vec{B} = \mu\vec{H},$

we can write:

$$\vec{E}.\frac{\partial\vec{D}}{\partial t} = \varepsilon\vec{E}.\frac{\partial\vec{E}}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{2}\vec{D}.\vec{E}\right)$$

$$\vec{H}.\frac{\partial\vec{B}}{\partial t} = \mu\vec{H}.\frac{\partial\vec{H}}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{2}\vec{B}.\vec{H}\right)$$

The trick is to pass from volume integral to a surface one:

$$\int_V \vec{\nabla}.\vec{A}dv = \oint_S \vec{A}.d\vec{S}$$

We get:

$$-\vec{\nabla}.\left(\vec{E}\wedge\vec{H}\right) = \frac{\partial}{\partial t}\left(\frac{1}{2}\vec{B}.\vec{H}\right) + \frac{\partial}{\partial t}\left(\frac{1}{2}\vec{D}.\vec{E}\right) + \vec{E}.\vec{J}$$

Integrated over given volume V:

$$-\int_V \vec{\nabla}.\left(\vec{E}\wedge\vec{H}\right).dv = \int_V \frac{\partial}{\partial t}\left(\frac{1}{2}\vec{B}.\vec{H}\right).dv + \int_V \frac{\partial}{\partial t}\left(\frac{1}{2}\vec{D}.\vec{E}\right) + \int_V \vec{E}.\vec{J}\,dv$$

III. Propagation of E.M Waves

8. Poynting's theorem and wave power

The new form of Poynting's equation:

$$-\underbrace{\oint_S (\vec{E}\wedge\vec{H}).d\vec{S}}_{\substack{\text{total E.M power}\\ \text{flowing out V through S}}} = \underbrace{\int_V \frac{\partial}{\partial t}\left(\frac{1}{2}\vec{B}.\vec{H}\right).dv}_{\substack{\text{total energy stored}\\ \text{in magnetic field}}} + \underbrace{\int_V \frac{\partial}{\partial t}\left(\frac{1}{2}\vec{D}.\vec{E}\right)}_{\substack{\text{Total energy stored}\\ \text{in electric field}}} + \underbrace{\int_V \vec{E}.\vec{J}\,dv}_{\substack{\text{Ohmic power}\\ \text{dissipated over V}}}$$

This theorem gives the time rates of increase of energy stored within the volume V, or the instantaneous power going to increase the stored energy.

The cross product of $\vec{E}$ and $\vec{H}$ define the Poynting's vector, indicating the power density flowing in the direction of $\vec{\mathcal{P}}$ at a given point. (homonym "Poynting" and "pointing" is accidentally "True")

$$\vec{\mathcal{P}}[W.m^{-2}] = \vec{E}\wedge\vec{H}$$

The measured value of Poynting value is an average value over a specific time (period) and could be obtained using general phasors:

$$\langle\vec{\mathcal{P}}\rangle \equiv \bar{\mathcal{P}} = \frac{1}{2}\Re e[\tilde{E}\wedge\tilde{H}^*] \propto \frac{1}{2|\eta|}E_0^2 e^{-2\beta z}$$

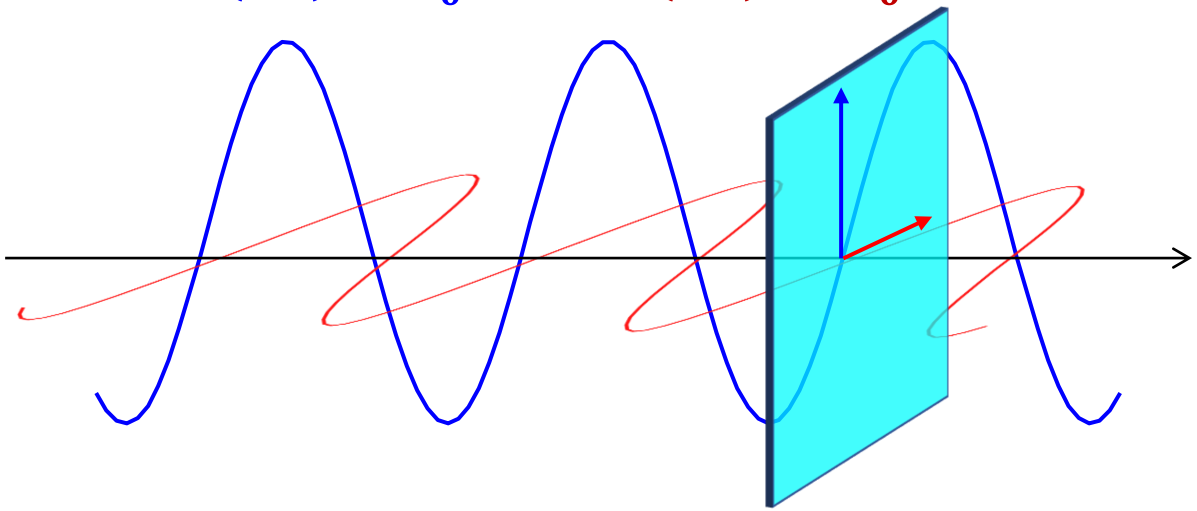
With: $\tilde{H}^*$ is the conjugate of $\tilde{H}$

III. Propagation of E.M Waves

9. Polarization of E.M wave

Let's consider a non attenuated plane E.M wave given by its electric and magnetic fields lying on the plane corresponding to the wave front, normal to the incidence direction (using space phasors):

$$\vec{E}(z,t) = \tilde{E}_0 \,.\, e^{\omega t};\ \vec{H}(z,t) = \tilde{H}_0 \,.\, e^{i\omega t}$$



In general, the electric field (and magnetic field) did not keep the same orientation on the wave plane, and it could vary with time and traces a curve by the tip of the field vector on the plane.

In such situation, the electric field (similarly the magnetic field), could be divided into two components on the wave front plane (x-y in this case) propagating in +z-direction, :

$$\tilde{E}(\mathrm{z}) = \tilde{E}_x(\mathrm{z})\vec{u}_x + \tilde{E}_y(\mathrm{z})\vec{u}_y$$

And we can set:

$$\tilde{E}_x(\mathrm{z}) = E_{x0}e^{-ikz};\tilde{E}_y(\mathrm{z}) = E_{y0}e^{-ikz}$$

III. Propagation of E.M Waves

9. Polarization of E.M wave

Both initial amplitudes E_{x0} and E_{y0} are in general complex numbers and could be written in exponential form:

$$E_{x0} = a_x e^{i\varphi_x}; E_{y0} = a_y e^{i\varphi_y}$$

With: $a_x = |E_{x0}| > 0; a_y = |E_{y0}| > 0$

Consequently, we can rewrite $\tilde{E}(z)$:

$$\tilde{E}(z) = a_x e^{-ikz} e^{i\varphi_x} \vec{u}_x + a_y e^{-ikz} e^{i\varphi_y} \vec{u}_y \rightarrow \tilde{E}(z) = e^{-ikz} e^{i\varphi_x} \left(a_x \vec{u}_x + a_y e^{i\varphi} \vec{u}_y \right)$$

With: $\varphi = \varphi_y - \varphi_x$ *called the phase difference between* $\tilde{E}_y(z)$ *and* $\tilde{E}_x(z)$

For the sake of simplicity, we can choose to take $\varphi_x = 0 \rightarrow \varphi = \varphi_y$: $\tilde{E}(z) = e^{-ikz} \left(a_x \vec{u}_x + a_y e^{i\varphi} \vec{u}_y \right)$

Taking the real part of the phasor, we will get the instantaneous electric field:

$$\vec{E}(z,t) = a_x . \cos(\omega t - kz)\, \vec{u}_x + a_y . \cos(\omega t - kz + \varphi)\, \vec{u}_x$$

III. Propagation of E.M Waves

9. Polarization of E.M wave

The specific cases of the E.M wave polarization could be discussed upon the values of phase difference φ, by analyzing the amplitude of $\vec{E}(z,t)$ and its direction:

The amplitude is given by:

$$\left|\vec{E}(z,t)\right| = \left[E_x^2(z,t) + E_y^2(z,t)\right]^{1/2}$$

$$= \left[a_x^2 cos^2(\omega t - kz) + a_y^2 cos^2(\omega t - kz + \varphi)\right]^{1/2}$$

The direction is dictated by the inclination angle:

$$\psi(z,t) = tan^{-1}\left(\frac{E_y(z,t)}{E_x(z,t)}\right)$$

a. Linear polarization $\varphi = 0 \; or \; \pi$:

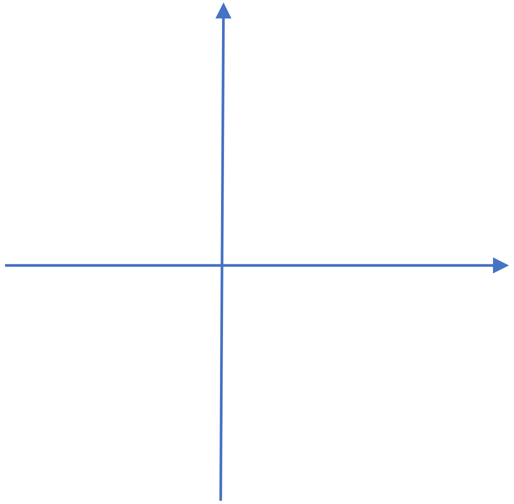
For $\varphi = 0$ (in-phase):

$$\vec{E}(z,t) = cos(\omega t - kz + \varphi)\left(a_x.\vec{u}_x + a_y.\vec{u}_x\right)$$

$$\left|\vec{E}(z,t)\right| = \left[a_x^2 + a_y^2\right]^{1/2}\left|cos(\omega t - kz)\right|$$

$$\psi(z,t) = tan^{-1}\left(\frac{a_y}{a_x}\right)$$

The amplitude is indeed function of z and t, whereas the direction is not (fixed direction).



III. Propagation of E.M Waves

9. Polarization of E.M wave

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$$= \left[a_x^2 cos^2(\omega t - kz) + a_y^2 cos^2(\omega t - kz + \varphi)\right]^{1/2}$$

The direction is dictated by the inclination angle:

$$\psi(z,t) = tan^{-1}\left(\frac{E_y(z,t)}{E_x(z,t)}\right)$$

a. Linear polarization $\varphi = 0 \; or \; \pi$:

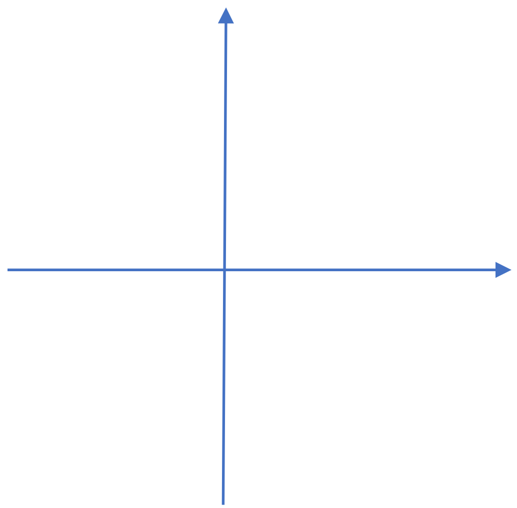
For $\varphi = \pi$ (out-phase):

$$\vec{E}(z,t) = cos(\omega t - kz + \varphi)\left(a_x.\vec{u}_x - a_y.\vec{u}_x\right)$$

$$\left|\vec{E}(z,t)\right| = \left[a_x^2 + a_y^2\right]^{1/2} |cos(\omega t - kz)|$$

$$\psi(z,t) = tan^{-1}\left(\frac{-a_y}{a_x}\right)$$

The amplitude is indeed function of z and t, whereas the direction is not (fixed direction).



III. Propagation of E.M Waves

9. Polarization of E.M wave

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The amplitude is given by:

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$$= \left[a_x^2 cos^2(\omega t - kz) + a_y^2 cos^2(\omega t - kz + \varphi)\right]^{1/2}$$

The direction is dictated by the inclination angle:

$$\psi(z,t) = tan^{-1}\left(\frac{E_y(z,t)}{E_x(z,t)}\right)$$

<u>b. Circular polarization</u> $\varphi = \pm \pi/2 \, , a_x = a_y = a$

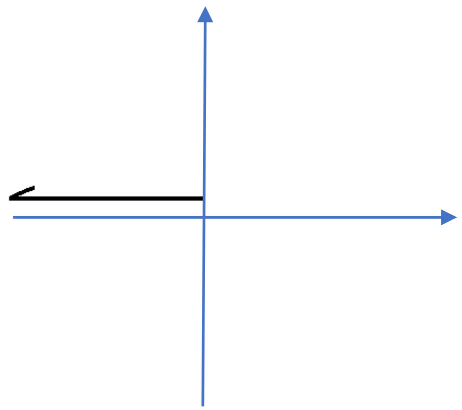
For $\varphi = \pi/2$ (Left Circular Polarization):

$$\vec{E}(z,t) = a(cos(\omega t - kz)\vec{u}_x - sin(\omega t - kz)\vec{u}_x)$$

$$\left|\vec{E}(z,t)\right| = a$$

$$\psi = tan^{-1}\left(\frac{-a.\,sin(\omega t - kz)}{a.\,cos(\omega t - kz)}\right) = -(\omega t - kz)$$

The direction is tracing a circular movement in counter-clockwise direction.



III. Propagation of E.M Waves

9. Polarization of E.M wave

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The amplitude is given by:

$$\left|\vec{E}(z,t)\right| = \left[E_x^2(z,t) + E_y^2(z,t)\right]^{1/2}$$

$$= \left[a_x^2 cos^2(\omega t - kz) + a_y^2 cos^2(\omega t - kz + \varphi)\right]^{1/2}$$

The direction is dictated by the inclination angle:

$$\psi(z,t) = tan^{-1}\left(\frac{E_y(z,t)}{E_x(z,t)}\right)$$

<u>b. Circular polarization</u> $\varphi = \pm \pi/_2, a_x = a_y = a$

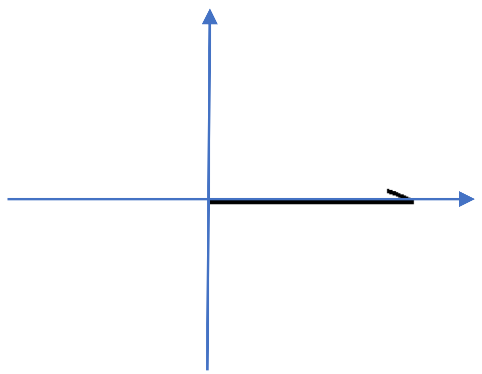
For $\varphi = -\pi/_2$ **(Right Circular Polarization):**

$$\vec{E}(z,t) = a(cos(\omega t - kz)\vec{u}_x + sin(\omega t - kz)\vec{u}_x)$$

$$\left|\vec{E}(z,t)\right| = a$$

$$\psi = tan^{-1}\left(\frac{a.sin(\omega t - kz)}{a.cos(\omega t - kz)}\right) = (\omega t - kz)$$

The direction is tracing a circular movement in counter-clockwise direction.



III. Propagation of E.M Waves

9. Polarization of E.M wave

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The amplitude is given by:

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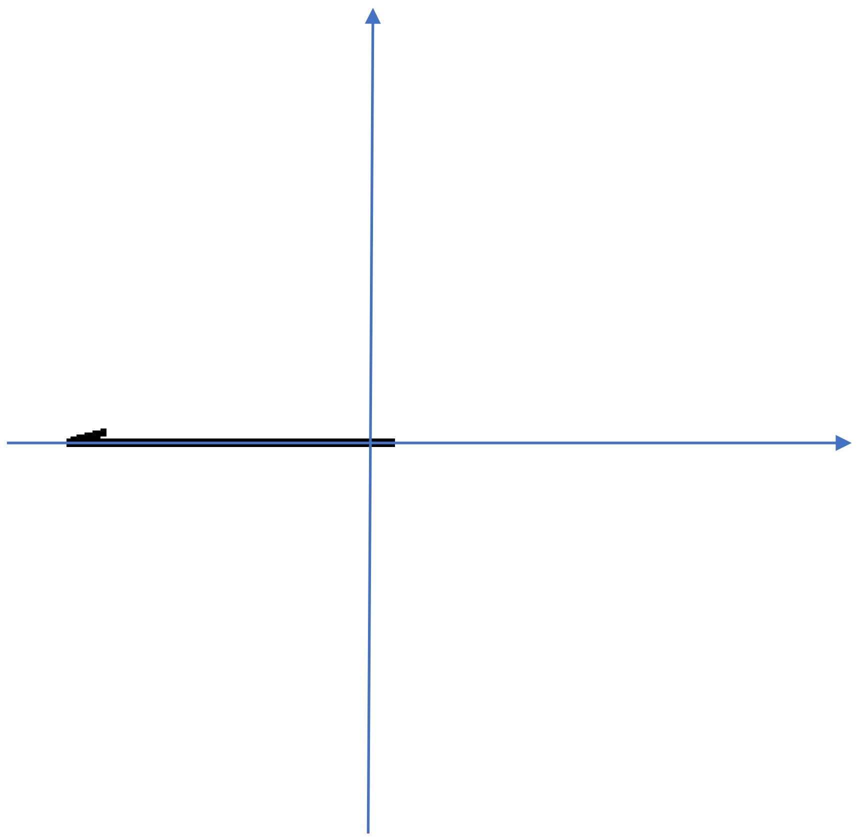
$$= \left[a_x^2 cos^2(\omega t - kz) + a_y^2 cos^2(\omega t - kz + \varphi)\right]^{1/2}$$

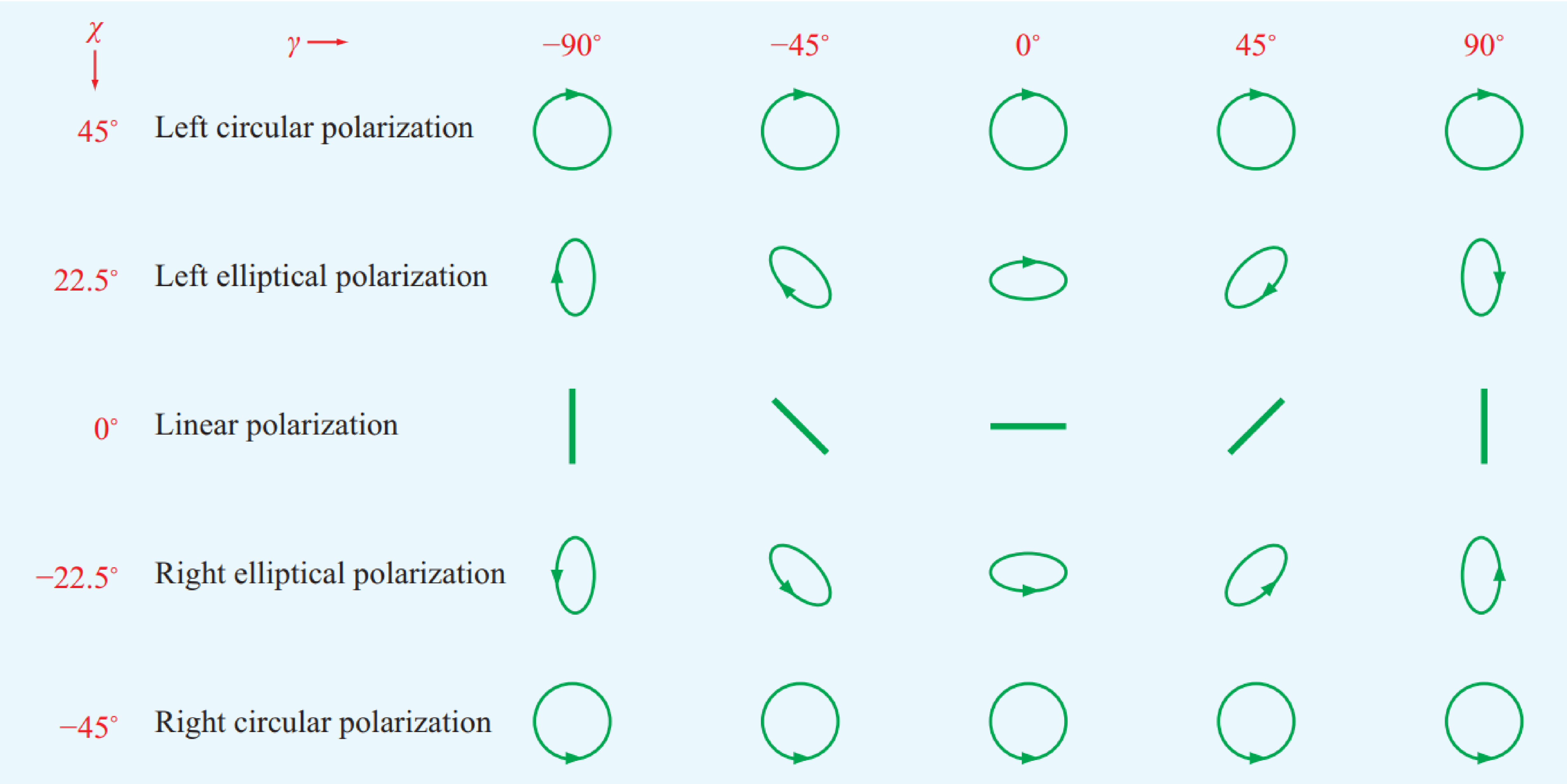
The direction is dictated by the inclination angle:

$$\psi(z,t) = tan^{-1}\left(\frac{E_y(z,t)}{E_x(z,t)}\right)$$

b. Elliptical polarization

$$0 < \varphi < {}^{\pi}/_{2}\,, a_x \neq a_y$$




χ
γ
−90°
−45°
0°
45°
90°
45°
Left circular polarization
22.5°
Left elliptical polarization
0°
Linear polarization
−22.5°
Right elliptical polarization
−45°
Right circular polarization

III. Propagation of E.M Waves

1. Normal incidence

When a travelling wave reaches an interface between two different regions, it is partly reflected and partly transmitted, with the magnitude of the two parts determined by the constants of the two regions:

Incident wave:

$$\tilde{E}^i(z) = E_0^i e^{-ik_1 z}\vec{u}_x;$$

$$\tilde{H}^i(z) = \frac{1}{\eta_1}\vec{u}_z \wedge \tilde{E}^i = \frac{E_0^i}{\eta_1} e^{-ik_1 z}\vec{u}_y$$

Reflected wave:

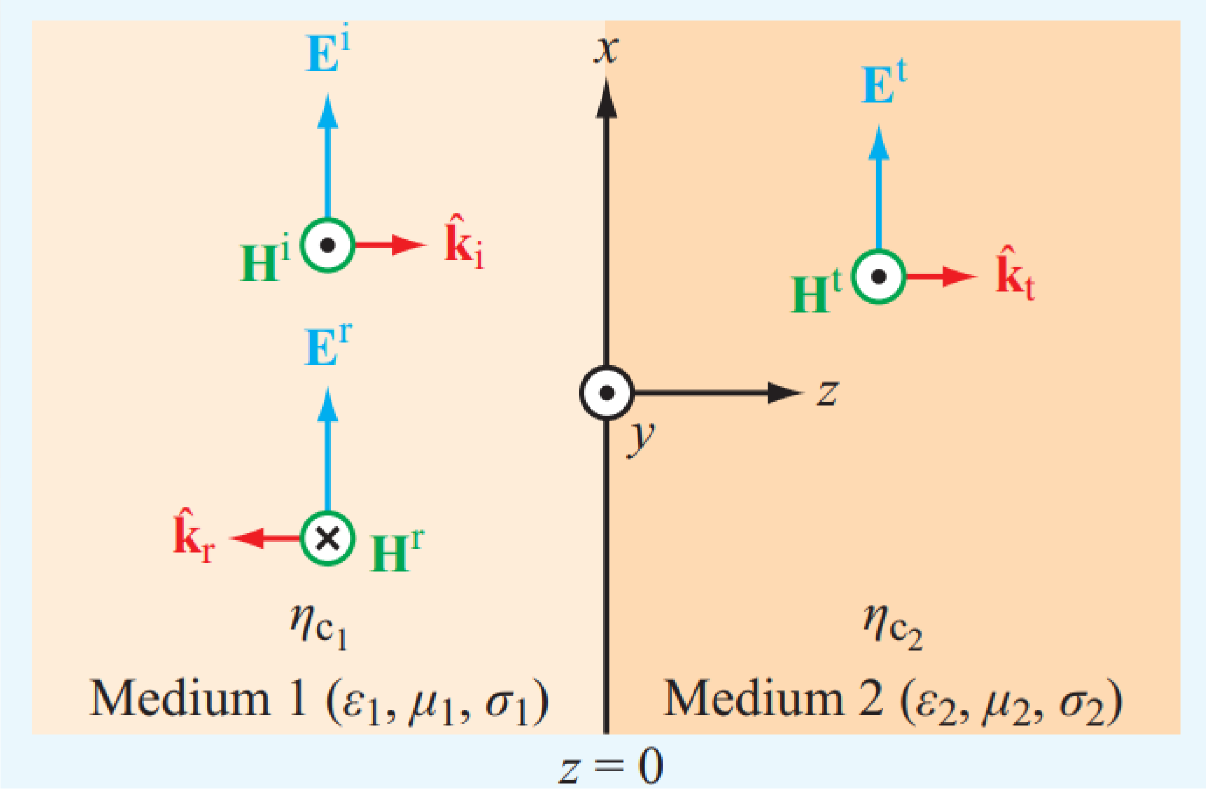
$$\tilde{E}^r(z) = E_0^r e^{ik_1 z}\vec{u}_x$$

$$\tilde{H}^r(z) = \frac{1}{\eta_1}(-\vec{u}_z) \wedge \tilde{E}^r = -\frac{E_0^r}{\eta_1} e^{ik_1 z}\vec{u}_y$$

Transmitted wave:

$$\tilde{E}^t(z) = E_0^t e^{-ik_2 z}\vec{u}_x$$

$$\tilde{H}^t(z) = \frac{1}{\eta_2}\vec{u}_z \wedge \tilde{E}^t = \frac{E_0^t}{\eta_2} e^{-ik_2 z}\vec{u}_y$$



1. Normal incidence

The total electric field $\tilde{E}_1(z)$ in medium 1 is the sum of the electric fields of the incident and reflected waves, and a similar statement applies to the magnetic field $\tilde{H}_1(z)$. Hence,

$$\tilde{E}_1(z) = \tilde{E}^i(z) + \tilde{E}^r(z) = \left(E_0^i e^{-ik_1 z} + E_0^r e^{ik_1 z}\right)\vec{u}_y$$

$$\tilde{H}_1(z) = \tilde{H}^i(z) + \tilde{H}^r(z) = \frac{\left(E_0^i e^{-ik_1 z} - E_0^r e^{ik_1 z}\right)}{\eta_1}\vec{u}_y$$

With only the transmitted wave present in medium 2, the total fields are

$$\tilde{E}_2(z) = \tilde{E}^t(z) = E_0^t e^{-ik_2 z}\vec{u}_x$$

$$\tilde{H}_1(z) = \tilde{H}^t(z) = \frac{E_0^t}{\eta_2} e^{-ik_2 z}\vec{u}_y$$

Assuming normal incidence, At the boundary ($z = 0$), the tangential components of the electric and magnetic fields are continuous. Hence,

$$\tilde{E}_1(0) = \tilde{E}_2(0) \rightarrow E_0^i + E_0^r = E_0^t$$

$$\tilde{H}_1(0) = \tilde{H}_2(0) \rightarrow \frac{E_0^i}{\eta_1} - \frac{E_0^r}{\eta_1} = \frac{E_0^t}{\eta_2}$$

Solving these equations for E_0^r and E_0^t in terms of E_0^i gives:

$$E_0^r = \left(\frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}\right) E_0^i = \Gamma E_0^i$$

$$E_0^t = \left(\frac{2\eta_2}{\eta_2 + \eta_1}\right) E_0^i = \tau E_0^i$$

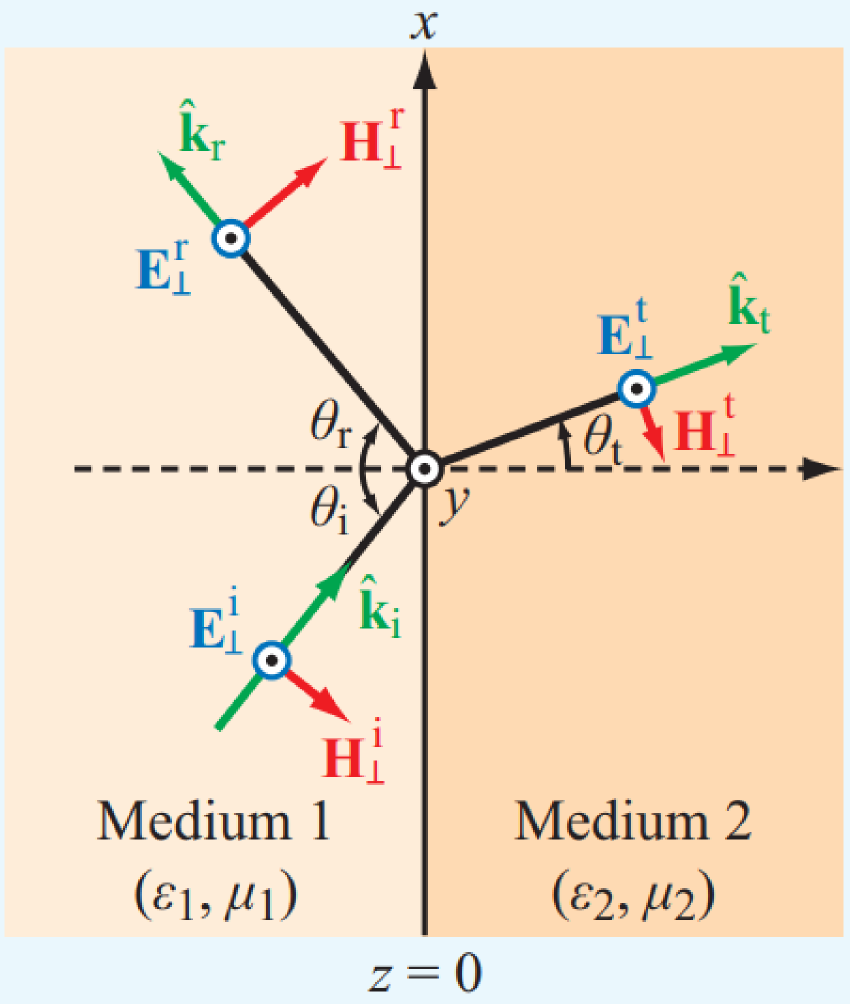
The quantities Γ and τ are called the reflection and transmission coefficients ($\tau = 1 + \Gamma$)

III. Propagation of E.M Waves

2. Oblique incidence

A wave of arbitrary polarization may be described as the superposition of two orthogonally polarized waves: one with its electric field parallel to the plane of incidence (*parallel polarization*) and the other with its electric field perpendicular to the plane of incidence (*perpendicular polarization*).

The perpendicular polarization where the plane of incidence is coincident with the $x-z$ plane $(y=0)$, is given with E perpendicular to the plane of incidence and it also called *transverse electric (TE)* polarization because E is perpendicular to the plane of incidence.



(a) Perpendicular polarization

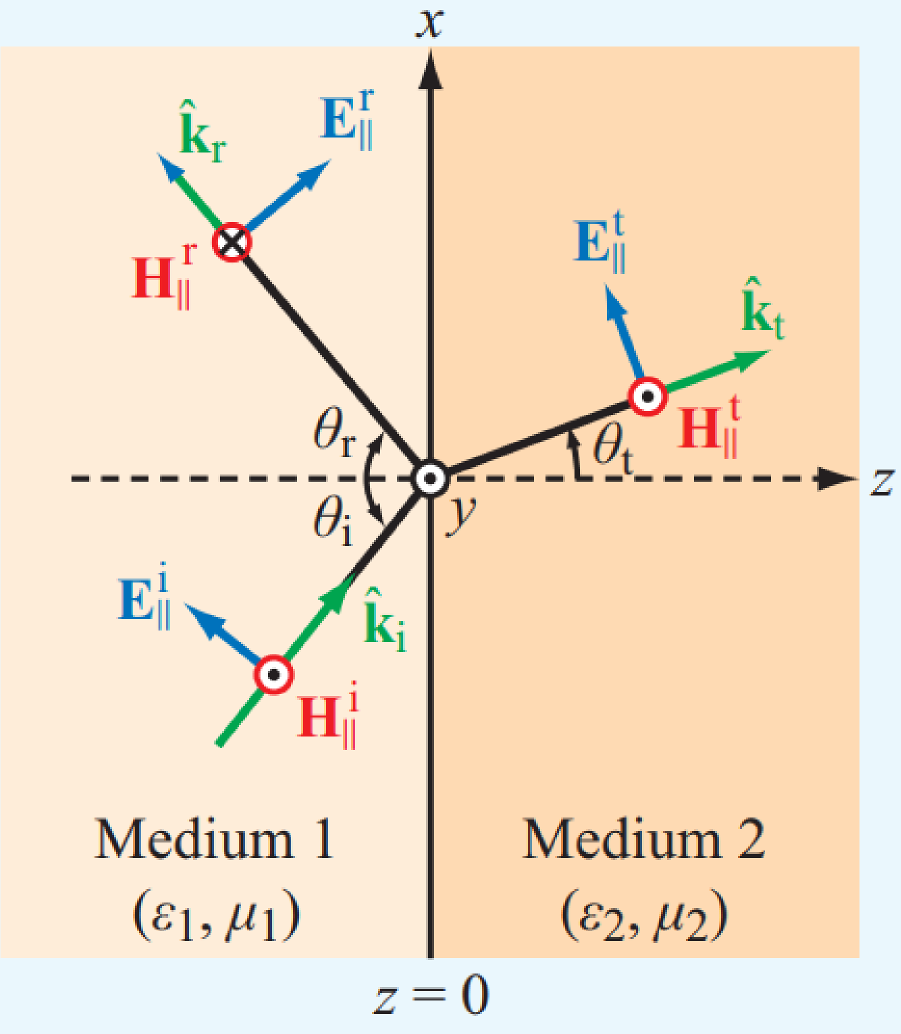
III. Propagation of E.M Waves

2. Oblique incidence

When E is parallel to the plane of incidence, the polarization is called *transverse magnetic (TM)* polarization because in that case it is the magnetic field that is perpendicular to the plane of incidence.

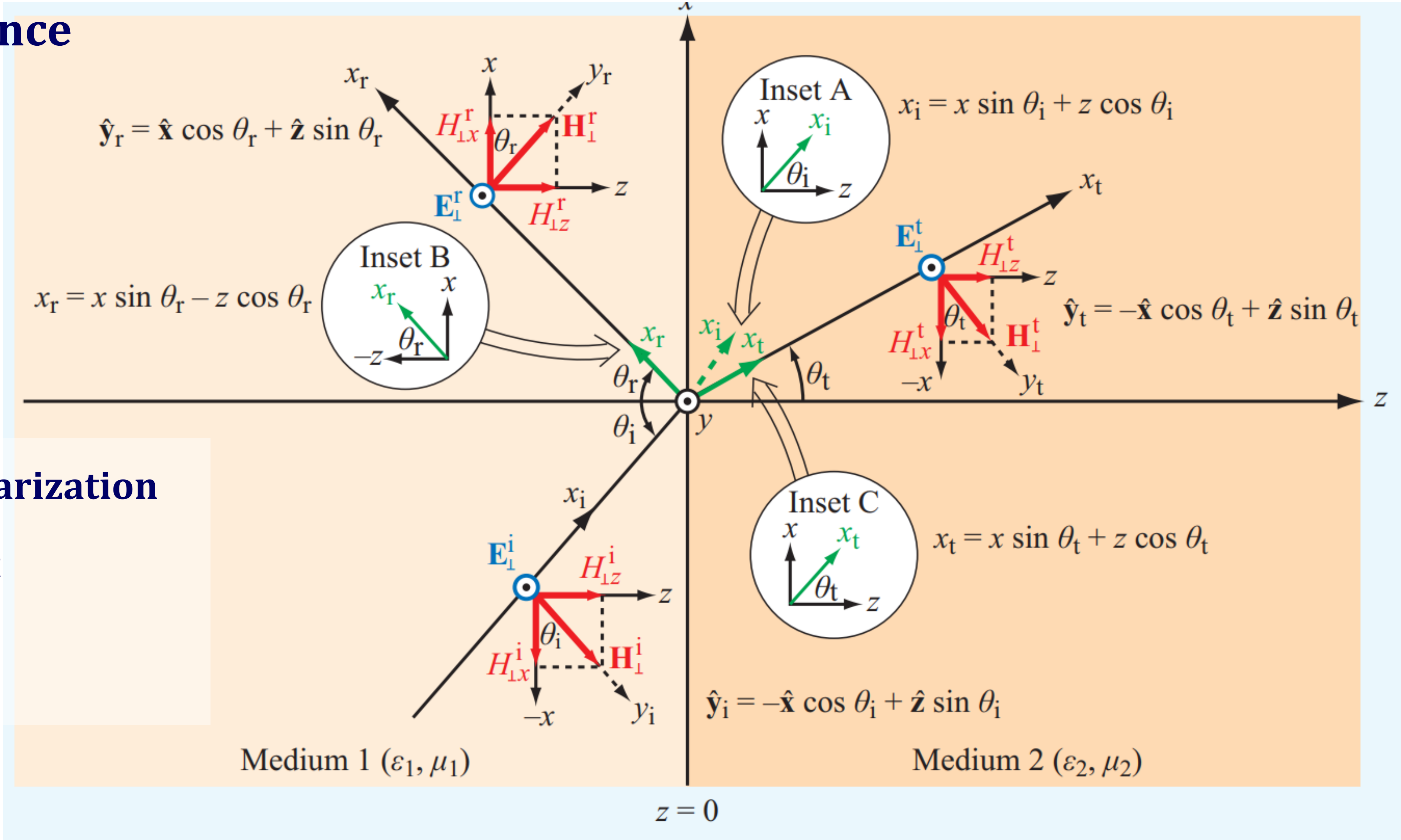
For the general case of a wave with an arbitrary polarization, it is common practice to decompose the incident wave $(\vec{E}^i, \vec{H}^i)$ into a perpendicularly polarized component $(\vec{E}^i_\perp, \vec{H}^i_\perp)$ and a parallel polarized component $(\vec{E}^i_\parallel, \vec{H}^i_\parallel)$.

Similar process is used to determine both reflected $(\vec{E}^r, \vec{H}^r)$ and transmitted $(\vec{E}^t, \vec{H}^t)$ waves.



(b) Parallel polarization

2. Oblique incidence



A. Perpendicular polarization

- Incidence angle θ_i
- Reflection angle θ_r
- Refraction angle θ_t

III. Propagation of E.M Waves

2. Oblique incidence

A. Perpendicular polarization

In this case, we will be interested in perpendicular components:

$$\tilde{E}^i_{\perp}(x,y) = E^i_{\perp 0} e^{-ik_1 x_i} \vec{u}_y$$

$$\tilde{H}^i_{\perp}(x,y) = \frac{E^i_{\perp 0}}{\eta_1} e^{-ik_1 x_i} \vec{u}_i$$

With:

$$x_i = x.\sin\theta_i + z.\cos\theta_i$$

$$\vec{u}_i = -\vec{u}_x.\cos\theta_i + \vec{u}_y.\sin\theta_i$$

The incident wave :

$$\tilde{E}^i_{\perp}(x,y) = E^i_{\perp 0} e^{-ik_1(x.\sin\theta_i + z.\cos\theta_i)} \vec{u}_y$$

$$\tilde{H}^i_{\perp}(x,y) = \frac{E^i_{\perp 0}}{\eta_1} e^{-ik_1(x.\sin\theta_i + z.\cos\theta_i)} \left(-\vec{u}_x.\cos\theta_i + \vec{u}_y.\sin\theta_i\right)$$

The reflected wave :

$$\tilde{E}^r_{\perp}(x,y) = E^r_{\perp 0} e^{-ik_1 x_r} \vec{u}_y = E^r_{\perp 0} e^{-ik_1(x.\sin\theta_r - z.\cos\theta_r)} \vec{u}_y$$

$$\tilde{H}^r_{\perp}(x,y) = \frac{E^r_{\perp 0}}{\eta_1} e^{-ik_1(x.\sin\theta_i - z.\cos\theta_i)} \left(\vec{u}_x.\cos\theta_i + \vec{u}_y.\sin\theta_i\right)$$

The transmitted wave :

$$\tilde{E}^t_{\perp}(x,y) = E^t_{\perp 0} e^{-ik_2 x_t} \vec{u}_y = E^t_{\perp 0} e^{-ik_2(x.\sin\theta_t + z.\cos\theta_t)} \vec{u}_y$$

$$\tilde{H}^t_{\perp}(x,y) = \frac{E^t_{\perp 0}}{\eta_2} e^{-ik_2(x.\sin\theta_i + z.\cos\theta_i)} \left(-\vec{u}_x.\cos\theta_i + \vec{u}_y.\sin\theta_i\right)$$

The following interface conditions are applied:

$$\left.\left(\tilde{E}^i_{\perp y} + \tilde{E}^r_{\perp y}\right)\right|_{z=0} = \left.\tilde{E}^t_{\perp y}\right|_{z=0}$$

$$\left.\left(\tilde{H}^i_{\perp x} + \tilde{H}^r_{\perp x}\right)\right|_{z=0} = \left.\tilde{H}^t_{\perp x}\right|_{z=0}$$

III. Propagation of E.M Waves

2. Oblique incidence

A. Perpendicular polarization

$$\left(\tilde{E}^{i}_{\perp y} + \tilde{E}^{r}_{\perp y}\right)\Big|_{z=0} = \tilde{E}^{t}_{\perp y}\Big|_{z=0} \rightarrow E^{i}_{\perp 0} e^{-ik_1(x.sin\theta_i)} + E^{r}_{\perp 0} e^{-ik_1(x.sin\theta_r)} = E^{t}_{\perp 0} e^{-ik_2(x.sin\theta_t)}$$

$$\left(\tilde{H}^{i}_{\perp x} + \tilde{H}^{r}_{\perp x}\right)\Big|_{z=0} = \tilde{H}^{t}_{\perp x}\Big|_{z=0} \rightarrow -cos\theta_i \frac{E^{i}_{\perp 0}}{\eta_1} e^{-ik_1(x.sin\theta_i)} + cos\theta_r \frac{E^{r}_{\perp 0}}{\eta_1} e^{-ik_1(x.sin\theta_r)} = cos\theta_t \frac{E^{t}_{\perp 0}}{\eta_2} e^{-ik_2(x.sin\theta_t)}$$

To satisfy the both equations for all possible values of x (i.e., all along the boundary), it follows that the arguments of all three exponentials must be equal. That is,

$$k_1(x.sin\theta_i) = k_1(x.sin\theta_r) = k_2(x.sin\theta_t) \leftrightarrow k_1 sin\theta_i = k_1 sin\theta_r = k_2 sin\theta_t$$

Which is known as the phase-matching condition.

The first equality leads to the Snell's law of reflection, while the second equality leads to the Snell's law of refraction:

$$\theta_i = \theta_r; \quad \frac{sin\theta_t}{sin\theta_i} = \frac{k_1}{k_2} = \frac{\omega\sqrt{\mu_1\varepsilon_1}}{\omega\sqrt{\mu_2\varepsilon_2}} = \frac{n_1}{n_2}$$

2. Oblique incidence

A. Perpendicular polarization

Using previous results, we get for both equations of E and H :

$$\left(\tilde{E}^i_{\perp y}+\tilde{E}^r_{\perp y}\right)\Big|_{z=0}=\tilde{E}^t_{\perp y}\Big|_{z=0}\rightarrow E^i_{\perp 0}+E^r_{\perp 0}=E^t_{\perp 0}$$

$$\left(\tilde{H}^i_{\perp x}+\tilde{H}^r_{\perp x}\right)\Big|_{z=0}=\tilde{H}^t_{\perp x}\Big|_{z=0}\rightarrow \cos\theta_i\left(-\frac{E^i_{\perp 0}}{\eta_1}+\frac{E^r_{\perp 0}}{\eta_1}\right)=\cos\theta_t\frac{E^t_{\perp 0}}{\eta_2}$$

These two equations can be solved simultaneously to yield the following expressions for the reflection and transmission coefficients in the perpendicular polarization case:

$$\Gamma_\perp=\frac{E^r_{\perp 0}}{E^i_{\perp 0}}=\frac{\eta_2\cos\theta_i-\eta_1\cos\theta_t}{\eta_2\cos\theta_i+\eta_1\cos\theta_t}$$

$$\tau_\perp=\frac{E^t_{\perp 0}}{E^i_{\perp 0}}=\frac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_i+\eta_1\cos\theta_t}$$

These two coefficients are known formally as the Fresnel reflection and transmission coefficients for perpendicular polarization and are related by :

$$\tau_\perp=1+\Gamma_\perp$$

If medium 2 is a perfect conductor ($\eta_2=0$), we get $\Gamma_\perp=-1$ and $\tau_\perp=0$, respectively, which means that the incident wave is totally reflected by the conducting medium.

III. Propagation of E.M Waves

B. Parallel polarization

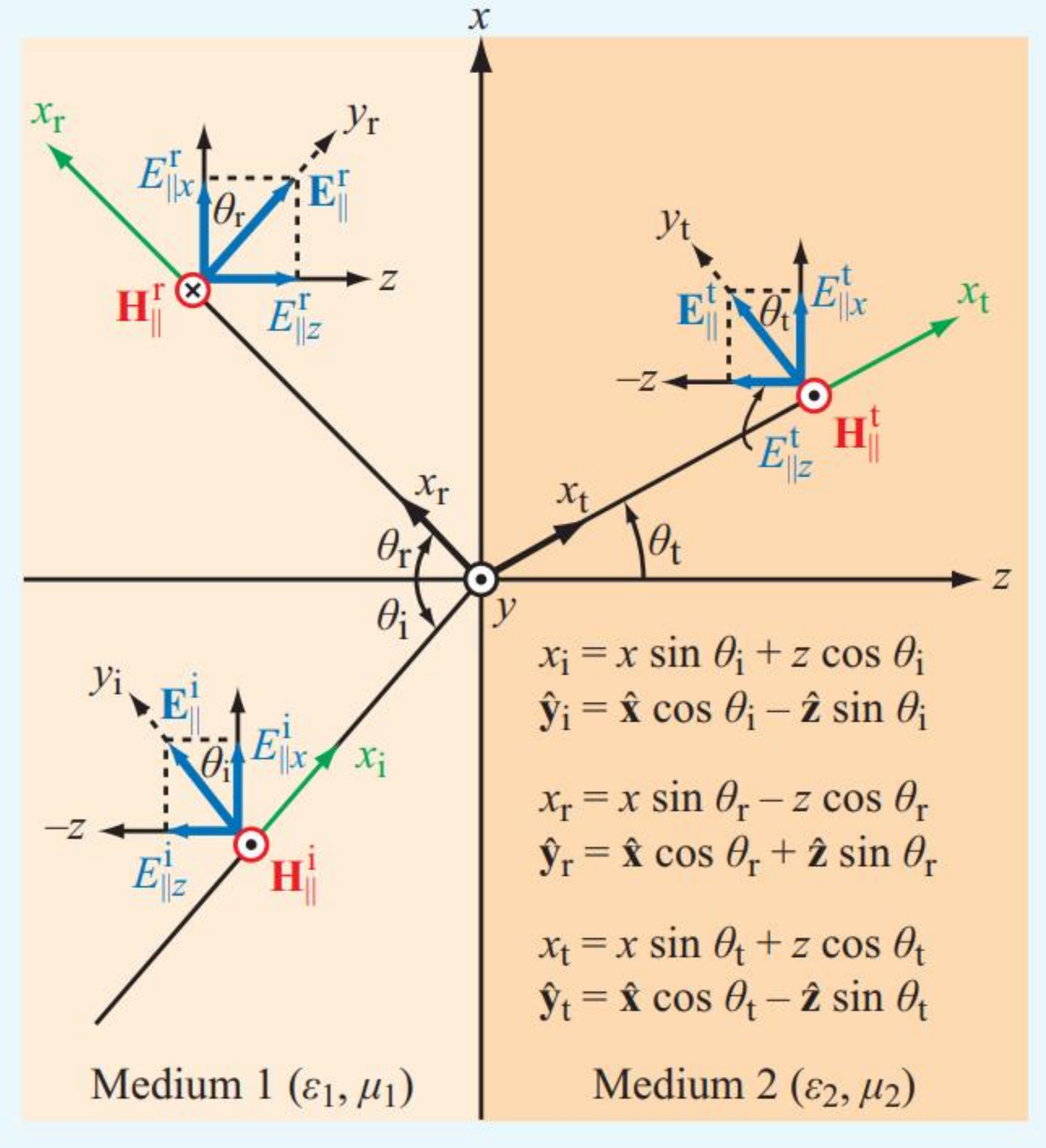
In this case the same reasoning and development are used for parallel polarization for all components of incident, reflected and transmitted wave, to obtain the Fresnel reflection and transmission coefficients for parallel polarization:

$$\Gamma_{\parallel} = \frac{E^{r}_{\parallel 0}}{E^{i}_{\parallel 0}} = \frac{\eta_2 \cos\theta_t - \eta_1 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i}$$

$$\tau_{\parallel} = \frac{E^{t}_{\parallel 0}}{E^{i}_{\parallel 0}} = \frac{2\eta_2 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i}$$

With the relation between both coefficients:

$$\tau_{\parallel} = (1 + \Gamma_{\parallel}) \frac{\cos\theta_i}{\cos\theta_t}$$



3. Brewster Angle

The Brewster angle θ_B is defined as the incidence angle θ_i at which the Fresnel reflection coefficient $\Gamma = 0$:

Perpendicular polarization:

$$\Gamma_\perp = 0 \rightarrow \eta_2 \cos\theta_i = \eta_1 \cos\theta_t$$

$$\sin\theta_i = \sin\theta_{\perp B} = \sqrt{\frac{1-(\mu_1\varepsilon_2/\mu_2\varepsilon_1)}{1-(\mu_1/\mu_2)^2}}$$

Parallel polarization:

$$\Gamma_\parallel = 0 \rightarrow \eta_2 \cos\theta_t = \eta_1 \cos\theta_i$$

$$\sin\theta_i = \sin\theta_{\parallel B} = \sqrt{\frac{1-(\mu_2\varepsilon_1/\mu_1\varepsilon_2)}{1-(\varepsilon_1/\varepsilon_2)^2}}$$