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Level : 1^{st} year LMD
Module: Algebra 2

Series 2: Linear maps

Exercise 1. Let $f:\mathbb{R}^4\to\mathbb{R}^3$ be the map defined for any $u=(x,y,z,t)\in\mathbb{R}^4$ by :

$$f(x,y,z,t)=(x+y,z+t,x+y+z+t)$$

1. Prove that f is a linear map.
2. Determine a basis of $\ker(f)$.
3. Determine a basis of $\operatorname{Im}(f)$.

Exercise 2. Let $u:\mathbb{R}^3\to\mathbb{R}^3$ the map defined by:

$$u(x_1,x_2,x_3)=(-2x_1+4x_2+4x_3,-x_1+x_3,-2x_1+4x_2+4x_3)$$

1. Prove that u is linear.
2. Determine a basis of $\ker(u)$ and a basis of $\operatorname{Im}(u)$.
3. Do we have $\ker(u)\oplus\operatorname{Im}(u)=\mathbb{R}^3$?

Exercise 3. Let $\beta=\{e_1,e_2,e_3\}$ be the canonical basis of $\mathbb{R}^3$. Let u be an endomorphism of $\mathbb{R}^3$ defined by :

$$u(e_1)=2e_1+e_2+3e_3;\quad u(e_2)=e_2-3e_3;\quad u(e_3)=-2e_2+2e_3.$$

1. Let $x=(x_1,x_2,x_3)\in\mathbb{R}^3$ a vector. Determine the image by u of the vector x. (Calculate $u(x)$).
2. Let $E=\{x\in\mathbb{R}^3,u(x)=2x\}$ and $F=\{x\in\mathbb{R}^3,u(x)=-x\}$. Prove that E and F are subspaces of $\mathbb{R}^3$.
3. Determine a basis of E and a basis of F.
4. Do we have $E\oplus F=\mathbb{R}^3$?

Exercise 4. Let $\beta=\{e_1,e_2,e_3\}$ be the canonical basis of $\mathbb{R}^3$. Let $f:\mathbb{R}^3\to\mathbb{R}^3$ be a linear map such that :

$$\begin{aligned}
f(e_1) &= -\frac{1}{3}e_1+\frac{2}{3}e_2+\frac{2}{3}e_3=\frac{1}{3}(-e_1+2e_2+2e_3),\\
f(e_2) &= \frac{2}{3}e_1-\frac{1}{3}e_2+\frac{2}{3}e_3=\frac{1}{3}(2e_1-e_2+2e_3)\ \text{and}\\
f(e_3) &= \frac{2}{3}e_1+\frac{2}{3}e_2-\frac{1}{3}e_3=\frac{1}{3}(2e_1+2e_2-e_3).
\end{aligned}$$

Let $E_{-1}=\{u\in\mathbb{R}^3\mid f(u)=-u\}$ and $E_1=\{u\in\mathbb{R}^3\mid f(u)=u\}$.

1. Prove that E_{-1} and E_1 are subspaces of $\mathbb{R}^3$.
2. Demonstrate that $e_1 - e_2$ and $e_1 - e_3$ are belong to E_{-1} and that $e_1 + e_2 + e_3$ is belong to E_1.
3. What can we deduce about the dimensions of E_{-1} and E_1 ?
4. Determine $E_{-1} \cap E_1$.
5. Do we have $E_{-1} \oplus E_1 = \mathbb{R}^3$?
6. Calculate $f^2 = f \circ f$ and deduce that f is bijective and determine f^{-1}.

<u>Exercise</u> 5. Let $\beta = \{e_1, e_2\}$ be the canonical basis of $\mathbb{R}^2$. Let u an endomorphism of $\mathbb{R}^2$ so that $u(e_1) = e_1 + e_2$ with $\dim(\ker(u)) = 1$

1. Determine $u(e_2)$ depending on a parameter $a \in \mathbb{R}$.
2. Determine the image of a vector $x = (x_1, x_2) \in \mathbb{R}$ depending on a.
3. Determine a basis of the kernel $\ker(u)$.

<u>Exercise</u> 6. Let $f : \mathbb{R}^4 \to \mathbb{R}$ be the map defined for any $x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ by:

$$f(x) = x_1 + x_2 + x_3 + x_4$$

Let $\beta = \{e_1, e_2, e_3, e_4\}$ be the canonical basis of $\mathbb{R}^4$.

1. Calculate the images of vectors of the canonical basis by f. Deduce the dimension of $\mathrm{Im}(f)$.
2. Determine the dimension of $\ker(f)$ and give a basis for it.

<u>Exercise</u> 7. Let $f : E \to F$ be a linear map. Prove that:

$$\ker(f) \cap \mathrm{im}(f) = f\left(\ker\left(f^2\right)\right).$$

<u>Exercise</u> 8. Let u be an endomorphism of a vector space E on a field $\Bbbk$.

1. Prove that $\ker(u) \subset \ker(u^2)$.
2. Prove that $\mathrm{Im}(u^2) \subset \mathrm{Im}(u)$.

<u>Exercise</u> 9. Let u be an endomorphism of a vector space E on a field $\Bbbk$.

Show that the following assertions are equivalent

(i) $\ker(u) \cap \mathrm{im}(u) = \{0_E\}$.

(ii) $\ker(u) = \ker(u \circ u)$.