Lagrange formulation of quantum filed theory

1.1 Recall the formalism of Lagrange

Lagrange's formalism is an extremely powerful tool for describing the evolution of a physical problem. Initially approached in the form of the principle of least action, it allows to determine the behavior of a system as soon as the expression of a physical quantity, the Lagrangian, is known. The aim of this reminder is to review the fundamental concepts of Lagrangian theory, first in the context of studying a massive particle, and then in the field theory.

1.1.1 Principle of least action

Given an initial state, a physical system has an infinite number of ways to evolve towards a final state:

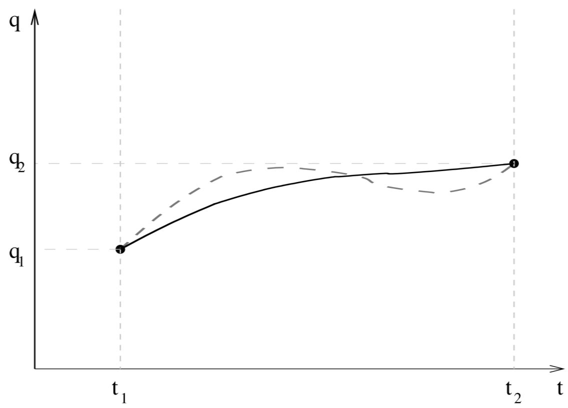

Figure 1.1: Conversion in the space of generalized coordinates

Therefore, during a real transformation, only one of these changes (evolutions) is actually carried out. How can we determine this preferred evolution and differentiate it from the others? This

question is answered by the principle of least action, which can be considered as one of the postulates of physics.
According to the principle of least action, there exists a quantity called "Action" defined by,

$$S[q] = \int_{t_1}^{t_2} dt\ L(q_i(t), \dot{q}_i(t), t) \qquad , \qquad i = 1 \longrightarrow N \tag{1.1}$$

The value of the system changes during its evolution and must remain minimal throughout the actual transformation. The action S is defined as the integral of a quantity known as the "Lagrangian," which is a function of the generalized coordinates q and the generalized velocities $\dot{q}(t) = \frac{dq}{dt}$.

1.1.2 Euler-Lagrange equations

Among all the paths that connect the two fixed points ($\delta q(t_1) = \delta q(t_2) = 0$) with generalized coordinates $Q_1 = q(t_1)$ and $Q_2 = q(t_2)$, the physical trajectories are those that minimize the action S, such that $\Delta S \simeq 0$.

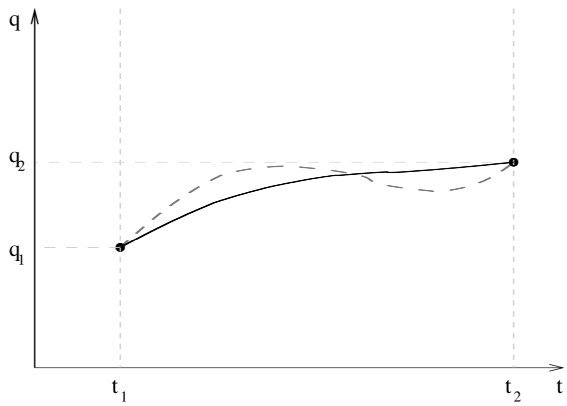

Figure 1.2: Transformation in the space of generalized coordinates

In case $\delta(q(t))$ is a infinitesimal function, then,

$$\Delta S[q] \simeq S(q + \delta q) - S(q) \tag{1.2}$$

On a

$$S[q] = \int_{t_1}^{t_2} dt\ L(q, \dot{q}, t) \quad \Longrightarrow \quad \Delta S[q] = \int_{t_1}^{t_2} dt[\ L(q + \delta q, \dot{q} + \delta \dot{q}, t) - \ L(q, \dot{q}, t)] \tag{1.3}$$

Or,

$$L(q+\delta q,\dot{q}+\delta\dot{q},t)=L(q,\dot{q},t)+\frac{\partial L(q,\dot{q},t)}{\partial q}\delta q+\frac{\partial L(q,\dot{q},t)}{\partial \dot{q}}\delta\dot{q} \tag{1.4}$$

Therefore,

$$\begin{aligned}\Delta S[q] &= \int_{t_1}^{t_2} dt\left[L(q,\dot{q},t)+\frac{\partial L(q,\dot{q},t)}{\partial q}\delta q+\frac{\partial L(q,\dot{q},t)}{\partial \dot{q}}\delta\dot{q}-\ L(q,\dot{q},t)\right]\\ &= \int_{t_1}^{t_2} dt\left[\frac{\partial L(q,\dot{q},t)}{\partial q}\delta q+\frac{\partial L(q,\dot{q},t)}{\partial \dot{q}}\delta\dot{q}\right]\simeq 0\end{aligned} \tag{1.5}$$

If we set that

$$\delta\dot{q}=\frac{d}{dt}(\delta q) \quad\Longrightarrow\quad \frac{\partial L(q,\dot{q},t)}{\partial \dot{q}}\delta\dot{q}=\frac{\partial L}{\partial \dot{q}}\frac{d}{dt}(\delta q) \tag{1.6}$$

We have also,

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}}\delta q\right]=\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}}\right]\delta q+\frac{\partial L}{\partial \dot{q}}\frac{d}{dt}(\delta q) \quad\Longrightarrow\quad \frac{\partial L}{\partial \dot{q}}\frac{d}{dt}(\delta q)=\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}}\delta q\right]-\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}}\right]\delta q \tag{1.7}$$

By substituting into equation (1.5), we find,

$$\begin{aligned}\Delta S[q] &= \int_{t_1}^{t_2} dt\left[\frac{\partial L}{\partial q}\delta q+\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}}\delta q\right]-\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}}\right]\delta q\right]\\ &= \int_{t_1}^{t_2} dt\delta q\left[\frac{\partial L}{\partial q}-\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}}\right]\right]+\int_{t_1}^{t_2} dt\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}}\delta q\right]\simeq 0\end{aligned} \tag{1.8}$$

where

$$\int_{t_1}^{t_2} dt\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}}\delta q\right]=\int_{t_1}^{t_2} d\left[\frac{\partial L}{\partial \dot{q}}\delta q\right]=\left[\frac{\partial L}{\partial \dot{q}}\delta q\right]=0 \tag{1.9}$$

Finally, the Euler-Lagrange equations are expressed as

$$\frac{\partial L}{\partial q_i}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)=0 \tag{1.10}$$

1.1.3 Lagrangian choice

The choice of the Lagrangian is not unique.

- If we replace the Lagrangian L with (αL), where α is a real number, then the equations of motion remain unchanged.

– If we replace the Lagrangian L with $(\beta + L)$, where β is a constant, then the equations of motion remain unchanged.

– If we replace the Lagrangian L with $(L + \frac{dF}{dt})$, where $F = F(q, \dot{q}, t)$ is a function, then the equations of motion remain unchanged.

Exercice 1 :
Show that the variation ΔS remains invariant under the change of the Lagrangian L to $L + \frac{dF}{dt}$.

1.1.4 Hamiltonian formulation

The Hamiltonian H is given by

$$H(p, q, t) = P_i \dot{q}_i - L \tag{1.11}$$

The generalized momentum is given by

$$P_i = \frac{\partial L}{\partial \dot{q}_i} \tag{1.12}$$

Exercice 2 :
Show that if the Lagrangian L does not explicitly depend on time t, then $\frac{dH}{dt} = 0$.
Solution 3:

$$\frac{dH}{dt} = p\frac{\partial \dot{q}}{\partial t} + \dot{q}\frac{\partial p}{\partial t} - \frac{\partial L}{\partial q}\frac{\partial q}{\partial t} - \frac{\partial L}{\partial \dot{q}}\frac{\partial \dot{q}}{\partial t} \tag{1.13}$$

Or, we have

$$p = \frac{\partial L}{\partial \dot{q}} \quad \text{et} \quad \frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0 \tag{1.14}$$

Therefore,

$$\frac{dH}{dt} = \left(\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q}\right)\frac{\partial q}{\partial t} = -\left(\frac{\partial L}{\partial q} - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}}\right)\right)\frac{\partial q}{\partial t} = 0 \tag{1.15}$$

1.2 Basic principle of quantum field theory

In a general case, a scalar field is associated with each particle that possesses zero spin. To characterize N particles, one defines N scalar fields. Consequently, the system comprising these N fields will be represented by a Lagrangian density of the following form,

$$\mathcal{L} = \mathcal{L}\left(\phi_1, \partial_\mu\phi_1, \phi_2, \partial_\mu\phi_2 \ldots \phi_N, \partial_\mu\phi_N, x_\mu\right) = \mathcal{L}\left(\phi_i, \partial_\mu\phi_i, x_\mu\right) \text{ avec } i = 1 \longrightarrow N \tag{1.16}$$

The motion of these N scalar fields will be described by the following N Euler-Lagrange equations,

$$\frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \right) = 0 \tag{1.17}$$

It is said that the scalar field $\phi(x_\mu)$ is a system with N degrees of freedom. According to its definition, the scalar field represents the most straightforward scenario. Its transformation occurs as follows,

$$\phi(x_\mu) = \phi'(x'_\mu) \tag{1.18}$$

- The scalar field (Klein-Gordon field) is used to describe the physics of zero-spin particles with relativistic speeds c.
- The scalar field can either be real $\phi(x_\mu) = \phi^*(x_\mu)$, or complex $\phi(x_\mu) \neq \phi^*(x_\mu)$.

1.2.1 Free scalar field

One possible form of the Lagrangian density that must be chosen to obtain the free Klein-Gordon equation is given by the following equation.

$$\left(\partial_\mu \partial_\mu - m^2\right) \phi(x_\mu) = 0 \tag{1.19}$$

Response: The selection is not singular. Our choice is as follows,

$$\mathcal{L}(\phi, \partial_\mu \phi, x_\mu) = -\frac{1}{2} \left(\partial_\mu \phi\right)^2 - \frac{1}{2} m^2 \phi^2 \tag{1.20}$$

Verification: Let us replace in the Euler-Lagrange equations, where $\phi_i = \phi = \phi^*$,

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = 0 \tag{1.21}$$

with $\dfrac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi, \quad \dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = -\partial_\mu \phi.$
By substituting into equation (1.21), we obtain the Klein-Gordon equation

$$\left(\partial_\mu \partial_\mu - m^2\right) \phi(x_\mu) = 0 \tag{1.22}$$

1.2.2 Free complex scalar field

If $\phi = \phi^*$, what is the general form of the Lagrangian density that must be selected in order to obtain the following two equations?

$$\left(\partial_\mu \partial_\mu - m^2\right) \phi(x_\mu) = 0 \ , \ \left(\partial_\mu \partial_\mu - m^2\right) \phi^*(x_\mu) = 0 \tag{1.23}$$

Response: Our choice is the following

$$\mathcal{L}(\phi, \partial_\mu \phi, \phi^*, \partial_\mu \phi^*, x_\mu) = -\left(\partial_\mu \phi\right)\left(\partial_\mu \phi^*\right) - m^2 \phi \phi^* \tag{1.24}$$

Verification: Let's substitute in both Euler-Lagrange equations for $\phi_i = \phi$, ϕ^*,

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right) = 0, \quad \frac{\partial \mathcal{L}}{\partial \phi^*} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)}\right) = 0 \tag{1.25}$$

with $\dfrac{\partial \mathcal{L}}{\partial \phi^*} = -m^2 \phi$, $\dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} = -\partial_\mu \phi$, $\dfrac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi^*$, $\dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = -\partial_\mu \phi^*$.
By substituting into equation (1.25), we obtain the following two equations,

$$\left(\partial_\mu \partial_\mu - m^2\right) \phi(x_\mu) = 0 \ , \ \left(\partial_\mu \partial_\mu - m^2\right) \phi^*(x_\mu) = 0 \tag{1.26}$$

1.2.3 Complex scalar field in the presence of an external electromagnetic field

What is the general form of the Lagrangian density that must be chosen to satisfy the following two equations?

$$\left[\left(\partial_\mu - iqA_\mu\right)\left(\partial_\mu - iqA_\mu\right) - m^2\right] \phi(x_\mu) = 0 \tag{1.27}$$

$$\left[\left(\partial_\mu + iqA_\mu\right)\left(\partial_\mu + iqA_\mu\right) - m^2\right] \phi^*(x_\mu) = 0 \tag{1.28}$$

Response: Our choice is as follows,

$$\mathcal{L}(\phi, \partial_\mu \phi, \phi^*, \partial_\mu \phi^*, x_\mu) = -\left(\partial_\mu + iqA_\mu\right) \phi^* \left(\partial_\mu - iqA_\mu\right) \phi - m^2 \phi \phi^* \tag{1.29}$$

1.2.4 Remark

- The complex Klein-Gordon field is equivalent to two real scalar fields ϕ_1 and ϕ_2. The latter is given by,

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2), \;\; \phi^* = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2) \tag{1.30}$$

- The Lagrangian density of the scalar field in the presence of an external electromagnetic field A_μ? can be written in the following form,

$$\mathcal{L}(\phi, \partial_\mu\phi, \phi^*, \partial_\mu\phi^*, x_\mu) = \mathcal{L}_o + \mathcal{L}_I \tag{1.31}$$

Note that $\mathcal{L}_o$? represents the Lagrangian density of the free complex scalar field. The latter is the sum of the kinetic term $(\left(\partial_\mu\phi^*\right)\left(\partial_\mu\phi\right)$ and the mass term $(m^2\phi\phi^*)$.

$$\mathcal{L}_o = -\left(\partial_\mu\phi^*\right)\left(\partial_\mu\phi\right) - m^2\phi\phi^* \tag{1.32}$$

While $\mathcal{L}_I$? represents the Lagrangian density due to the interaction of the complex scalar field (ϕ, ϕ^*) with the external electromagnetic field A_μ?.

$$\mathcal{L}_I = -iqA_\mu\phi^*\left(\partial_\mu - iqA_\mu\right)\phi + iqA_\mu\phi\left(\partial_\mu + iqA_\mu\right)\phi^* \tag{1.33}$$