Exercises

Exercice 8 :

1. Provide the explicit forms of the following Dirac matrices: γ^1, γ^2, γ^3, and γ^4.
2. Demonstrate that

$$(\gamma^\mu)^+ = \gamma^\mu, \quad (\gamma^\mu)^1 = 1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \{\gamma^\mu, \gamma^\nu\} = 2\delta_{\mu\nu} \tag{6.1}$$

Exercice 9 :

1^o/ In the presence of an external electromagnetic field A_μ, the dynamics of a relativistic particle with charge q, mass m, and non-zero spin can be described by the following Lagrangian density of the spinor field.

$$\mathcal{L}_2 = -\bar{\psi}(\partial\!\!\!/ - iq \; A\!\!\!/ + m)\psi = -\psi^\dagger \gamma^4 (\partial_\mu \gamma^\mu - iqA_\mu \gamma^\mu + m)\psi$$

1. Derive the equations of motion by utilizing the Euler-Lagrange equations.

2^o/ In the absence of the electromagnetic field, the dynamics of a free particle can be described by

$$\mathcal{L}_3 = -\bar{\psi} \; \partial\!\!\!/ \psi = -\psi^\dagger \gamma^4 \partial_\mu \gamma^\mu \psi$$

1. Demonstrate that this Lagrangian density remains invariant under the following phase transformation:

$$\begin{cases} \psi(x) \longrightarrow \psi'(x) = e^{-i\theta\gamma^5}\psi(x) \\ \bar{\psi}(x) \longrightarrow \bar{\psi}'(x) = \bar{\psi}(x)e^{-i\theta\gamma^5} \end{cases}, \quad \theta \text{ is a constant.}$$

   where $\gamma^5 = \gamma^1\gamma^2\gamma^3\gamma^4$ and verify the following relations: $\{\gamma^5, \gamma^\mu\} = 0$, $\left(\gamma^5\right)^\dagger = \gamma^5$ et $(\gamma^5)^2 = 1$.

2. Employ the Noether Theorem to identify the conserved quantities associated with this transformation.

3^o / If we set:

$$\begin{cases} \psi_L(x) = \left(\frac{1+\gamma^5}{2}\right)\psi(x) \\ \psi_R(x) = \left(\frac{1-\gamma^5}{2}\right)\psi(x) \end{cases}$$

1. Rewrite the expression for the Lagrangian density $\mathcal{L}_3$ in terms of ψ_L and ψ_R.
2. Examine the invariance of the Lagrangian density $\mathcal{L}_3$ under the following phase transformation.

$$\begin{cases} \psi_L(x) \longrightarrow \psi_L^{'}(x) = \psi_L(x)e^{-i\alpha} \\ \psi_R(x) \longrightarrow \psi_R^{'}(x) = \psi_R(x) \quad , \quad \alpha \text{ is a constant.} \end{cases}$$