Symmetries and conservation laws

In this section, it will be assumed that the Lagrangian density does not depend explicitly on (x_μ). It will also be assumed that the equations of motion (and hence the action) remain unchanged during an infinitesimal (continuous) transformation defined by,

$$\begin{cases} x_\mu \longrightarrow x'_\mu = x_\mu + \delta x_\mu \\ \phi(x_\mu) \longrightarrow \phi'(x'_\mu) = \phi(x_\mu) + \delta\phi(x_\mu) \end{cases} \tag{3.1}$$

with,

$$\begin{cases} x_\mu \longrightarrow \text{position spatio-temporelle (coordonnées)} \\ \delta x_\mu \longrightarrow \text{variation infinitisimale (deplacement l'espace et dans le temps)} \\ \phi(x_\mu) \longrightarrow \text{champ scalaire (variable)} \\ \delta\phi(x_\mu) \longrightarrow \text{variation de phase (dûe à une rotation)} \end{cases}$$

3.1 Example of transformation

3.1.1 Space-time transformation

A space-time transformation is defined by

$$\begin{cases} x_\mu \longrightarrow x'_\mu = x_\mu + a_\mu \ , \quad (a_\mu = \delta x_\mu) \\ \phi(x_\mu) \longrightarrow \phi'(x'_\mu) = \phi(x_\mu) \ , \quad (\delta\phi(x_\mu) = 0) \end{cases} \tag{3.2}$$

Where a_μ represents the quadri-vector displacement in space-time.
According to the infinitesimal transformation given in equation (3.2),

$$\phi'(x'_\mu) = \phi'(x_\mu + a_\mu) = \phi(x_\mu) \tag{3.3}$$

therefore;

$$\phi'(x_\mu + a_\mu) = \phi(x_\mu) \tag{3.4}$$

3.1.2 Global phase transformation $(\phi(x_\mu) \neq \phi^*(x_\mu))$

This transformation is given by,

$$\begin{cases} x_\mu \longrightarrow x'_\mu = x_\mu \ , \quad (\delta x_\mu = 0) \\ \phi(x_\mu) \longrightarrow \phi'(x'_\mu) = \phi(x_\mu) + \delta\phi(x_\mu) = e^{-iq\theta(x_\mu)}\phi(x_\mu) \end{cases} \tag{3.5}$$

Where $theta(x_\mu)$ is a real scalar.
According to the infinitesimal transformation given in equation (3.5),

$$\phi'(x'_\mu) = \phi'(x_\mu) = \phi(x_\mu) + \delta\phi(x_\mu) = e^{-iq\theta(x_\mu)}\phi(x_\mu) \tag{3.6}$$

therefor,

$$\phi'^*(x_\mu) = e^{+iq\theta(x_\mu)}\phi^*(x_\mu) \tag{3.7}$$

3.1.3 Local phase transformation $(\phi(x_\mu) = \phi^*(x_\mu))$

This transformation is given by,

$$\begin{cases} x_\mu \longrightarrow x'_\mu = x_\mu \ , \quad (\delta x_\mu = 0) \\ \phi(x_\mu) \longrightarrow \phi'(x'_\mu) = \phi(x_\mu) + \delta\phi(x_\mu) = e^{-iq\theta(x_\mu)}\phi(x_\mu) \end{cases} \tag{3.8}$$

Where $theta(x_\mu)$ is a real scalar.
According to the infinitesimal transformation given in equation (3.8),

$$\phi'(x'_\mu) = \phi'(x_\mu) = \phi(x_\mu) + \delta\phi(x_\mu) = e^{-iq\theta(x_\mu)}\phi(x_\mu) \tag{3.9}$$

therefor,

$$\phi'^*(x_\mu) = e^{+iq\theta(x_\mu)}\phi(x_\mu) \tag{3.10}$$

3.2 Noether's theorem

3.2.1 Statement

For any continuous transformation of the action S, there is a current J_μ satisfying the equation

$$\partial_\mu J_\mu = 0 \tag{3.11}$$

This implies that there is a self-preserving charge, defined by

$$Q = \int \rho \, d^3x \tag{3.12}$$

3.2.2 Demonstration

The equations of motion are said to be invariant if the action S is stationary.

$$\delta S = S^{'} - S \simeq 0 \tag{3.13}$$

We have

$$S = \int d^4x \, \mathcal{L}(\phi, \partial_\mu \phi) \Rightarrow S^{'} = \int d^4x^{'} \, \mathcal{L}(\phi^{'}, \partial^{'}_\mu \phi^{'}) \tag{3.14}$$

Given $\mathcal{L} = \mathcal{L}(\phi, \partial_\mu \phi)$ (where the Lagrangian density does not have explicit dependence on x_μ). Let us consider infinitesimal transformations of the form,

$$\begin{cases} x_\mu \longrightarrow x^{'}_\mu = x_\mu + \delta x_\mu \\ \phi(x_\mu) \longrightarrow \phi^{'}(x^{'}_\mu) = \phi(x_\mu) + \delta\phi(x_\mu) \end{cases} \tag{3.15}$$

where

$$\delta\phi(x) = \phi^{'}(x^{'}) - \phi(x) \tag{3.16}$$

The symbol $\delta\phi(x_\mu)$ represents the variation of the field due to both the transformation of the field (variable) and the transformation of the coordinates (x_μ).
Thus, the change at a specific point in 4-dimensional space is determined by

$$\delta_o\phi(x) = \phi^{'}(x) - \phi(x) \ , \quad \text{pour } x^{'} = x \tag{3.17}$$

The relationship between the spacetime derivatives is expressed by

$$d^4x^{'} = [1 + \partial_\mu(\delta x_\mu)] d^4x \tag{3.18}$$

Let's now examine the relationship between the field variation at two different points $\delta\phi$ and the field variation at a fixed point $\delta_o\phi$.
The variation of the field at two different points is given by

$$\delta\phi(x) = \phi^{'}(x^{'}) - \phi(x) = \phi^{'}(x^{'}) - \phi^{'}(x) + \phi^{'}(x) - \phi(x) \tag{3.19}$$

$$\delta\phi(x) = \phi^{'}(x) + (\partial_\nu\phi)\delta x_\nu - \phi^{'}(x) + \delta_o\phi(x) \tag{3.20}$$

with

$$\phi^{'}(x^{'}) = \phi^{'}(x_\mu + \delta x_\mu) = \phi^{'}(x_\mu) + (\partial_\nu\phi)\delta x_\nu = \phi^{'}(x) + (\partial_\nu\phi)\delta x_\nu \tag{3.21}$$

Therefor,

$$\delta\phi(x) = \delta_o\phi(x) + (\partial_\nu\phi)\delta x_\nu \tag{3.22}$$

Let us calculate the term $\partial^{'}_\mu\phi^{'}$
We have

$$\partial^{'}_\mu\phi^{'}(x^{'}) = \partial^{'}_\mu(\phi + \delta\phi) = \frac{\partial}{\partial x^{'}_\mu}(\phi + \delta\phi) \tag{3.23}$$

$$= \frac{\partial}{\partial x_\nu}\frac{\partial x_\nu}{\partial x^{'}_\mu}(\phi + \delta\phi) = \frac{\partial}{\partial x_\nu}(\phi + \delta\phi)\frac{\partial x_\nu}{\partial x^{'}_\mu} \tag{3.24}$$

We have also

$$x^{'}_\nu = x_\nu + \delta x_\nu \Rightarrow x_\nu = x^{'}_\nu - \delta x_\nu \tag{3.25}$$

Therefor

$$\frac{\partial x_\nu}{\partial x^{'}_\mu} = \frac{\partial x^{'}_\nu}{\partial x^{'}_\mu} + \frac{\partial}{\partial x^{'}_\mu}(\delta x_\nu) \tag{3.26}$$

Finally, we get

$$\frac{\partial x_\nu}{\partial x^{'}_\mu} = \delta_{\mu\nu} - \partial_\mu(\delta x_\nu) \tag{3.27}$$

By substituting the equation (**??**) into equation (3.23), we obtain

$$\partial^{'}_\mu\phi^{'}(x^{'}) = \frac{\partial}{\partial x_\nu}(\phi + \delta\phi)\frac{\partial x_\nu}{\partial x^{'}_\mu} \tag{3.28}$$

$$= \left(\frac{\partial\phi}{\partial x_\nu} + \frac{\partial}{\partial x_\nu}(\delta\phi)\right)(\delta_{\mu\nu} - \partial_\mu(\delta x_\nu)) \tag{3.29}$$

$$= (\partial_\nu\phi + \partial_\nu(\delta\phi))(\delta_{\mu\nu} - \partial_\mu(\delta x_\nu)) \tag{3.30}$$

$$= (\partial_\nu\phi)\delta_{\mu\nu} - (\partial_\nu\phi)\partial_\mu(\delta x_\nu) + \partial_\nu(\delta\phi)\delta_{\mu\nu} - \partial_\nu(\delta\phi)\partial_\mu(\delta x_\nu) \tag{3.31}$$

$$\partial'_\mu \phi'(x') = (\partial_\mu \phi) - (\partial_\nu \phi)\partial_\mu(\delta x_\nu) + \partial_\mu(\delta\phi) \tag{3.32}$$

The term $\partial_\nu(\delta\phi)\partial_\mu(\delta x_\nu)$ is neglected, as it is a higher-order term.
The Lagrangian density does not explicitly depend on x_μ, which implies that $\mathcal{L} = \mathcal{L}(\phi, \partial_\mu\phi)$.
Therefor,

$$\mathcal{L}(\phi', \partial'_\mu \phi') = \mathcal{L}(\phi + \delta\phi, (\partial_\mu\phi) - (\partial_\nu\phi)\partial_\mu(\delta x_\nu) + \partial_\mu(\delta\phi)) \tag{3.33}$$

$$= \mathcal{L}(\phi, \partial_\mu\phi) + \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}[\partial_\mu(\delta\phi) - (\partial_\nu\phi)\partial_\mu(\delta x_\nu)] \tag{3.34}$$

we get

$$\mathcal{L}(\phi', \partial'_\mu \phi') = \mathcal{L}(\phi, \partial_\mu\phi) + \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu(\delta\phi) - \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}(\partial_\nu\phi)\partial_\mu(\delta x_\nu) \tag{3.35}$$

By substituting the equation (3.18) into the equation (3.13), one arrives at the following result

$$\delta S = \int d^4x' \, \mathcal{L}(\phi', \partial'_\mu \phi') - \int d^4x \, \mathcal{L}(\phi, \partial_\mu\phi) \simeq 0 \tag{3.36}$$

$$= \int [1 + \partial_\mu(\delta x_\mu)] d^4x \, \mathcal{L}(\phi', \partial'_\mu \phi') - \int d^4x \, \mathcal{L}(\phi, \partial_\mu\phi) \simeq 0 \tag{3.37}$$

$$\delta S = \int [\mathcal{L}(\phi', \partial'_\mu \phi') - \mathcal{L}(\phi, \partial_\mu\phi) + \partial_\mu(\delta x_\mu)\mathcal{L}] d^4x \simeq 0 \tag{3.38}$$

Let us calculate the following term: $\mathcal{L}(\phi', \partial'_\mu \phi') - \mathcal{L}(\phi, \partial_\mu\phi)$

$$\mathcal{L}(\phi', \partial'_\mu \phi') - \mathcal{L}(\phi, \partial_\mu\phi) = \mathcal{L}(\phi, \partial_\mu\phi) + \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu(\delta\phi) - \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}(\partial_\nu\phi)\partial_\mu(\delta x_\nu) - \mathcal{L}(\phi, \partial_\mu\phi)$$

$$\mathcal{L}(\phi', \partial'_\mu \phi') - \mathcal{L}(\phi, \partial_\mu\phi) = \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu(\delta\phi) - \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}(\partial_\nu\phi)\partial_\mu(\delta x_\nu) \tag{3.39}$$

According to the Euler-Lagrange equations,

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\right) = 0$$

then

$$\frac{\partial \mathcal{L}}{\partial \phi} = \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\right) \tag{3.40}$$

We have also

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right) = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\mu(\delta\phi)$$

Therefor

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\mu(\delta\phi) = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right) - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \delta\phi \tag{3.41}$$

By replacing equations (3.40) and (3.41) in equation (3.39), we obtain

$$\mathcal{L}(\phi', \partial'_\mu \phi') - \mathcal{L}(\phi, \partial_\mu \phi) = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \delta\phi + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right) - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \delta\phi - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} (\partial_\nu \phi) \partial_\mu (\delta x_\nu)$$

$$\mathcal{L}(\phi', \partial'_\mu \phi') - \mathcal{L}(\phi, \partial_\mu \phi) = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right) - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} (\partial_\nu \phi) \partial_\mu (\delta x_\nu) \tag{3.42}$$

We have

$$\delta\phi = \delta_o \phi + (\partial_\nu \phi) \delta x_\nu$$

Then

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right) = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} (\delta_o \phi + (\partial_\nu \phi)(\delta x_\nu)) \right)$$

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right) = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta_o \phi \right) + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} (\partial_\nu \phi)(\delta x_\nu) \right) \tag{3.43}$$

Let us calculate the term $\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} (\partial_\nu \phi)(\delta x_\nu) \right)$:

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} (\partial_\nu \phi)(\delta x_\nu) \right) = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) (\partial_\nu \phi)(\delta x_\nu) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\mu \left((\partial_\nu \phi) \right) (\delta x_\nu) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} (\partial_\nu \phi) \partial_\mu \left(\delta x_\nu \right)$$

By neglecting higher order terms, one can find

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} (\partial_\nu \phi)(\delta x_\nu) \right) = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) (\partial_\nu \phi)(\delta x_\nu) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} (\partial_\nu \phi) \partial_\mu \left(\delta x_\nu \right) \tag{3.44}$$

Therefor

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right) = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta_o \phi \right) + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) (\partial_\nu \phi)(\delta x_\nu) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} (\partial_\nu \phi) \partial_\mu \left(\delta x_\nu \right) \tag{3.45}$$

By inserting equation (3.45) into equation (3.42), we get

$$\mathcal{L}(\phi',\partial'_\mu\phi') - \mathcal{L}(\phi,\partial_\mu\phi) = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\right) - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}(\partial_\nu\phi)\partial_\mu(\delta x_\nu)$$

$$= \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta_o\phi\right) + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)(\partial_\nu\phi)(\delta x_\nu) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}(\partial_\nu\phi)\partial_\mu\,(\delta x_\nu) - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}(\partial_\nu\phi)\partial_\mu\,(\delta x_\nu)$$

So,

$$\mathcal{L}(\phi',\partial'_\mu\phi') - \mathcal{L}(\phi,\partial_\mu\phi) = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta_o\phi\right) + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)(\partial_\nu\phi)(\delta x_\nu)$$

Calculating the term $\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)(\partial_\nu\phi)(\delta x_\nu)$:

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)(\partial_\nu\phi)(\delta x_\nu) = \frac{\partial\mathcal{L}}{\partial\phi}(\partial_\nu\phi)(\delta x_\nu) = \frac{\partial\mathcal{L}}{\partial x_\mu}\frac{\partial x_\mu}{\partial\phi}\frac{\partial\phi}{\partial x_\nu}\delta x_\nu$$

$$= \frac{\partial\mathcal{L}}{\partial x_\mu}\frac{\partial x_\mu}{\partial\partial x_\nu}\delta x_\nu = \frac{\partial\mathcal{L}}{\partial x_\mu}\delta_{\mu\nu}\delta x_\nu = \partial_\mu\mathcal{L}\;\delta x_\mu$$

Finally, we get

$$\mathcal{L}(\phi',\partial'_\mu\phi') - \mathcal{L}(\phi,\partial_\mu\phi) = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta_o\phi\right) + \partial_\mu\mathcal{L}\;\delta x_\mu \tag{3.46}$$

The variation of the action in the equation (3.38) becomes

$$\delta S = \int\left[\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta_o\phi\right) + \partial_\mu\mathcal{L}\;\delta x_\mu + \partial_\mu(\delta x_\mu)\mathcal{L}\right]d^4x \simeq 0$$

We have

$$\partial_\mu\mathcal{L}\;\delta x_\mu + \partial_\mu(\delta x_\mu)\mathcal{L} = \partial_\mu(\mathcal{L}\,\delta x_\mu)$$

Then,

$$\delta S = \int\left[\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta_o\phi\right) + \partial_\mu(\mathcal{L}\,\delta x_\mu)\right]d^4x \simeq 0$$

$$\delta S = \int\partial_\mu\left[\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta_o\phi\right) + \mathcal{L}\,\delta x_\mu\right]d^4x \simeq 0$$

$$\Rightarrow \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta_o\phi + \mathcal{L}\,\delta x_\mu\right] = 0$$

The final equation can be expressed in the following form

$$\partial_\mu J_\mu = 0$$

with

$$J_\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta_o \phi + \mathcal{L}\, \delta x_\mu \longrightarrow \text{ Courant de Noether}$$

3.3 Energy-Momentum Tensor of the scalar field

Since the Lagrangian density $\mathcal{L}$ does not explicitly depend on the four-position vector x_μ, its derivative with respect to x_μ is as follows

$$\partial_\mu \mathcal{L} = \partial_\mu \mathcal{L}(\phi, \partial_\mu \phi) \quad \text{où} \quad \partial_\mu = \frac{\partial}{\partial x_\mu} \tag{3.47}$$

Therefor

$$\partial_\mu \mathcal{L} = \frac{\partial \mathcal{L}}{\partial x_\mu} \tag{3.48}$$

We have,

$$\partial_\mu \mathcal{L} = \frac{\partial \mathcal{L}}{\partial x_\mu} = \frac{\partial \mathcal{L}}{\partial \phi} \frac{\partial \phi}{\partial x_\mu} + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} \frac{\partial(\partial_\nu \phi)}{\partial x_\mu} \tag{3.49}$$

According to the Euler-Lagrange equation, we have

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} \right) = 0 \quad \Rightarrow \frac{\partial \mathcal{L}}{\partial \phi} = \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} \right) \quad \text{pour} \qquad \mu = \nu \tag{3.50}$$

Therefor,

$$\partial_\mu \mathcal{L} = \frac{\partial \mathcal{L}}{\partial x_\mu} = \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} \right) \partial_\mu \phi + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} \partial_\mu \left(\partial_\nu \phi \right) \tag{3.51}$$

By setting,

$$\partial_\mu \left(\partial_\nu \phi \right) = \partial_\nu \left(\partial_\mu \phi \right) \tag{3.52}$$

we found that,

$$\partial_\mu \mathcal{L} = \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} \right) \partial_\mu \phi + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} \partial_\nu \left(\partial_\mu \phi \right) = \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} \partial_\mu \phi \right) \tag{3.53}$$

The expression $\partial_\mu \mathcal{L}$ can also be represented in the following way:

$$\partial_\mu \mathcal{L} = \frac{\partial \mathcal{L}}{\partial x_\mu} = \frac{\partial \mathcal{L}}{\partial x_\nu}\frac{\partial x_\nu}{\partial x_\mu} = (\partial_\nu \mathcal{L})\, \delta_{\mu\nu} = \partial_\nu \left(\mathcal{L}\delta_{\mu\nu}\right) \tag{3.54}$$

Comparing equations (3.53) and (3.54), we can see that

$$\partial_\mu \mathcal{L} = \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)}\partial_\mu \phi\right) = \partial_\nu \left(\mathcal{L}\delta_{\mu\nu}\right) \tag{3.55}$$

Therefor,

$$\partial_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)}\partial_\mu \phi - \mathcal{L}\delta_{\mu\nu}\right) = 0 \tag{3.56}$$

Now, if we replace ν by μ

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\partial_\nu \phi - \mathcal{L}\delta_{\mu\nu}\right) = 0 \tag{3.57}$$

The letter can be rewritten in the following form,

$$\partial_{\mu\nu} T_{\mu\nu} = 0 \;\text{ avec }\; T_{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\partial_\nu \phi - \mathcal{L}\delta_{\mu\nu} \tag{3.58}$$

The tensor $T_{\mu\nu}$ denotes the energy-momentum tensor of the scalar field.