Dirac equation

We will now attempt to develop a relativistic theory for particles with non-zero spin. Initially, we will consider a scenario in which the electromagnetic field is not taken into account.

5.1 Dirac's Hamiltonian

To prevent the use of particles with negative energies, as was the case with the Hamiltonian (the total energy) from which the Klein-Gordon equation for a free particle was derived, Paul Dirac suggested in 1928 that the general form of the Hamiltonian be expressed as follows:

$$H_{Dirac} = \overrightarrow{\alpha}.\overrightarrow{p}c + \beta mc^2 = \sum_{i=1}^{3} \alpha_i.p_i c + \beta mc^2 = \alpha_i.p_i c + \beta mc^2 \tag{5.1}$$

where the coefficients β and α_i are constants that do not commute.

- We are seeking the values of these two constants.

By calculating the square of the Dirac Hamiltonian H^2_{Dirac}, one arrives at the following expression

$$H^2 = \left(\alpha_i.p_i c + \beta mc^2\right)\left(\alpha_j.p_j c + \beta mc^2\right) = \overrightarrow{p}^2 c^2 + m^2 c^4 \tag{5.2}$$

$$H^2 = p_i p_j \alpha_i \alpha_j c^2 + \beta^2 mc^2 c^4 + mc^3 p_i \left(\beta\alpha_j + \alpha_j \beta\right) = \overrightarrow{p}^2 c^2 + m^2 c^4 \tag{5.3}$$

- It is observed through comparison that

$$\beta^2 = 1 \Longrightarrow \beta\beta^{-1} = 1 \Longrightarrow \beta = \beta^{-1} \tag{5.4}$$

$$\beta\alpha_j + \alpha_j \beta = 0 \tag{5.5}$$

$$p_i p_j \alpha_i \alpha_j = p^2 \tag{5.6}$$

for $i = j = 1, 2, 3$ we can get:

$$p_i p_j \alpha_i \alpha_j = p_1^2 \alpha_1^2 + p_2^2 \alpha_2^2 + p_1 p_2 (\alpha_1 \alpha_2 + \alpha_2 \alpha_1) + p_1 p_3 (\alpha_1 \alpha_3 + \alpha_3 \alpha_1) + p_2 p_3 (\alpha_2 \alpha_3 + \alpha_3 \alpha_2) \tag{5.7}$$

$$p_i p_j \alpha_i \alpha_j = p_1^2 + p_2^2 + p_3^2 \tag{5.8}$$

For (5.7) to be equal to (5.8), it is necessary that

$$\alpha_1^2 = \alpha_2^2 = \alpha_3^2 = 1 \tag{5.9}$$

$$\alpha_1 \alpha_2 + \alpha_2 \alpha_1 = \alpha_1 \alpha_3 + \alpha_3 \alpha_1 = \alpha_2 \alpha_3 + \alpha_3 \alpha_2 = 0 \tag{5.10}$$

Therefor, if we suppose that $\alpha_i^2 = 1$ où $i = 1, 2, 3$ then

$$\{\alpha_i, \alpha_j\} = 2\delta_{ij} \tag{5.11}$$

In this context, $\{A, B\} = AB + BA$ represents the anti-commutator of the two quantities A and B. Finally, the dimensionless constants α_i and β satisfy the following anti-commutation relations

$$\beta^2 = 1 \tag{5.12}$$

$$\{\beta, \alpha_i\} = 0 \tag{5.13}$$

$$\{\alpha_i, \alpha_j\} = 2\delta_{ij} \tag{5.14}$$

$$\alpha_i^2 = 1 \tag{5.15}$$

(5.16)

Therefor,

$$\alpha_1^2 = \alpha_2^2 = \alpha_3^2 = \beta^2 = 1, \tag{5.17}$$

$$\{\alpha_1, \alpha_2\} = \{\alpha_1, \alpha_3\} = \{\alpha_2, \alpha_3\} = \{\beta, \alpha_1\} = \{\beta, \alpha_2\} = \{\beta, \alpha_3\} = 0, \tag{5.18}$$

5.2 The characteristics of Dirac matrices

Prior to formulating the Dirac equation that describes particles with non-zero spin, it is essential to ascertain the order of the matrices present in the expression of the Dirac Hamiltonian. Establishing the order of the matrices β and α_i will facilitate the determination of the number of components in the spinor that characterizes the state of such a particle in the relativistic context. To achieve this:

1. The eigenvalues of matrices are determined. $\beta\,,\alpha_i : i = 1,2,3$.
   The eigenvalue equation, pertaining to β (and similarly to the α_i), is expressed in the following form.
$$\beta \overrightarrow{X} = \lambda \overrightarrow{X}.$$
   A second application of β (or the α_i) yields, taking into account (**??**):
$$\begin{array}{rcl} \beta^2 \overrightarrow{X} = \lambda \beta \overrightarrow{X} & \Rightarrow & 1.\overrightarrow{X} = \lambda^2 \overrightarrow{X} \\ \lambda^2 = 1 & \Rightarrow & \lambda = \pm 1. \end{array}$$
   Therefore, the eigenvalues of the matrices β and α_i are either $+1$ or -1.
2. It is subsequently demonstrated that the traces $Tr(\beta) = Tr(\alpha_i) = 0$. To achieve this, we will utilize, on one hand, the anti-commutation of the matrices in question, and on the other hand, the well-known properties.
$$\begin{aligned} Tr(A\,B) &= Tr(B\,A), \\ Tr(\lambda\,A) &= \lambda\,Tr(A). \end{aligned} \tag{5.19}$$
   Indeed,
$$\begin{aligned} Tr(\alpha_i) &= Tr(1.\alpha_i) = Tr(\beta^2\,\alpha_i) = Tr\big[\beta(\beta\,\alpha_i)\big] = Tr\big[\beta(-\alpha_i\,\beta)\big] \\ &= -Tr\big[\beta(\alpha_i\,\beta)\big] = -Tr\big[(\alpha_i\,\beta)\beta\big] = -Tr\big[\alpha_i\,\beta^2\big] \\ &= -Tr(\alpha_i) \\ &\Rightarrow \quad Tr(\alpha_i) = 0. \end{aligned} \tag{5.20}$$
   A similar demonstration can be conducted to illustrate that $Tr(\beta) = 0$.
3. We will utilize the property that Hermitian matrices M are diagonalizable, meaning there

exists an invertible matrix S such that.

$$S\,M\,S^{-1} = M_D = \begin{pmatrix} \lambda_1 & ... & 0 & ... & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & ... & \lambda_i & ... & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & ... & 0 & ... & \lambda_n \end{pmatrix}, \tag{5.21}$$

where the λ_i represent the eigenvalues of M. Additionally, the equality of the traces of the two matrices M and M_D is also utilized. Indeed,

$$Tr(M) = Tr\big[S^{-1}(M_D S)\big] = Tr\big[(M_D S)S^{-1}\big] = Tr(M_D) \tag{5.22}$$

Since the matrices β and α_i are Hermitian, it is possible to apply the aforementioned properties, which can be expressed in the context of β and α_i as follows

$$\begin{aligned} Tr(\beta) = Tr(\alpha_i) = 0 \qquad &\Rightarrow \qquad Tr(\beta_D) = Tr\big[(\alpha_i)_D\big] = 0 \\ &\Rightarrow \qquad \sum_{i=1}^{n} \lambda_i = 0 \\ &\Rightarrow \qquad \underbrace{(1+1-1+..-1+1)}_{\text{n termes}} = 0. \end{aligned}$$

In order to achieve a sum of zero, it is necessary for the +1 and -1 values to completely offset each other. This condition is met only when the dimensions of the matrices β_D, $(\alpha_i)_D$, or alternatively, β and α_i, are even, specifically when $n = 2p$.

For $n = 2$, A basis for the complex matrices $M_{2\times 2}$ consists of the set of Pauli matrices, along with the identity matrix $\{\sigma_1, \sigma_2, \sigma_3, 1\}$. In this scenario, there is no solution, as equating the α_i with the σ_i necessitates that $\beta = 1$. However, β has a trace that differs from 1 ($Tr(1) = 2$), which is contradictory.

for $n = 4$, Solutions do exist. They can be expressed in standard representation in the following form.

$$\overrightarrow{\alpha} = \begin{pmatrix} 0 & \overrightarrow{\sigma} \\ \overrightarrow{\sigma} & 0 \end{pmatrix} \quad , \quad \beta = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}, \tag{5.23}$$

where $\mathbf{1}$ represents the identity matrix of size (2×2) and $\overrightarrow{\sigma} = \overrightarrow{e_1}\,\sigma_1 + \overrightarrow{e_2}\,\sigma_2 + \overrightarrow{e_3}\,\sigma_3$. The three Pauli matrices are defined as follows

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5.24}$$

In conclusion, it can be stated that the matrices β and α_i present in the Dirac Hamiltonian are of order 4×4. Consequently, the wave function that characterizes the state of a particle with non-zero spin is a four-component spinor. This spinor is capable of describing both the particle and its non-zero spin antiparticle. In standard representation, it is customary to employ the following condensed notation.

$$\psi = \begin{pmatrix} \varphi \\ \chi \end{pmatrix}, \tag{5.25}$$

In this context, φ and χ represent two-component spinors, which correspond to the particle and its antiparticle, respectively.

5.3 Standard representation

The representation of Dirac matrices in the standard form is provided by

$$\gamma^k = \begin{pmatrix} \mathbb{O} & -i\sigma_k \\ i\sigma_k & \mathbb{O} \end{pmatrix} \tag{5.26}$$

$$\gamma^4 = \begin{pmatrix} \mathbb{I} & \mathbb{O} \\ \mathbb{O} & -\mathbb{I} \end{pmatrix} \tag{5.27}$$

where σ_k represents the Pauli matrices (which are 2×2 matrices), defined as follows.

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{5.28}$$

and

$$\mathbb{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \longrightarrow \text{matrice unitaire}, \qquad \mathbb{O} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{5.29}$$

5.4 Free Dirac equation

In the following discussion, we will attempt to derive the Dirac equation from the Schrödinger evolution equation,

$$i\hbar\frac{\partial\psi}{\partial t} = H_{Shrdinger}\psi, \quad \text{avec} \qquad H_{Shrodinger} = -\frac{\hbar^2}{2m}\overrightarrow{\nabla}^2 \tag{5.30}$$

The Dirac's Hamiltonian is given by,

$$H_{Dirac} = \alpha_i.p_i c + \beta\, mc^2 \tag{5.31}$$

and

$$\overrightarrow{p} = -i\hbar\overrightarrow{\nabla} = -i\hbar\overrightarrow{\partial} = -i\,\hbar\,\partial_i \tag{5.32}$$

We have also,

$$\partial_4 = \frac{-i}{c}\frac{\partial}{\partial t} \Longrightarrow i\frac{\partial}{\partial t} = -c\,\partial_4 \tag{5.33}$$

By substituting into (5.30), one obtains.

$$i\hbar\frac{\partial\psi}{\partial t} = \left(\alpha_i.p_i c + \beta\, mc^2\right)\psi \Longrightarrow -c\,\hbar\,\partial_4\psi = \left(-i\,\hbar\,\alpha_i\,\partial_i\, c + \beta\, mc^2\right)\psi \tag{5.34}$$

If we divide both sides of the equation (5.34) by c, we obtain

$$-\hbar\,\partial_4\psi = \left(-i\,\hbar\,\alpha_i\,\partial_i + \beta\, m\, c\right)\psi \tag{5.35}$$

At this point, if we divide both sides of the equation (5.35) by β, we obtain

$$-\beta\,\hbar\,\partial_4\psi = \left(-i\,\beta\,\hbar\,\alpha_i\,\partial_i + m\,c\right)\psi \text{avec} \quad \beta = \beta^{-1} \tag{5.36}$$

$$\left(\partial_4\,\beta + \partial_i\left(-i\,\beta\,\alpha_i\right) + \frac{m\,c}{\hbar}\right)\psi = 0 \tag{5.37}$$

$$\left(\partial_4\,\gamma^4 + \partial_i\,\gamma^i + \frac{m\,c}{\hbar}\right)\psi = 0 \tag{5.38}$$

with

$$\gamma^4 = \beta \tag{5.39}$$
$$\gamma^i = -i\,\beta\,\alpha_i \tag{5.40}$$

Finally, we found,

$$\left(\partial_4\,\gamma^4 + \partial_i\,\gamma^i + m\right)\psi = 0 \quad \text{avec} \quad \hbar = c = 1 \tag{5.41}$$

By employing the representation of the two quadri-vectors.

$$\partial_\mu = (\partial_i, \partial_4) \tag{5.42}$$
$$\gamma^\mu = \left(\gamma^i, \gamma^4\right) \tag{5.43}$$

where,

$$(\partial_i, \partial_4)\cdot\left(\gamma^i, \gamma^4\right) = \partial_4\,\gamma^4 + \partial_i\,\gamma^i \tag{5.44}$$

This equation can be rewritten as follow,

$$\left(\partial_\mu\,\gamma^\mu + m\,\right)\psi = 0 \tag{5.45}$$

The last equation represents the Dirac equation for a free particle.
If we make the following assumption,

$$\partial\!\!\!/ = \partial_\mu\,\gamma^\mu \tag{5.46}$$

We get,

$$(\,\partial\!\!\!/ + m\,)\,\psi(x) = 0 \quad \text{avec} \quad \psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \\ \psi_4(x) \end{pmatrix} \longrightarrow \quad \text{spineur de dirac} \tag{5.47}$$

Therefore, the Dirac matrices exhibit the following properties for subscripts $\mu, \nu = 1, 2, 3, 4$

$$(\gamma^\mu)^+ = \gamma^\mu \tag{5.48}$$
$$(\gamma^\mu)^2 = 1 \tag{5.49}$$
$$\{\gamma^\mu, \gamma^\nu\} = 2\delta_{\mu\nu} \tag{5.50}$$

5.5 Physical interpretation of the negatives energies

The advantage of utilizing component vectors (spinors) lies in their ability to represent fermions (such as electrons). Specifically, two components of the Dirac spinor are employed to characterize the two spin states ($\pm\frac{1}{2}$) of the particle, which possesses an energy of ($\sqrt{p^2c^2 + m^2c^4}$). The remaining two components of the spinor are used to describe the spin state of the antiparticle, which has an energy of ($-\sqrt{p^2c^2 + m^2c^4}$).
The antiparticle simply represents the absence of matter (a void).
For instance, when a particle transitions from a lower energy level to a higher energy level, the vacancy created by the particle, known as a hole, is regarded as the antiparticle of energy ($E = -\sqrt{p^2c^2 + m^2c^4}$), commonly referred to as a positron. A positron has the same mass as an electron but carries a positive charge ($+q$).

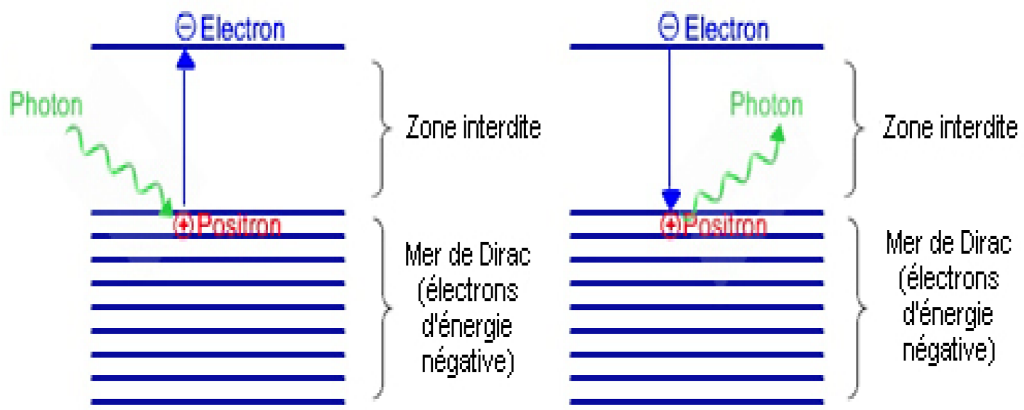

Figure 5.1: Diagram of the Dirac Sea.

When an electron returns to its initial state, it emits a photon of energy ($h\nu$)

$$e^- + e^+ \longrightarrow \gamma \tag{5.51}$$

This process is referred to as the annihilation phenomenon. It can be observed in particle accelerators, where electrons and positrons are accelerated to speeds approaching that of light, subsequently colliding to produce new particles (such as pions, mesons, etc.) that possess extremely short lifetimes.

5.6 Current of free Dirac equation

We seek the expression of the current associated with the Dirac equation, which satisfies the given continuity equation.

$$\frac{\partial \rho}{\partial t} + \overrightarrow{\nabla} \cdot \overrightarrow{j} = 0 \Longrightarrow \partial_\mu J_\mu = 0 \quad \text{avec} \quad \mu = 1, 2, 3, 4 \tag{5.52}$$

The free Dirac equation is given by,

$$(\not{\partial} + m)\, \psi(x) = 0 \Longrightarrow (\partial_\mu \gamma^\mu + m)\, \psi(x) = 0 \tag{5.53}$$

- By calculating the conjugate of the Dirac equation, we arrive at the following result

$$[(\partial_\mu \gamma^\mu + m)\, \psi(x)]^* = 0 \Longrightarrow \psi^+(x) \left(\partial_\mu^* \, (\gamma^\mu)^+ + m\right) = 0 \tag{5.54}$$

We have

$$\partial_\mu = (\partial_i, \partial_4) \Longrightarrow \partial_\mu^* = (\partial_i^*, \partial_4^*) \tag{5.55}$$

with

$$\partial_i^* = \partial_i, \quad \partial_4^* = -\partial_4 \tag{5.56}$$

Therefor,

$$\partial_\mu^* = (\partial_i, -\partial_4) \tag{5.57}$$

and

$$\gamma^\mu = \left(\gamma^i, \gamma^4\right) \Longrightarrow (\gamma^\mu)^+ = \gamma^\mu = \left(\gamma^i, \gamma^4\right) \tag{5.58}$$

Therefor,

$$\partial_\mu^* \, (\gamma^\mu)^+ = \partial_i \gamma^i - \partial_4 \gamma^4 \tag{5.59}$$

Substituting (5.59) in (5.54) we get,

$$\psi^+(x) \left(\partial_i \gamma^i - \partial_4 \gamma^4 + m\right) = 0 \tag{5.60}$$

By multiplying both sides of the equation (5.60) by (γ^4), one arrives at the following result.

$$\left[\psi^+(x) \left(\partial_\mu^* \, (\gamma^\mu)^+ + m\right) = 0\right] \times \gamma^4 \tag{5.61}$$

$$\psi^{+}(x)\left(\partial_i\,\gamma^i\gamma^4-\partial_4\,\gamma^4\gamma^4+m\,\gamma^4\right)=0 \tag{5.62}$$

Or

$$\{\gamma^\mu,\gamma^\nu\}=2\,\delta_{\mu\nu}\Longrightarrow\{\gamma^1,\gamma^4\}=\gamma^1\gamma^4+\gamma^4\gamma^1=0\Longrightarrow\gamma^1\gamma^4=-\gamma^4\gamma^1 \tag{5.63}$$

Therefor,

$$\psi^{+}\left(-\gamma^4\,\partial_i\,\gamma^i-\gamma^4\,\partial_4\,\gamma^4+\gamma^4\,m\right)=0\Longrightarrow \tag{5.64}$$

$$\psi^{+}\,\gamma^4\left(-\partial_i\,\gamma^i-\partial_4\,\gamma^4+m\right)=0\Longrightarrow\psi^{+}\,\gamma^4\,(-\partial_\mu\,\gamma^\mu+m)=0 \tag{5.65}$$

If we define $\overline{\psi}=\psi^{+}\,\gamma^4$, the adjoint equation of the free Dirac equation is transformed

$$\overline{\psi}\,(-\partial_\mu\,\gamma^\mu+m)=0\Longrightarrow\overline{\psi}\,(\partial_\mu\,\gamma^\mu-m)=0 \tag{5.66}$$

It can be expressed in the following final form,

$$\overline{\psi}\left(\overleftarrow{\not\partial}-m\right)=0 \tag{5.67}$$

By multiplying equation (5.53) by $\overline{\psi}$ and equation (5.67) by ψ, we obtain the following results

$$\overline{\psi}\,(\partial_\mu\,\gamma^\mu+m\,)\,\psi=0 \tag{5.68}$$

$$\overline{\psi}\,(\partial_\mu\,\gamma^\mu-m)\,\psi=0 \tag{5.69}$$

By calculating the sum of the two equations (5.68) and (5.69), one finds that

$$\overline{\psi}\,(\partial_\mu\,\gamma^\mu+m\,)\,\psi+\overline{\psi}\,(\partial_\mu\,\gamma^\mu-m)\,\psi=0\Longrightarrow \tag{5.70}$$

$$\overline{\psi}\,\overleftarrow{\partial_\mu}\,\gamma^\mu\psi+m\overline{\psi}\,\psi+\overline{\psi}\,\overrightarrow{\partial_\mu}\,\gamma^\mu\psi-m\overline{\psi}\psi=0\Longrightarrow \tag{5.71}$$

$$\partial_\mu\left(\overline{\psi}\,\gamma^\mu\,\psi\right)=0\Longrightarrow\partial_\mu J_\mu^{Dirac}=0 \tag{5.72}$$

with

$$J^{Dirac}=k\,\overline{\psi}\,\gamma^\mu\,\psi=i\,\overline{\psi}\,\gamma^\mu\,\psi\quad avec\quad k=i \tag{5.73}$$

5.6.1 vector current and total charge

Let us compute the expressions for the components of the momentum vector of j_4 and j_i

$$J_4 = i\,\overline{\psi}\,\gamma^4\,\psi = i\,\psi^+\,\gamma^4\,\gamma^4\,\psi = i\,\psi^+\,\psi = \rho \tag{5.74}$$

$$J_i = i\,\overline{\psi}\,\gamma^i\,\psi = i\,\psi^+\,\gamma^4\,\gamma^i\,\psi \tag{5.75}$$

Or

$$\gamma^i = -i\,\beta\,\alpha_i, \quad \beta = \gamma^4 \Longrightarrow \gamma^i = -i\,\gamma^4\,\alpha_i \Longrightarrow \tag{5.76}$$

$$\gamma^4\,\gamma^i = -i\,\gamma^4\,\gamma^4\,\alpha_i \Longrightarrow \alpha_i = i\,\gamma^4\,\gamma^i \tag{5.77}$$

Therefor,

$$J_i = \psi^+\,\alpha_i\,\psi \Longrightarrow \overrightarrow{J} = \psi^+\,\overrightarrow{\alpha}\,\psi \tag{5.78}$$

Finally, the total charge is given by,

$$Q = \int d^3x\,\rho = i\int d^3x\,\psi^+\,\psi \tag{5.79}$$

5.7 Dirac equation in the presence of an external electromagnetic field

To recover the Dirac equation in the presence of an external electromagnetic field A_μ, the method of minimal coupling is employed

$$\partial_\mu \rightarrow \partial_\mu - iqA_\mu \tag{5.80}$$

$$(\,\partial\!\!\!/ + m\,)\,\psi(x) = 0 \Longrightarrow \big(\partial_\mu\,\gamma^\mu + m\,\big)\,\psi(x) = 0 \tag{5.81}$$

Substituting (5.80) in (5.81) we get,

$$\big(\big(\partial_\mu - iqA_\mu\big)\;\gamma^\mu + m\,\big)\,\psi(x) = 0 \Longrightarrow \big(\partial_\mu\,\gamma^\mu - iqA_\mu\,\gamma^\mu + m\,\big)\,\psi(x) = 0 \tag{5.82}$$

$$(\,\partial\!\!\!/ - iq\;A\!\!\!/ + m\,)\,\psi(x) = 0 \tag{5.83}$$

This is the Dirac equation in the presence of an external electromagnetic field A_μ.

5.8 Lagrangian of the complex spinor field

It is possible to derive the Dirac equation and the adjoint Dirac equation by employing the Lagrangian formulation. Our selection of the Lagrangian is as follows

$$\mathcal{L}(\psi, \partial_\mu \psi, \overline{\psi}, \partial_\mu \overline{\psi}, x_\mu) = -\overline{\psi}\,(\partial\!\!\!/ + m\,)\,\psi \tag{5.84}$$

Verification: Let us verify that this Lagrangian density enables us to obtain the equations of motion for the free complex spinor field $(\psi, \overline{\psi})$. To conduct this verification, it is necessary to substitute the expression of the Lagrangian density into the Euler-Lagrange equations for a field,

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = 0 \quad \text{avec} \quad \phi_i = \psi = \overline{\psi} \tag{5.85}$$

Therefore, each value of ϕ_i corresponds to a motion equation

$$\frac{\partial \mathcal{L}}{\partial \psi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \right) = 0 \longrightarrow \text{This equation allows for the derivation of the adjoint Dirac equation.} \tag{5.86}$$

$$\frac{\partial \mathcal{L}}{\partial \overline{\psi}} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \overline{\psi})} \right) = 0 \longrightarrow \text{This equation allows for the derivation of the Dirac equation,} \tag{5.87}$$

1- Let us revisit the Dirac adjoint equation

$$\frac{\partial \mathcal{L}}{\partial \overline{\psi}} = -\partial_\mu \gamma^\mu - m\,\psi, \quad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \overline{\psi})} = 0 \tag{5.88}$$

So,

$$\frac{\partial \mathcal{L}}{\partial \overline{\psi}} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \overline{\psi})} \right) = 0 \Longrightarrow -\left(\partial_\mu\, \gamma^\mu + m\,\right)\psi = 0 \Longrightarrow (\,\partial\!\!\!/ + m\,)\,\psi = 0 \tag{5.89}$$

2- Let us revisit the Dirac equation

$$\frac{\partial \mathcal{L}}{\partial \psi} = -m\,\overline{\psi}, \quad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} = -\overline{\psi}\gamma^\mu \tag{5.90}$$

Therefor

$$\frac{\partial \mathcal{L}}{\partial \psi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \right) = 0 \Longrightarrow -m\,\overline{\psi} + \overline{\psi}\gamma^\mu \Longrightarrow \overline{\psi}\left(\overleftarrow{\partial_\mu}\,\gamma^\mu - m \right) = 0 \Longrightarrow \overline{\psi}\left(\overleftarrow{\not\partial} + m \right) = 0 \quad (5.91)$$

Therefore, the Lagrangian density of the free complex spinor field is expressed as

$$\mathcal{L} = -\overline{\psi}\,(\not\partial + m\,)\,\psi = -\overline{\psi}\,(\partial_\mu\,\gamma^\mu + m\,)\,\psi = -\overline{\psi}\overrightarrow{\partial_\mu}\,\gamma^\mu\psi - m\overline{\psi}\psi \quad (5.92)$$

5.9 Lagrangian of the complex spinor field in the presence of an external electromagnetic field.

To derive the two equations of motion for the spinor fields ψ and $\overline{\psi}$ in the presence of an external electromagnetic field A_μ, the following Lagrangian density is employed

$$\mathcal{L} = -\overline{\psi}\,(\not\partial - iq\,\not A + m\,)\,\psi = -\overline{\psi}\,(\partial_\mu\,\gamma^\mu - iq\,A_\mu\gamma^\mu + m\,)\,\psi \quad (5.93)$$

That we can write in the following form,

$$\mathcal{L} = -\overline{\psi}\overrightarrow{\partial_\mu}\,\gamma^\mu\psi - iq\,A_\mu\,\gamma^\mu\,\overline{\psi}\psi + m\overline{\psi}\psi \quad (5.94)$$

It is important to recall that the Dirac equations and the adjoint Dirac equation in the presence of an external electromagnetic field are expressed as follows,

$$(\not\partial - iq\,\not A + m\,)\,\psi(x) = 0 \quad (5.95)$$

$$\overline{\psi}\left(\overleftarrow{\not\partial} + iq\,\not A + m \right)\psi(x) = 0 \quad (5.96)$$

Verification: Let us verify that this Lagrangian density enables us to derive the equations of motion for the complex spinor field in the presence of an electromagnetic field. To conduct this verification, it is necessary to substitute the expression of the Lagrangian density into the Euler-Lagrange equations for a field,

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = 0 \quad \text{avec} \quad \phi_i = \psi = \overline{\psi} \quad (5.97)$$

Therefore, each value of ϕ_i corresponds to a motion equation

$$\frac{\partial \mathcal{L}}{\partial \psi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \right) = 0 \tag{5.98}$$

$$\frac{\partial \mathcal{L}}{\partial \overline{\psi}} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \overline{\psi})} \right) = 0 \tag{5.99}$$

1- Let us revisit the adjoint equation

$$\frac{\partial \mathcal{L}}{\partial \overline{\psi}} = -\,(\partial\!\!\!/ - iq \;A\!\!\!/ + m\,)\,\psi, \quad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \overline{\psi})} = 0 \tag{5.100}$$

So

$$\frac{\partial \mathcal{L}}{\partial \overline{\psi}} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \overline{\psi})} \right) = 0 \Longrightarrow -\,(\partial\!\!\!/ - iq \;A\!\!\!/ + m\,)\,\psi = 0 \Longrightarrow (\partial\!\!\!/ - iq \;A\!\!\!/ + m\,)\,\psi = 0 \tag{5.101}$$

2- Let us revisit the Dirac equation

$$\frac{\partial \mathcal{L}}{\partial \psi} = iq\,A_\mu\,\gamma^\mu\,\overline{\psi} - m\,\overline{\psi}, \quad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} = -\overline{\psi}\gamma^\mu \tag{5.102}$$

So

$$\frac{\partial \mathcal{L}}{\partial \psi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \right) = 0 \Longrightarrow iq\,A_\mu\,\gamma^\mu\,\overline{\psi} - m\,\overline{\psi} + \overline{\psi}\gamma^\mu = 0 \Longrightarrow \overline{\psi}\left(\overleftarrow{\partial\!\!\!/} + iq\,A_\mu\,\gamma^\mu - m \right) = 0 \tag{5.103}$$