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Calculus I

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Unit 8. Taylor Expansions

8. Taylor Expansions

8.1 Landau's 'o' notation

As we did for sequences, when it comes to functional limits comparing functions is worthwhile. Over time, a standard notation has been created that makes algebraic manipulations of functions easy. We are going to introduce and discuss that notation.

Definition 8.1.1 We say that function f is **negligible** with respect to g when $x \to a$ (where $-\infty \leqslant a \leqslant \infty$) if

$$\lim_{x\to a} \frac{f(x)}{g(x)} = 0.$$

We denote this as $f = o(g)$ $(x \to a)$, and read it "f is *small* o of g as x goes to a".

The intuitive meaning of $f = o(g)$ $(x \to a)$ is that the numerical value of $f(x)$ is much smaller than that of $g(x)$, the more so the closer is x to a.

This symbol bears a set of basic algebraic properties with which we can easily manipulate it:

Proposition 8.1.1 For given $-\infty \leqslant a \leqslant \infty$,

(a) if $f = o(g)$ $(x \to a)$ and $\lambda \in \mathbb{R}$, then $\lambda f = o(g)$ **scaling**
(b) if $f_1 = o(g)$ $(x \to a)$ and $f_2 = o(g)$ $(x \to a)$, then $f_1 + f_2 = o(g)$ $(x \to a)$ **additive**
(c) if $f_1 = o(g_1)$ $(x \to a)$ and $f_2 = o(g_2)$ $(x \to a)$, then $f_1 f_2 = o(g_1 g_2)$ $(x \to a)$ **multiplicative**
(d) if $f = o(g)$ $(x \to a)$ and $g = o(h)$ $(x \to a)$, then $f = o(h)$ $(x \to a)$. **transitive**
(e) if $f = o(g)$ $(x \to a)$ then $hf = o(hg)$ $(x \to a)$ **factorisation**

R The short-hand version of these properties is

(a) $\lambda\, o(g) = o(g)$,
(b) $o(g) + o(g) = o(g)$,
(c) $o(g_1)\, o(g_2) = o(g_1 g_2)$,
(d) $o\big(o(h)\big) = o(h)$,
(e) $h\, o(g) = o(hg)$,

where it is implicitly understood $x \to a$. These are the common manipulations of the o symbol.

Proof. (a) We are given that

$$\lim_{x\to a}\frac{f(x)}{g(x)}=0;$$

therefore

$$\lim_{x\to a}\frac{\lambda f(x)}{g(x)}=\lambda\cdot\lim_{x\to a}\frac{f(x)}{g(x)}=0.$$

(b) We are given that

$$\lim_{x\to a}\frac{f_1(x)}{g(x)}=\lim_{x\to a}\frac{f_2(x)}{g(x)}=0;$$

therefore

$$\lim_{x\to a}\frac{f_1(x)+f_2(x)}{g(x)}=\lim_{x\to a}\frac{f_1(x)}{g(x)}+\lim_{x\to a}\frac{f_2(x)}{g(x)}=0.$$

(c) In this case

$$\lim_{x\to a}\frac{f_1(x)}{g_1(x)}=\lim_{x\to a}\frac{f_2(x)}{g_2(x)}=0;$$

therefore

$$\lim_{x\to a}\frac{f_1(x)f_2(x)}{g_1(x)g_2(x)}=\lim_{x\to a}\frac{f_1(x)}{g_1(x)}\cdot\lim_{x\to a}\frac{f_2(x)}{g_2(x)}=0.$$

(d) Now we know that

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{g(x)}{h(x)}=0;$$

therefore

$$\lim_{x\to a}\frac{f(x)}{h(x)}=\lim_{x\to a}\frac{f(x)g(x)}{g(x)h(x)}=\lim_{x\to a}\frac{f(x)}{g(x)}\cdot\lim_{x\to a}\frac{g(x)}{h(x)}=0.$$

(e) Finally, if

$$\lim_{x\to a}\frac{f(x)}{g(x)}=0,$$

then

$$\lim_{x\to a}\frac{h(x)f(x)}{h(x)g(x)}=\lim_{x\to a}\frac{f(x)}{g(x)}=0.$$

∎

■ Example 8.1

(a) If $a,\alpha,\gamma\in\mathbb{R}$ and $\alpha>0$, then $(x-a)^{\gamma+\alpha}=o\big((x-a)^{\gamma}\big)$ $(x\to a)$ because

$$\lim_{x\to a}\frac{(x-a)^{\gamma+\alpha}}{(x-a)^{\gamma}}=\lim_{x\to a}(x-a)^{\alpha}=0.$$

(b) If $\alpha, \gamma \in \mathbb{R}$, and $\alpha > 0$, then $\frac{1}{x^{\gamma+\alpha}} = o\left(\frac{1}{x^\gamma}\right)$ $(x \to \pm\infty)$, because

$$\lim_{x\to\pm\infty} \frac{x^{-\gamma-\alpha}}{x^{-\gamma}} = \lim_{x\to\pm\infty} \frac{1}{x^\alpha} = 0.$$

(c) $\sin x = o(\sqrt{x})$ $(x \to 0^+)$ because

$$\lim_{x\to 0^+} \frac{\sin x}{\sqrt{x}} = \lim_{x\to 0^+} \sqrt{x}\frac{\sin x}{x} = \underbrace{\lim_{x\to 0^+} \sqrt{x}}_{=0} \cdot \underbrace{\lim_{x\to 0^+} \frac{\sin x}{x}}_{=1} = 0.$$

(d) $1-\cos x = o(x)$ $(x \to 0)$ because

$$\lim_{x\to 0} \frac{1-\cos x}{x} = \lim_{x\to 0} x\frac{1-\cos x}{x^2} = \underbrace{\lim_{x\to 0} x}_{=0} \cdot \underbrace{\lim_{x\to 0} \frac{1-\cos x}{x^2}}_{=\frac{1}{2}} = 0.$$

(e) Clearly, when $x \to \infty$,

$$(\log x)^r = o(x^q), \qquad x^q = o(a^x), \qquad a^x = o(x^x),$$

provided $r, q > 0$ and $a > 1$.

■

■ **Example 8.2** To illustrate how to manipulate expressions involving o terms let us calculate

$$[1+x+x^2+o(x^2)]^2.$$

By expanding this espression we obtain

$$\begin{aligned}[1+x+x^2+o(x^2)]^2 =& [1+x+x^2+o(x^2)][1+x+x^2+o(x^2)] \\ =& 1+x+x^2+o(x^2)+x+x^2+x^3+x\cdot o(x^2)+x^2+x^3+x^4+x^2\cdot o(x^2) \\ &+o(x^2)+o(x^2)\cdot x+o(x^2)\cdot x^2+o(x^2)\cdot o(x^2).\end{aligned}$$

Our first simplification is to add equal powers and to transform products like $x\cdot o(x^2) = o(x^3)$ (rule (e)) or $o(x^2)\cdot o(x^2) = o(x^4)$ (rule (c)). Then we get

$$\begin{aligned}[1+x+x^2+o(x^2)]^2 =& 1+2x+3x^2+2x^3+x^4+2\cdot o(x^2)+2\cdot o(x^3)+3\cdot o(x^4) \\ =& 1+2x+3x^2+2x^3+x^4+o(x^3)+o(x^2)+o(x^4).\end{aligned}$$

Now we apply rule (d) and transform $o(x^3) = o(x^2)$ and $o(x^4) = o(x^2)$, so

$$[1+x+x^2+o(x^2)]^2 = 1+2x+3x^2+2x^3+x^4+3\cdot o(x^2) = 1+2x+3x^2+2x^3+x^4+o(x^2).$$

Finally, $2x^3 = o(x^2)$ and $x^4 = o(x^2)$, thus

$$[1+x+x^2+o(x^2)]^2 = 1+2x+3x^2+3\cdot o(x^2) = 1+2x+3x^2+o(x^2).$$

This is the simplest form of the result.

For a practical implementation of this calculation we do not really need to take all these intermediate steps. Once an $o(x^2)$ in present in the expression every other one, or every $o(x^n)$ with $n > 2$, or every power x^m with $m > 2$, can be simply neglected. ■

R Note that $f = o(1)$ $(x \to a)$ is another way to express that $\lim_{x\to a} f(x) = 0$.

Strongly related to the notion of negligible function is that of **equivalent** functions when $x \to a$. We will say that $f_1 = f_2 + o(g)$ $(x \to a)$ if $f_1 - f_2 = o(g)$ $(x \to a)$. The idea this represents is that functions f_1 and f_2 differ in a amount that is negligible compared to g when x approaches a.

8.2 Taylor's polynomial

Using the o notation we can re-express some features of a function f. For example, if we know that f is continuous at a we know that

$$\lim_{x\to a} f(x) = f(a).$$

But we can rewrite this expression as

$$\lim_{x\to a}[f(x) - f(a)] = 0,$$

which in terms of the o notation can be stated as

$$f(x) = f(a) + o(1) \quad (x \to a). \tag{8.1}$$

From this viewpoint, a continuous function at a can be approximated by its value at a if we are close enough to a.

We can go a step beyond. Let us assume that a function is differentiable at a. Then

$$\lim_{x\to a} \frac{f(x) - f(a)}{x-a} = f'(a).$$

But this can be rewritten as

$$0 = \lim_{x\to a}\left[\frac{f(x) - f(a)}{x-a} - f'(a)\right] = \lim_{x\to a} \frac{f(x) - f(a) - f'(a)(x-a)}{x-a},$$

in other words,

$$f(x) = f(a) + f'(a)(x-a) + o(x-a) \quad (x \to a). \tag{8.2}$$

Since $y = f(a) + f'(a)(x-a)$ is the equation of the tangent to $f(x)$ at $x = a$, the equation above expresses the fact that differentiable functions at a point a can be approximated by their tangent if x is sufficiently close to a. The $o(x-a)$, compared to the previous $o(1)$ of continuous functions, means that the approximation is better.

We can try to push the idea a bit further and see if this sort of approximations can be improved. Inspired by the two equations (8.1) (8.2) we may try to seek for an expression like

$$f(x) = f(a) + f'(a)(x-a) + c_2(x-a)^2 + o\big((x-a)^2\big) \quad (x \to a),$$

i.e., we can try to use a parabola to approximate better the function near a. For this expression to hold we must have

$$0 = \lim_{x\to a} \frac{f(x) - f(a) - f'(a)(x-a) - c_2(x-a)^2}{(x-a)^2} = \lim_{x\to a}\left[\frac{f(x) - f(a) - f'(a)(x-a)}{(x-a)^2} - c_2\right].$$

But this is equivalent to

$$\lim_{x\to a} \frac{f(x) - f(a) - f'(a)(x-a)}{(x-a)^2} = c_2.$$

Now, to calculate the limit we can apply l'Hôpital's rule (we know that f is differentiable at a). This yields

$$c_2 = \lim_{x\to a} \frac{f'(x) - f'(a)}{2(x-a)} = \frac{1}{2} \lim_{x\to a} \frac{f'(x) - f'(a)}{x-a}.$$

But the last limit is the definition of the derivative of f' at $x = a$, so if we assume that f is twice differentiable at a we conclude that

$$c_2 = \frac{f''(a)}{2}$$

and arrive to the formula

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + o\big((x-a)^2\big) \quad (x \to a). \tag{8.3}$$

Let us do it once more: let us look for an improved approximation of the form

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + c_3(x-a)^3 + o\big((x-a)^3\big) \quad (x \to a).$$

Thus,

$$\begin{aligned} 0 &= \lim_{x\to a} \frac{f(x) - f(a) - f'(a)(x-a) - \frac{f''(a)}{2}(x-a)^2 - c_3(x-a)^3}{(x-a)^3} \\ &= \lim_{x\to a} \left[\frac{f(x) - f(a) - f'(a)(x-a) - \frac{f''(a)}{2}(x-a)^2}{(x-a)^3} - c_3 \right], \end{aligned}$$

or equivalently

$$c_3 = \lim_{x\to a} \frac{f(x) - f(a) - f'(a)(x-a) - \frac{f''(a)}{2}(x-a)^2}{(x-a)^3}.$$

To compute the limit we apply l'Hôpital's rule twice (we know that f is twice differentiable at a) and get

$$c_3 = \lim_{x\to a} \frac{f''(x) - f''(a)}{3\cdot 2(x-a)} = \frac{1}{3!} \lim_{x\to a} \frac{f''(x) - f''(a)}{x-a}.$$

Again this limit is the expression of the derivative of f'' at $x = a$, so assuming f is three times differentiable at a

$$c_3 = \frac{f'''(a)}{3!}$$

(because $3 \cdot 2 = 3!$), which yields

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + o\big((x-a)^3\big) \quad (x \to a) \tag{8.4}$$

(note that $2 = 2!$).

By now we can easily put forth a conjecture: if we define the nth degree polynomial

$$P_{n,a}(x) \equiv f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n, \tag{8.5}$$

which we will refer to as the nth order **Taylor polynomial** of function f at the point a, the previous findings suggest the following theorem:

Theorem 8.2.1 — Taylor's theorem (I). If f is n times differentiable at a, then

$$f(x) = P_{n,a}(x) + o\big((x-a)^n\big) \quad (x \to a). \tag{8.6}$$

Proof. In order to prove that

$$f(x) = P_{n,a}(x) + o\big((x-a)^n\big) \quad (x \to a),$$

we need to calculate the limit

$$\lim_{x\to a} \frac{f(x) - P_{n,a}(x)}{(x-a)^n}$$

and show that it is zero. To do that we apply $n-1$ times l'Hôpital's rule, so that we need to calculate the limit

$$\lim_{x\to a} \frac{f^{(n-1)}(x) - P_{n,a}^{(n-1)}(x)}{n!(x-a)}.$$

But differentiating $n-1$ times $P_{n,a}(x)$ all powers smaller than $n-1$ disappear, and there remain the last two terms. Now,

$$\frac{d^{n-1}}{dx^{n-1}}(x-a)^{n-1} = (n-1)!, \qquad \frac{d^{n-1}}{dx^{n-1}}(x-a)^n = n!(x-a),$$

therefore

$$P_{n,a}^{(n-1)}(x) = f^{(n-1)}(a) + f^{(n)}(a)(x-a).$$

Then the limit becomes

$$\lim_{x\to a} \frac{f^{(n-1)}(x) - f^{(n-1)}(a) - f^{(n)}(a)(x-a)}{n!(x-a)} = \frac{1}{n!}\lim_{x\to a}\left[\frac{f^{(n-1)}(x) - f^{(n-1)}(a)}{x-a} - f^{(n)}(a)\right] = 0$$

because

$$\lim_{x\to a} \frac{f^{(n-1)}(x) - f^{(n-1)}(a)}{x-a}$$

is the definition of the derivative of $f^{(n-1)}$ at a. This completes the proof. ■

R We will write $P_{n,a}(x)$ in a more compact way as

$$P_{n,a}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k, \tag{8.7}$$

where we define $f^{(0)}(a) = f(a)$ and $0! = 1$.

■ **Example 8.3** Consider the function $f(x) = (1+x)^\alpha$, where $\alpha \in \mathbb{R}$. Then $f(0) = 1$ and

$$\begin{aligned}
f'(x) &= \alpha(1+x)^{\alpha-1}, & f'(0) &= \alpha,\\
f''(x) &= \alpha(\alpha-1)(1+x)^{\alpha-2}, & f''(0) &= \alpha(\alpha-1),\\
f'''(x) &= \alpha(\alpha-1)(\alpha-2)(1+x)^{\alpha-3}, & f'''(0) &= \alpha(\alpha-1)(\alpha-2),\\
&\vdots & &\vdots\\
f^{(n)}(x) &= \alpha(\alpha-1)\cdots(\alpha-n+1)(1+x)^{\alpha-n}, & f^{(n)}(0) &= \alpha(\alpha-1)\cdots(\alpha-n+1).
\end{aligned}$$

Therefore

$$(1+x)^\alpha = 1+\alpha x+\frac{\alpha(\alpha-1)}{2}x^2+\cdots+\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n+o(x^n) \quad (x\to 0).$$

There is an interesting notation for this expression derived from the formula for the binomial coefficients. From equation (B.4), if $\alpha\in\mathbb{N}$,

$$\binom{\alpha}{n}=\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}.$$

Since this formula is meaningful even if $\alpha\in\mathbb{R}$, we use it as a definition and thus write

$$(1+x)^\alpha=\sum_{k=0}^{n}\binom{\alpha}{k}x^k+o(x^n) \quad (x\to 0).$$

This is the famous binomial formula as it was first obtained by Newton in 1665. ■

8.3 Calculating limits

Taylor's theorem can be applied to calculating complicated limits. As a matter of fact, it is more powerful than l'Ĥopital's rule in dealing with indeterminacies. A few examples will illustrate the procedure.

■ **Example 8.4** Suppose we want to calculate the limit

$$\lim_{x\to 0}\frac{\cos x-e^x+x}{x^2}.$$

All we need to do is to use the Taylor expansions

$$\cos x = 1-\frac{x^2}{2}+o(x^2), \qquad e^x=1+x+\frac{x^2}{2}+o(x^2).$$

Then

$$\cos x-e^x+x=1-\frac{x^2}{2}+o(x^2)-1-x-\frac{x^2}{2}+o(x^2)+x=-x^2+o(x^2).$$

Now

$$\lim_{x\to 0}\frac{\cos x-e^x+x}{x^2}=\lim_{x\to 0}\frac{-x^2+o(x^2)}{x^2}=-1+\lim_{x\to 0}\frac{o(x^2)}{x^2}.$$

But by definition the last limit is 0, so

$$\lim_{x\to 0}\frac{\cos x-e^x+x}{x^2}=-1.$$

■

■ **Example 8.5** In Example 7.12 we calculated the limit

$$\lim_{x\to 0}\frac{e^x-x-\cos x}{\sin x^2},$$

by applying twice l'Hôpital's rule. Let us do the same using Taylor expansion. As for the denominator, since $\sin y = y+o(y)$,

$$\sin x^2=x^2+o(x^2).$$

$f(x)$	$P_{k,0}(x)$	$o(x^k)$
$(1+x)^\alpha$	$1+\alpha x+\dfrac{\alpha(\alpha-1)}{2}x^2+\cdots+\dfrac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n$	$o(x^n)$
$\log(1+x)$	$x-\dfrac{x^2}{2}+\dfrac{x^3}{3}-\dfrac{x^4}{4}+\cdots+(-1)^{n+1}\dfrac{x^n}{n}$	$o(x^n)$
e^x	$1+x+\dfrac{x^2}{2!}+\dfrac{x^3}{3!}\cdots+\dfrac{x^n}{n!}$	$o(x^n)$
$\sin x$	$x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}-\dfrac{x^7}{7!}+\cdots+(-1)^n\dfrac{x^{2n+1}}{(2n+1)!}$	$o(x^{2n+2})$
$\sinh x$	$x+\dfrac{x^3}{3!}+\dfrac{x^5}{5!}+\dfrac{x^7}{7!}+\cdots+\dfrac{x^{2n+1}}{(2n+1)!}$	$o(x^{2n+2})$
$\cos x$	$1-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}-\dfrac{x^6}{6!}+\cdots+(-1)^n\dfrac{x^{2n}}{(2n)!}$	$o(x^{2n+1})$
$\cosh x$	$1+\dfrac{x^2}{2!}+\dfrac{x^4}{4!}+\dfrac{x^6}{6!}+\cdots+\dfrac{x^{2n}}{(2n)!}$	$o(x^{2n+1})$
$\arcsin x$	$x+\dfrac{x^3}{2\cdot 3}+\dfrac{1\cdot 3}{2\cdot 4\cdot 5}x^5+\dfrac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7}x^7+\cdots+\dfrac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots(2n)(2n+1)}x^{2n+1}$	$o(x^{2n+2})$
$\operatorname{arcsinh} x$	$x-\dfrac{x^3}{2\cdot 3}+\dfrac{1\cdot 3}{2\cdot 4\cdot 5}x^5-\dfrac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7}x^7+\cdots+(-1)^n\dfrac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots(2n)(2n+1)}x^{2n+1}$	$o(x^{2n+2})$
$\arctan x$	$x-\dfrac{x^3}{3}+\dfrac{x^5}{5}-\dfrac{x^7}{7}+\cdots+(-1)^n\dfrac{x^{2n+1}}{2n+1}$	$o(x^{2n+2})$
$\operatorname{arctanh} x$	$x+\dfrac{x^3}{3}+\dfrac{x^5}{5}+\dfrac{x^7}{7}+\cdots+\dfrac{x^{2n+1}}{2n+1}$	$o(x^{2n+2})$
$\tan x$	$x+\dfrac{1}{3}x^3+\dfrac{2}{15}x^5+\dfrac{17}{315}x^7$	$o(x^8)$
$\tanh x$	$x-\dfrac{1}{3}x^3+\dfrac{2}{15}x^5-\dfrac{17}{315}x^7$	$o(x^8)$

Table 8.1: Taylor polynomials of some elementary functions as $x \to 0$. (Here $\alpha \in \mathbb{R}$.)

This suggests expanding the numerator up to x^2. Thus,

$$e^x - x - \cos x = 1 + x + \frac{x^2}{2} + o(x^2) - x - \left[1 - \frac{x^2}{2} + o(x^2)\right] = x^2 + o(x^2)$$

Therefore

$$\lim_{x\to 0} \frac{e^x - x - \cos x}{\sin x^2} = \lim_{x\to 0} \frac{x^2 + o(x^2)}{x^2 + o(x^2)} = \lim_{x\to 0} \frac{1 + o(1)}{1 + o(1)} = 1.$$

■

■ **Example 8.6** Here is a complicated limit:

$$\lim_{x\to\infty} x^2 \left(1 - \sec \frac{1}{x}\right).$$

To calculate this limit it is convenient to change to the variable $t = 1/x$, so that

$$\lim_{x\to\infty} x^2\left(1-\sec\frac{1}{x}\right) = \lim_{t\to 0^+}\frac{1}{t^2}\left(1-\frac{1}{\cos t}\right).$$

Now, $\cos t = 1-\frac{t^2}{2}+o(t^2)$ $(t\to 0)$, therefore the limit becomes

$$\lim_{t\to 0^+}\frac{1}{t^2}\left(1-\frac{1}{1-\frac{t^2}{2}+o(t^2)}\right) = \lim_{t\to 0^+}\frac{1}{t^2}\frac{1-\frac{t^2}{2}+o(t^2)-1}{1-\frac{t^2}{2}+o(t^2)} = \lim_{t\to 0^+}\frac{-\frac{1}{2}+o(1)}{1+o(1)} = -\frac{1}{2}.$$

Here we have used that $o(t^2)/t^2 = o(1)$, $-\frac{t^2}{2} = o(1)$, and $o(t^2) = o(1)$, as $t\to 0$. ■

8.4 Remainder and Taylor's theorem

The difference between the function $f(x)$ and its Taylor polynomial $P_{n,a}(x)$ is called the remainder, and denoted $R_{n,a}(x)$. It is the error we make when approximating $f(x)$ by its Taylor polynomial. So far we only know that $R_{n,a}(x) = o\left((x-a)^n\right)$ as $x\to a$, but it would be interesting to have a quantitative estimate of that error. As a matter of fact, Cauchy's mean value theorem can help us in deriving such an expression. The result is a second version of Taylor's theorem that yields an explicit form for the remainder.

Theorem 8.4.1 — Taylor's theorem (II). Let f be a function $n+1$ times differentiable in an interval I, and let $a\in I$. Then, for every $x\in I$ there exists some c between a and x such that

$$f(x) = P_{n,a}(x)+\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}. \tag{8.8}$$

Proof. For each $b\in I$ let us define two functions:

$$F(x) = f(b)-P_{n,x}(b), \qquad G(x) = (b-x)^{n+1}, \qquad x\in I.$$

Note that

$$P_{n,x}(b) = \sum_{k=0}^{n}\frac{f^{(k)}(x)}{k!}(b-x)^k$$

is no longer a polynomial, but a complicated combination of the function $f(x)$ and its n first derivatives. Also note that both functions, F and G, are differentiable in I.

Under this conditions Cauchy's mean value theorem states that there exists c between a and b such that

$$\frac{F(b)-F(a)}{G(b)-G(a)} = \frac{F'(c)}{G'(c)}.$$

Let us now compute all these terms and see what this expression amounts to.

First of all $P_{n,b}(b) = f(b)$, hence $F(b) = 0$. Then, $F(a) = f(b)-P_{n,a}(b)$. Now, $G(b) = 0$ and $G(a) = (b-a)^{n+1}$. Thus,

$$\frac{F(b)-F(a)}{G(b)-G(a)} = \frac{F(a)}{G(a)} = \frac{f(b)-P_{n,a}(b)}{(b-a)^{n+1}}.$$

On the other hand $G'(x) = -(n+1)(b-x)^n$, and

$$\begin{aligned}\frac{d}{dx}P_{n,x}(b) &= f'(x) + \sum_{k=1}^{n}\left[\frac{f^{(k+1)}(x)}{k!}(b-x)^k + \frac{f^{(k)}(x)}{k!}k(b-x)^{k-1}(-1)\right] \\ &= f'(x) + \sum_{k=1}^{n}\left[\frac{f^{(k+1)}(x)}{k!}(b-x)^k - \frac{f^{(k)}(x)}{(k-1)!}(b-x)^{k-1}\right].\end{aligned}$$

The sum in the last expression is telescoping. Thus,

$$\sum_{k=1}^{n}\left[\frac{f^{(k+1)}(x)}{k!}(b-x)^k - \frac{f^{(k)}(x)}{(k-1)!}(b-x)^{k-1}\right] = \frac{f^{(n+1)}(x)}{n!}(b-x)^n - f'(x),$$

which implies

$$F'(x) = -\frac{d}{dx}P_{n,x}(b) = -\frac{f^{(n+1)}(x)}{n!}(b-x)^n.$$

Accordingly,

$$\frac{F'(c)}{G'(c)} = \frac{-\frac{f^{(n+1)}(c)}{n!}(b-c)^n}{-(n+1)(b-c)^n} = \frac{f^{(n+1)}(c)}{(n+1)!}.$$

The theorem thus reads

$$\frac{f(b)-P_{n,a}(b)}{(b-a)^{n+1}} = \frac{f^{(n+1)}(c)}{(n+1)!},$$

or equivalently

$$f(b) = P_{n,a}(b) + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}.$$

Changing the name of b to x this is the stament of the theorem. ■

The theorem provides an expression for the remainder that bears the name **Lagrange's remainder.** Given that $c = (1-\theta)a + \theta x$ for some $0 < \theta < 1$, it is customary to write it in the form

$$R_{n,a}(x) = \frac{f^{(n+1)}\big((1-\theta)a+\theta x\big)}{(n+1)!}(x-a)^{n+1}, \qquad 0 < \theta < 1. \tag{8.9}$$

This is not the only possible form in which we can write the remainder. For instance, in the proof of the theorem Cauchy chose to apply Cauchy's mean value theorem to the same function $F(x)$ but to $G(x) = b - x$. The expression this yields for the remainder is

$$R_{n,a}(x) = \frac{f^{(n+1)}(c)}{n!}(x-c)^n(x-a).$$

So we have **Cauchy's remainder** that can be expressed as

$$R_{n,a}(x) = \frac{f^{(n+1)}\big((1-\theta)a+\theta x\big)}{n!}(1-\theta)^n(x-a), \qquad 0<\theta<1. \tag{8.10}$$

In further chapters we will see yet another form for the remainder involving integrals.

Corollary 8.4.2 $P_n(x) = P_{n,a}(x)$ for any polynomial $P_n(x)$ of degree n and every $a \in \mathbb{R}$.

Proof. If $f(x) = P_n(x)$ then $R_{n,a}(x) = 0$ no matter what a is because $f^{n+1}(a) = 0$ for every $a \in \mathbb{R}$. Hence Taylor formula is exact. ∎

▪ **Example 8.7** To write down the polynomial $P(x) = 1-2x^2+x^3$ in powers of $x-1$ all we need to do is to obtain its Taylor's polynomial $P_{3,1}(x)$. Since

$$\begin{aligned}
P(x) &= 1-2x^2+x^3, & P(1) &= 0,\\
P'(x) &= -4x+3x^2, & P'(1) &= -1,\\
P''(x) &= -4+6x, & P''(1) &= 2,\\
P'''(x) &= 6, & P'''(1) &= 6,
\end{aligned}$$

we have $P(x) = P_{3,1}(x) = -(x-1)+(x-1)^2+(x-1)^3$. (Check that it is indeed the same polynomial by expanding the two binomials and simplifying.) ▪

8.5 Taylor series

Suppose we have a function f that can be differentiated infinitely often in an interval containing a. For this function we have a formula

$$f(x) = \sum_{n=0}^{k} \frac{f^{(n)}(a)}{n!}(x-a)^n + R_{k,a}(x)$$

for each $k \in \mathbb{N}$ and every x in the interval. We can then take the limit when $k \to \infty$ in this expression. Since the left-hand side does not depend on k we will obtain

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n + \lim_{k\to\infty} R_{k,a}(x). \tag{8.11}$$

Those functions for which

$$\lim_{k\to\infty} R_{k,a}(x) = 0 \tag{8.12}$$

for every x in the interval are given by their **Taylor series**

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n, \tag{8.13}$$

as long as the series converges.

The expression we have just obtained is a particular case of a class of series referred to as **power series.** These are series of the form

$$\sum_{n=0}^{\infty} a_n(x-a)^n, \qquad a_n \in \mathbb{R}. \tag{8.14}$$

We can assess the absolute convergence of these series using the root test. Accordingly, one such series will converge absolutely if

$$\lim_{n\to\infty} \sqrt[n]{|a_n||x-a|^n} < 1 \qquad \Leftrightarrow \qquad \left(\lim_{n\to\infty} \sqrt[n]{|a_n|}\right)|x-a| < 1. \tag{8.15}$$

We can define the number $\rho > 0$ by the formula

$$\frac{1}{\rho} \equiv \lim_{n\to\infty} \sqrt[n]{|a_n|}. \tag{8.16}$$

We refer to ρ as the **convergence radius** of the series because condition (8.15) holds for every x such that

$$|x-a| < \rho. \tag{8.17}$$

In other words, the series (8.14) converges absolutely in the interval $(a-\rho, a+\rho)$.

What happens outside this interval? If $|x-a| > \rho$ then

$$\lim_{n\to\infty} \sqrt[n]{|a_n||x-a|^n} = \frac{|x-a|}{\rho} \equiv \ell > 1.$$

Hence, for any $\varepsilon > 0$,

$$\ell - \varepsilon < \sqrt[n]{|a_n||x-a|^n} < \ell + \varepsilon \qquad \Rightarrow \qquad (\ell-\varepsilon)^n < |a_n||x-a|^n$$

if n is large enough. As we can take ε so that $\ell - \varepsilon > 1$ (e.g., $\varepsilon = (\ell-1)/2$), then

$$\lim_{n\to\infty} (\ell-\varepsilon)^n = \infty \qquad \Rightarrow \qquad \lim_{n\to\infty} |a_n||x-a|^n = \infty \qquad \Rightarrow \qquad \lim_{n\to\infty} a_n(x-a)^n \neq 0.$$

Therefore the power series does not converge.

In summary, the power series converges if $x \in (a-\rho, a+\rho)$ and diverges otherwise, except maybe at $x = a \pm \rho$ (where the root test yields a limit 1 and does not decide). At these two points the analysis has to be done on a case-by-case basis.

R As a consequence of Corollary 3.4.4, the convergence radius can also be obtained as

$$\frac{1}{\rho} = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

provided this limit exists.

■ **Example 8.8** Consider the Taylor expansion of $f(x) = e^x$ with remainder. Given that $f^{(n)}(x) = e^x$ we will have

$$e^x = \sum_{k=0}^{n} \frac{x^n}{n!} + R_{n,0}(x), \qquad R_{n,0}(x) = e^{\theta x}\frac{x^{n+1}}{(n+1)!}, \qquad 0 < \theta < 1.$$

Since the exponential is an increasing function, $e^{\theta x} < \max\{1, e^x\}$ —that includes the cases $x > 0$ and $x < 0$. Therefore

$$0 < R_{n,0}(x) < \max\{1, e^x\}\frac{x^n}{n!} \xrightarrow[n\to\infty]{} 0$$

for any given $x \in \mathbb{R}$. Hence

$$e^x = \sum_{k=0}^{\infty} \frac{x^n}{n!}.$$

■

Table 8.2 is a version of Table 8.1 containing a list with the Taylor series of some elementary fractions, along with their convergence radii. For some of these series is easy to prove the equality with the function —as the case of the exponential—, for other it is more difficult, but for all of them it can be proven that $R_{n,0}(x) \to 0$ as $n \to \infty$.

A nice —and very useful— property of power series is that they can be differentiated within their interval of convergence. The precise statement of this property is:

Theorem 8.5.1 Let $f(x)$ be an infinitely often differentiable function such that

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n, \qquad |x-a| < \rho.$$

Then

$$f'(x) = \sum_{n=1}^{\infty} n a_n (x-a)^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} (x-a)^n,$$

and the convergence radius of this series is also ρ.

Aside from the practical applications of this theorem, there is a very important consequence that we can extract from it:

Corollary 8.5.2 The power series of $f(x)$ in Theorem 8.5.1 is unique.

Proof. According to Theorem 8.5.1, the series for f can be differentiated term by term infinitely often, and

$$f^{(k)}(x) = \sum_{n=0}^{\infty} (n+k)(n+k-1)\cdots(n+1) a_{n+k} (x-a)^n,$$

so $f^{(k)}(a) = k! a_k$, which implies $a_k = f^{(k)}(a)/k!$—hence the coefficients are uniquely determined by the function and its derivatives at $x = a$. ■

■ **Example 8.9** The differentiability of power series helps finding some series. For instance, suppose we want to obtain a power series, in powers of x, of $f(x) = \arctan x$. We know of this function that is odd, so its powers series can only contain odd powers. In other words,

$$\arctan x = \sum_{n=0}^{\infty} a_n x^{2n+1}.$$

But we also know that

$$(\arctan x)' = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)},$$

so

$$\sum_{n=0}^{\infty} (2n+1) a_n x^{2n} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}, \qquad |x| < 1.$$

As power series are unique, the coefficients of the two series that represent the same function must be the same, therefore $a_n = (-1)^n/(2n+1)$ and

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}, \qquad |x| < 1.$$

■

$f(x)$	Taylor series	ρ
$(1+x)^\alpha$	$\displaystyle\sum_{n=0}^{\infty}\binom{\alpha}{n}x^n$	1
$\log(1+x)$	$\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}$	1
e^x	$\displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!}$	∞
$\sin x$	$\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$	∞
$\sinh x$	$\displaystyle\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}$	∞
$\cos x$	$\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}$	∞
$\cosh x$	$\displaystyle\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}$	∞
$\arcsin x$	$\displaystyle\sum_{n=0}^{\infty}\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots(2n)(2n+1)}x^{2n+1}$	1
$\operatorname{arcsinh} x$	$\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots(2n)(2n+1)}x^{2n+1}$	1
$\arctan x$	$\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$	1
$\operatorname{arctanh} x$	$\displaystyle\sum_{n=0}^{\infty}\frac{x^{2n+1}}{2n+1}$	1

Table 8.2: Taylor series of some elementary functions as powers of x, along with their convergence radii ρ. (Here $\alpha \in \mathbb{R}$.)

8.6 Numerical approximations

With the expression of the remainder we can find bounds to the error that we incur when approximating a function by its Taylor polynomial of a certain degree. This allows us to obtain numerical values of transcendental functions —which would otherwise be difficult to obtain. Some examples illustrate the method.

■ **Example 8.10** We know that

$$\sin x = x - \frac{x^3}{6} + R_{4,0}(x), \qquad R_{4,0}(x) = \frac{\cos(\theta x)}{120}x^5, \quad 0 < \theta < 1.$$

We of course ignore the value of θ (otherwise $\sin x$ could be exactly computed), but we know that irrespective of θ and x, $|\cos(\theta x)| \leqslant 1$. Thus,

$$|R_{4,0}(x)| \leqslant \frac{|x|^5}{120}.$$

Suppose we want to compute $\sin(0.1)$. From the previous inequality $|R_{4,0}(x)| \leqslant 8.3333 \times 10^{-8}$. Now compare:

$$\sin(0.1) = \mathbf{0.09983}41664\ldots, \qquad P_{4,0}(0.1) = \mathbf{0.09983}33333\ldots$$

The error incurred using this simple approximation is 8.3313×10^{-8}, very close to our estimate.

Suppose we do not want our error to be larger than 10^{-5}. What is the largest x for which we can use this approximation? To answer this question we simply set the estimate to the error tolerance and find $|x|$:

$$\frac{|x|^5}{120} = 10^{-5} \qquad \Rightarrow \qquad |x| = \sqrt[5]{120}\times 0.1 \approx 0.26.$$

■

■ **Example 8.11** Imagine that we want to compute $\sqrt{3.8}$. We can do it by expanding the function $\sqrt{4-x}$ around $x=0$. Thus,

$$\begin{aligned}
f(x) &= \sqrt{4-x}, & f(0) &= \sqrt{4} = 2,\\
f'(x) &= \frac{-1}{2\sqrt{4-x}}, & f'(0) &= \frac{-1}{2\sqrt{4}} = -\frac{1}{4},\\
f''(x) &= \frac{-1}{4(4-x)^{3/2}}, & f''(0) &= \frac{-1}{4\cdot 4^{3/2}} = -\frac{1}{32},\\
f'''(x) &= \frac{-3}{8(4-x)^{5/2}}.
\end{aligned}$$

Then

$$\sqrt{4-x} = 2 - \frac{x}{4} - \frac{x^2}{64} + R_{2,0}(x), \qquad R_{2,0}(x) = \frac{-1}{16(4-\theta x)^{5/2}}x^3, \quad 0<\theta<1.$$

If $x>0$,

$$|R_{2,0}(x)| < \frac{x^3}{16(4-x)^{5/2}} = \frac{x^3}{16\left(\sqrt{4-x}\right)^5}.$$

Now we can estimate

$$\sqrt{3.8} = P_{2,0}(0.2) = 2 - \frac{0.2}{4} - \frac{(0.2)^2}{64} = 1.949375\ldots$$

and use this estimation in the error bound

$$|R_{2,0}(x)| < \frac{(0.2)^3}{16(1.949375\ldots)^5} \approx 1.78\times10^{-5}.$$

As a matter of fact,

$$\sqrt{3.8} = \mathbf{1.949}\mathbf{3}58869\ldots, \qquad P_{2,0}(0.2) = \mathbf{1.9493}75,$$

the difference being 1.61×10^{-5}. ■

8.7 Local behaviour of functions

We saw in Corollary 7.3.4 that the sign of $f'(x)$ determines wether the function is increasing (positive) or decreasing (negative) at x, and Theorem 7.3.1 showed that at local extrema the function satisfies $f'(x)=0$ (provided it is differentiable). In its second formulation —with the remainder— Taylor's theorem provides a more detailed information about the local behaviour of a function which has higher order derivatives.

Before getting into it, we need to characterise another qualitative feature of functions: whether their slope increases or decreases. This feature is called *convexity.*

We say that f is **convex** at $x = a$ if it is locally *above* its tangent at that point, i.e., $f(x) > f(a) + f'(a)(x-a)$ for all $0 < |x-a| < \varepsilon$, for some $\varepsilon > 0$.

Likewise, we say that it is **concave** at $x = a$ if it is locally *below* its tangent a that point, i.e., $f(x) < f(a) + f'(a)(x-a)$ for all $0 < |x-a| < \varepsilon$, for some $\varepsilon > 0$.

Finally, we say that f has an **inflection point** at $x = a$ if the sign of $f(x) - f(a) - f'(a)(x-a)$ is different for $x < a$ and for $x > a$.

Figure 8.1 illustrates these three behaviours.

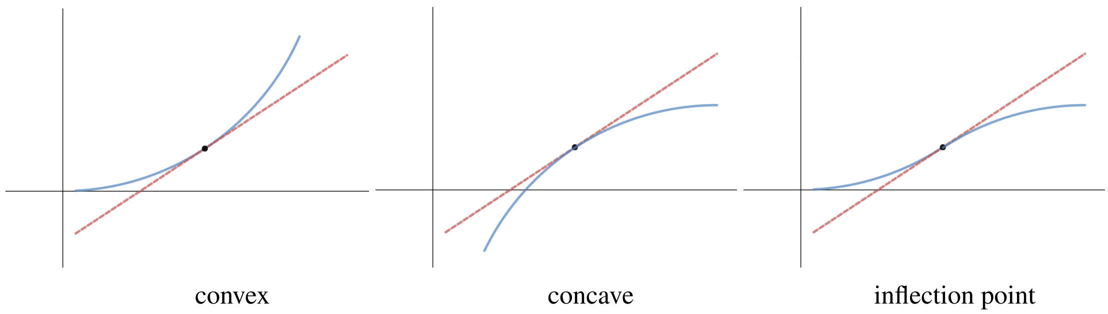

Figure 8.1: Local behaviour of a function with respect to its tangent at a point (convexity).

Suppose that a function f can be differentiated several times (posibly infinitely many) in a certain interval and that the first nonzero derivative beyond the first at $x = a$ is $f^{(n)}(a)$. We can use Taylor's theorem —with Lagrange's remainder— to write

$$f(x) = f(a) + f'(a)(x-a) + \frac{f^{(n)}(c)}{n!}(x-a)^n,$$

where $c = (1-\theta)a + \theta x$ with $0 < \theta < 1$. One important point to stress here is that, since $f^{(n)}(a) \neq 0$ —so it is either positive or negative—, when x is sufficiently close to a —and so is c— $f^{(n)}(c)$ will have the same sign as $f^{(n)}(a)$. This is key for the argument to come.

Since we can write the Taylor expansion as

$$f(x) - f(a) - f'(a)(x-a) = \frac{f^{(n)}(c)}{n!}(x-a)^n,$$

the sign of the left-hand side —which decides the convexity— will be determined by sign of the product $f^{(n)}(c)(x-a)^n$ or, given what we have just argued, by the sign of the product $f^{(n)}(a)(x-a)^n$.

Now, if n is odd, the sign of $f^{(n)}(a)$ is irrelevant because $(x-a)^n$ has a different sign for $x < a$ and for $x > a$. Therefore a will be an *inflection point.*

If n is even then $(x-a)^n > 0$ for all $x \neq a$. Then the sign is determined by that of $f^{(n)}(a)$. We will then have two possibilities:

(a) $f^{(n)}(a) > 0$, and then the function is *convex,* or

(b) $f^{(n)}(a) < 0$, and then the function is *concave.*

If added to that we have that $f'(a) = 0$, then for n odd nothing changes —hence $x = a$ still is an inflection point—, but for n even the point $x = a$ is a local extremum. A convex extremum ($f^{(n)}(a) > 0$) is a *local minimum* and a concave extremum ($f^{(n)}(a) < 0$) is a *local maximum.*

All these results are summarised in Table 8.3.

n	sign of $f^{(n)}(a)$	$f'(a) \neq 0$	$f'(a) = 0$
odd	$+/-$	inflection point	inflection point
even	$+$	convex	local minimum
even	$-$	convex	local maximum

Table 8.3: Classification of the local behaviour of a function according to the sign of the first nonzero derivative $f^{(n)}(a)$ with $n > 1$.

8.8 Function graphing

All the local information provided by the derivatives can be gathered to sketch a qualitative graph of any function $f(x)$. The steps to follow in graphing a function are these (some of them might not be necessary):

1. **Domain:** Determine precisely the set of points where the function $f(x)$ is defined.
2. **Symmetries:** It is helpful to know whether the function has one of these symmetries:
   (a) *Even:* $f(-x) = f(x)$.
   (b) *Odd:* $f(-x) = -f(x)$.
   (c) *Periodic:* $f(x+c) = f(x)$ for some $c > 0$.
   In the first two cases it is enough to represent the function for $x \geqslant 0$ (for $x < 0$ it is represented using the symmetry). In the last case it is enough to represent the function in the interval $[0, c]$ (or any other interval of the same lenght) and then reproduce its graph periodically. Other symmetries might be possible (e.g., $f(a+x) = \pm f(a-x)$, i.e., f is even/odd around the vertical axis $x = a$).
3. **Continuity and differentiability:** Discontinuities ("jumps") and points where $f'(x)$ does not exists ("cusps") are relevant features of the function, and might be useful in detecting local extrema.
4. **Zeroes:** Finding the solutions of $f(x) = 0$ determines where f crosses the X axis. These points separate regions where the sign of f remains constant.
5. **Growth:** Finding the solutions of $f'(x) = 0$ determines the regions where f increases ($f'(x) > 0$) or decreases ($f'(x) > 0$). Usually this is enough to locate the extrema of f.
6. **Convexity:** The convex/concave regions are usually determined by the sign of $f''(x)$. Inflections points can be inferred from that information (as points where the concavity changes).
7. **Asymptotes:** These are known curves (usually straight lines) which $f(x)$ approaches when it gets close to some points or to $\pm\infty$. The main ones are:
   (a) *Vertical asymptotes:* These are the vertical straight lines through the points $x = a$ where $\lim_{x \to a^{\pm}} f(x) = \pm\infty$.
   (b) *Horizontal asymptotes:* These are the horizontal straight lines $y = \ell$ where ℓ is such that $\lim_{x \to \pm\infty} f(x) = \ell$.
   (c) *Inclined asymptotes:* We say that $y = mx + b$ is an asymptote of $f(x)$ when $x \to \pm\infty$ if
   $$m = \lim_{x \to \pm\infty} \frac{f(x)}{x}, \qquad b = \lim_{x \to \pm\infty} [f(x) - mx].$$
   (In other words, $f(x) = mx + b + o(1)$ $(x \to \pm\infty)$.)
   Other types of asymptote are possible. In general, the curve $y = g(x)$ is an asymptote of f when $x \to \pm\infty$ if $f(x) = g(x) + o(1)$ $(x \to \pm\infty)$.

■ **Example 8.12** Sketch the graph of

$$f(x) = \frac{3x^2 + x + 1}{x + 2}.$$

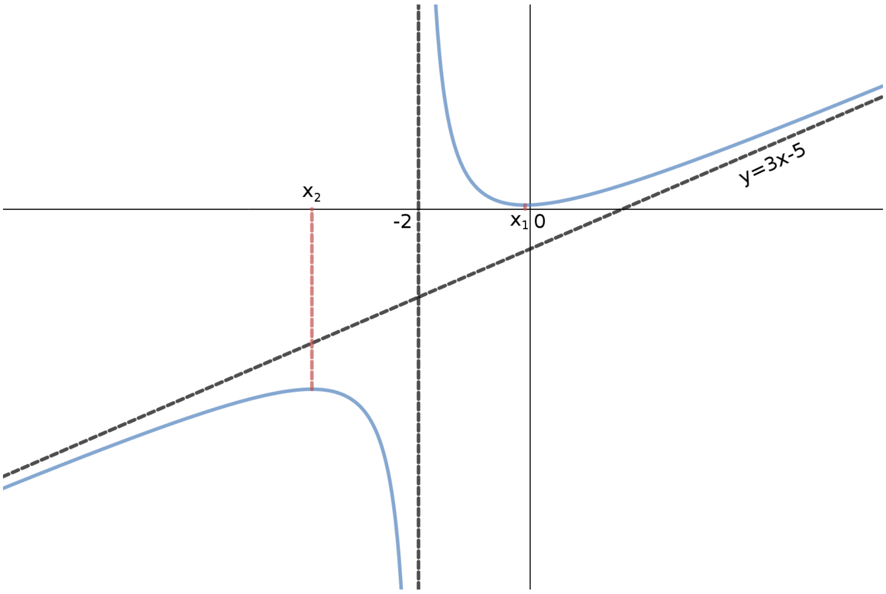

Figure 8.2: Sketch of $f(x) = \dfrac{3x^2+x+1}{x+2}$.

The domain of this function is $\mathbb{R} - \{-2\}$ (because the denominator vanishes at that point.) It has no obvious symmetries and, being a rational function, it is continuous and differentiable (an infinite number of times) in all its domain.

We can obtain the derivative as

$$f'(x) = \frac{(6x+1)(x+2)-(3x^2+x+1)}{(x+2)^2} = \frac{6x^2+13x+2-3x^2-x-1}{(x+2)^2} = \frac{3x^2+12x+1}{(x+2)^2}.$$

This derivative vanishes when $3x^2+12x+1=0$. The roots of this parabola are $x=-2\pm\sqrt{11/3}$, i.e., $x_1 \approx -0.085$, $x_2 \approx -3.91$. For $x < x_2$ and $x > x_1$ function f increases ($f' > 0$) and for $x_2 < x < x_1$ it decreases ($f' < 0$).

f has no zeros because $3x^2+x+1 > 0$ for all $x \in \mathbb{R}$ (the parabola has no roots). So $f(x) < 0$ for $x < -2$ and $f(x) > 0$ for $x > -2$.

It is not necessary to analyse the concavity, as it can be inferred from all the other information, including that of the asymptotes. We know there is a vertical asymptote at $x=-2$ because

$$\lim_{x\to-2^-} f(x) = -\infty, \qquad \lim_{x\to-2^+} f(x) = +\infty.$$

There are no horizontal asymptotes because f diverges when $x \to \pm\infty$. However, we can express the polynomial $P(x) = 3x^2+x+1$ in powers of $x+2$ using Taylor's polynomial, because $P_{2,-2}(x) = P(x)$. As

$$\begin{aligned} P(x) &= 3x^2+x+1, & P(-2) &= 11,\\ P'(x) &= 6x+1, & P'(-2) &= -11,\\ P''(x) &= 6, & P''(-2) &= 6, \end{aligned}$$

we have $P(x) = 11 - 11(x+2) + 3(x+2)^2$. Therefore

$$\begin{aligned} f(x) &= \frac{3x^2+x+1}{x+2} = \frac{11-11(x+2)+3(x+2)^2}{x+2} = \frac{11}{x+2} - 11 + 3(x+2) = \frac{11}{x+2} - 5 + 3x\\ &= 3x - 5 + o(1) \quad (x \to \pm\infty), \end{aligned}$$

i.e., $y = 3x - 5$ is an inclined asymptote both when $x \to \pm\infty$.

$f(x)$ is represented in Figure 8.2. ■

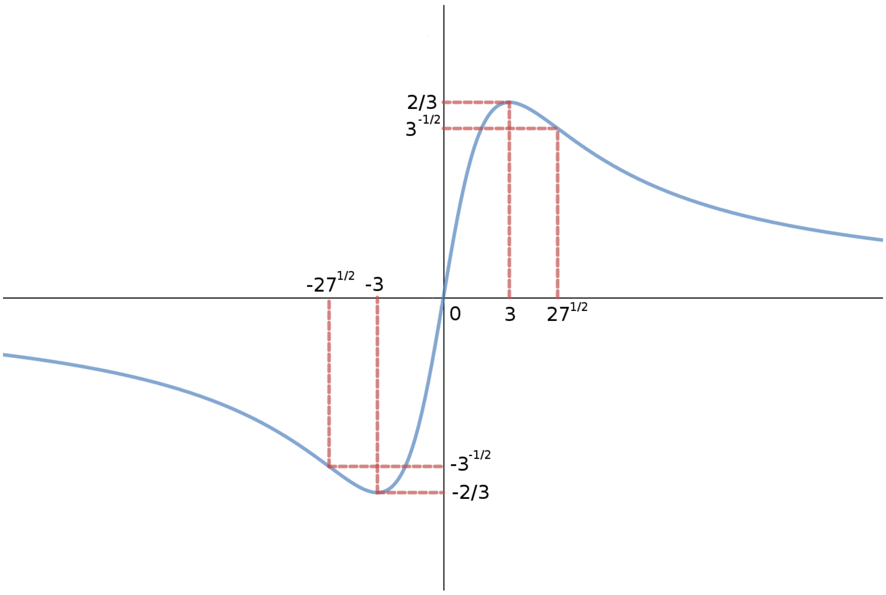

Figure 8.3: Sketch of $f(x) = \dfrac{4x}{x^2+9}$.

■ **Example 8.13** Sketch the graph of

$$f(x) = \frac{4x}{x^2+9}.$$

The domain of this function is $\mathbb{R}$, and it is continuous and differentiable everywhere. It is an odd function because

$$f(-x) = \frac{4(-x)}{(-x)^2+9} = -\frac{4x}{x^2+9} = -f(x),$$

so we only need to care about the region $x \geqslant 0$. As every odd continuous function $f(0) = 0$, and this is the only point where f croses the X axis. Besides $f(x) > 0$ for $x > 0$.

Its derivative is

$$f'(x) = \frac{4(x^2+9) - 4x \cdot 2x}{(x^2+9)^2} = \frac{4x^2+36-8x^2}{(x^2+9)^2} = \frac{4(9-x^2)}{(x^2+9)^2}.$$

Thus, in $x \geqslant 0$ we have $f'(x) > 0$ for $x < 3$ and $f'(x) < 0$ for $x > 3$. The function grows up to $x = 3$, where it has a local maximum, and then decreases beyond that point.

As for the second derivative,

$$\begin{aligned} f''(x) &= \frac{-8x(x^2+9)^2 - (36-4x^2)2(x^2+9)2x}{(x^2+9)^4} = \frac{-8x(x^2+9) - (36-4x^2)4x}{(x^2+9)^3} \\ &= \frac{8x^3 - 216x}{(x^2+9)^3} = \frac{8x(x^2-27)}{(x^2+9)^3}, \end{aligned}$$

so f is concave ($f'' < 0$) for $x < \sqrt{27} = 3\sqrt{3}$ and convex ($f'' > 0$) for $x > 3\sqrt{3}$. At $x = 3\sqrt{3}$ there is an inflection point.

Finally, there are no vertical asymptotes (f is defined in the whole $\mathbb{R}$), but since $\lim_{x\to\infty} f(x) = 0$, the X axis is a horizontal asymptote.

$f(x)$ is represented in Figure 8.3. ■

Problems

Problem 8.1 Write the Taylor polynomial $P_{5,0}(x)$ for these functions:

(i) $e^x \sin x$;
(ii) $e^{-x^2}\cos 2x$;
(iii) $\sin x \cos 2x$;
(iv) $e^x \log(1-x)$;
(v) $\sin^2 x$;
(vi) $\dfrac{1}{1-x^3}$.

Problem 8.2 Write the polynomial $x^4 - 5x^3 + x^2 - 3x + 4$ in powers of $x-4$.

Problem 8.3 Write the Taylor polynomial $P_{n,a}(x)$ for these functions around the specified a:

(i) $f(x) = 1/x$ around $a = -1$;
(ii) $f(x) = xe^{-2x}$ around $a = 0$;
(iii) $f(x) = (1+e^x)^2$ around $a = 0$;
(iv) $f(x) = \sin x$ around $a = \pi$.

Problem 8.4 Consider the function

$$f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0, \\ 0 & x = 0. \end{cases}$$

(i) Prove by induction that $f^{(n)}(x) = Q_n(1/x)e^{-1/x^2}$ for $x \neq 0$, where $Q_n(t)$ is some polynomial.
(ii) Prove by induction that $f^{(n)}(0) = 0$ for all $n \in \mathbb{N}$.
(iii) Write the Taylor polynomial $P_{n,0}(x)$ of $f(x)$. What can you conclude from that?

Problem 8.5 Prove that

(i) $\sin x = o(x^\alpha)$ $(x \to 0)$ for all $\alpha < 1$;
(ii) $\log(1+x^2) = o(x)$ $(x \to 0)$;
(iii) $\log x = o(x)$ $(x \to \infty)$;
(iv) $\tan x - \sin x = o(x^2)$ $(x \to 0)$.

Problem 8.6 Calculate the following limits using Taylor's theorem:

(i) $\displaystyle\lim_{x\to 0} \frac{e^x - \sin x - 1}{x^2}$;
(ii) $\displaystyle\lim_{x\to 0} \frac{\sin x - x + x^3/6}{x^5}$;
(iii) $\displaystyle\lim_{x\to 0} \frac{\cos x - \sqrt{1-x}}{\sin x}$;
(iv) $\displaystyle\lim_{x\to 0} \frac{\tan x - \sin x}{x^3}$;
(v) $\displaystyle\lim_{x\to 0} \frac{x - \sin x}{x(1-\cos 3x)}$;
(vi) $\displaystyle\lim_{x\to 0} \frac{\cos x + e^x - x - 2}{x^3}$;
(vii) $\displaystyle\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{\sin x}\right)$;
(viii) $\displaystyle\lim_{x\to 0} \frac{1}{x}\left(\frac{1}{x} - \cot x\right)$;
(ix) $\displaystyle\lim_{x\to \infty} x^{3/2}\left(\sqrt{x+1} + \sqrt{x-1} - 2\sqrt{x}\right)$;
(x) $\displaystyle\lim_{x\to \infty} \left[x - x^2 \log\left(1 + \frac{1}{x}\right)\right]$.

Problem 8.7 If $f(x) = -\dfrac{x}{2} - \dfrac{x^2}{4} + o(x^2)$ $(x \to 0)$, calculate

$$\lim_{x\to 0} \frac{\log[1+f(x)] + x/2}{x^2}.$$

Problem 8.8 Prove that the function

$$f(x) = \begin{cases} \dfrac{1}{x} - \dfrac{1}{e^x - 1}, & \text{if } x \neq 0, \\ \dfrac{1}{2}, & \text{if } x = 0, \end{cases}$$

is differentiable at $x = 0$ by calculating $f'(0)$ from the definition.

Problem 8.9 Determine the first nonzero order in the Taylor expansion of the following functions:

(i) $f(x) = \tan(\sin x) - \sin(\tan x)$;

(ii) $f(x) = \dfrac{1}{R^2} - \dfrac{1}{(R+x)^2}$;

(iii) $f(x) = \sqrt[3]{\dfrac{1+x}{1-x}} - \sqrt[3]{\dfrac{1-x}{1+x}}$.

Problem 8.10 Consider the function

$$f(x) = \frac{1-\cos x}{1+\cos x}.$$

This function is even and $f(0) = 0$, so its Taylor expansion up to 7th order will be

$$f(x) = Ax^2 + Bx^4 + Cx^6 + o(x^7), \quad (x \to 0).$$

Then

$$1 - \cos x = \left[Ax^2 + Bx^4 + Cx^6 + o(x^7)\right](1+\cos x).$$

Using the Taylor expansion of $\cos x$ up to 7th order find the coefficients A, B, and C from this equation.

Problem 8.11 Find coefficients a and b so that

(i) $x - (a + b\cos x)\sin x = o(x^4)$ $(x \to 0)$;

(ii) $\cot x - \dfrac{1+ax^2}{x+bx^3} = o(x^4)$ $(x \to 0)$.

Problem 8.12 Find constants $a, b, c, d \in \mathbb{R}$ such that

$$e^x = \frac{1+ax+bx^2}{1+cx+dx^2} + o(x^4) \quad (x \to 0).$$

Problem 8.13 Given that $\sqrt{1+x} = 1 + \frac{x}{2} + o(x)$ $(x \to 0)$, prove:

(i) $\lim\limits_{n\to\infty} \sin\left(\pi\sqrt{1+n^2}\right) = 0$;

(ii) $\sum\limits_{n=0}^{\infty} \sin^2\left(\pi\sqrt{1+n^2}\right) < \infty$.

Problem 8.14 Calculate the Taylor polynomial $P_{4,0}(x)$ for $f(x) = 1 + x^3\sin x$. Given the result, does f have a local maximum, minimum or inflection point at $x = 0$?

Problem 8.15 Use a Taylor polynomial of the specified degree to provide an approximation to these numbers, and give an upper bound for the error incurred:

(i) $\dfrac{1}{\sqrt{1.1}}$, degree 3; (ii) $\sqrt[3]{28}$, degree 2.

Problem 8.16 Given the function $f(x) = \cos x + e^x$,

(i) find its Taylor polynomial $P_{3,0}(x)$;

(ii) estimate an upper bound for the error incurred if $-1/4 \leqslant x \leqslant 1/4$.

Problem 8.17 What is the smallest degree Taylor polynomial necessary to approximate the function $f(x) = e^x$ in $[-1, 1]$ with at least three exact decimal places?

Problem 8.18 Determine the convergence radius of the following power series, and specify the interval where they converge absolutely:

(i) $\sum_{n=1}^{\infty} \frac{x^n}{2^n n^2}$; (iii) $\sum_{n=1}^{\infty} \frac{x^n}{n 10^{n-1}}$; (v) $\sum_{n=0}^{\infty} (3-2x)^n$;

(ii) $\sum_{n=1}^{\infty} \frac{n! x^n}{n^n}$; (iv) $\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}$; (vi) $\sum_{n=1}^{\infty} \frac{(x-2)^n}{\sqrt{2n}}$.

Problem 8.19 Expand the function $f(x) = \frac{1}{(1-x)^k}$ for $k = 1$, 2, and 3.

Problem 8.20 Consider the power series

$$\frac{1}{x^2+x+1} = \sum_{n=0}^{\infty} a_n x^n.$$

What are the values of the coefficients a_{300}, a_{301}, and a_{302}?
HINT: Recall that $1-x^3 = (1-x)(x^2+x+1)$.

Problem 8.21 Calculate the derivatives $f^{(100)}(0)$ and $f^{(231)}(0)$ of the function $f(x) = \log\left(1+x^2\right)$.

Problem 8.22 Determine the convergence radius of the following power series, and calculate their sums:

(i) $\sum_{n=1}^{\infty} \frac{x^n}{n}$; (ii) $\sum_{n=0}^{\infty} (n+1) 2^{-n} x^n$.

Problem 8.23 Expand in power series the following functions, specifying the domain of validity of those expansions:

(i) $f(x) = \sin^2 x$; (iii) $f(x) = \frac{x}{a+bx}$; (v) $f(x) = \frac{1+x-(1-x)e^{2x}}{e^x}$.

(ii) $f(x) = \log\sqrt{\frac{1+x}{1-x}}$; (iv) $f(x) = \frac{1}{2-x^2}$;

Problem 8.24 Sum the following series:

(i) $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!}$; (iii) $\sum_{n=1}^{\infty} \frac{1}{n 2^n}$;

(ii) $\sum_{n=1}^{\infty} \frac{n}{2^n}$; (iv) $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$.

Problem 8.25 Given the function $f(x) = \sum_{n=1}^{\infty} \frac{n^x}{n!}$, compute the values $f(0)$, $f(1)$, and $f(2)$.

Problem 8.26 Find a function $f(x)$ that can be expaded in power series of x and such that it satisfies the equation $f'(x) = f(x) + x$ with the condition $f(0) = 2$.

Problem 8.27 Prove that if f and g are twice differentiable, convex functions, and f is increasing, then $h = f \circ g$ is convex.

Problem 8.28 Discuss the convexity of the following functions:

(i) $f(x) = (x-2)x^{2/3}$; (ii) $f(x) = |x|e^{|x|}$; (iii) $f(x) = \log(x^2-6x+8)$.

Problem 8.29

(i) Sketch the graph of the function $f(x) = x + \log|x^2-1|$.

(ii) Based on the previous graph, plot function $g(x) = |x| + \log|x^2-1|$ and $h(x) = \left|x + \log|x^2-1|\right|$.

Problem 8.30 Sketch a plot of the following functions:

(i) $f(x) = e^x \sin x$;

(ii) $f(x) = \sqrt{x^2 - 1} - 1$;

(iii) $f(x) = xe^{1/x}$;

(iv) $f(x) = x^2 e^x$;

(v) $f(x) = (x-2)x^{2/3}$;

(vi) $f(x) = (x^2 - 1)\log\left(\dfrac{1+x}{1-x}\right)$;

(vii) $f(x) = \dfrac{x}{\log x}$;

(viii) $f(x) = \dfrac{x^2-1}{x^2+1}$;

(ix) $f(x) = \dfrac{e^{1/x}}{1-x}$;

(x) $f(x) = \log[(x-1)(x-2)]$;

(xi) $f(x) = \dfrac{e^x}{x(x-1)}$;

(xii) $f(x) = 2\sin x + \cos 2x$;

(xiii) $f(x) = \dfrac{x-2}{\sqrt{4x^2+1}}$;

(xiv) $f(x) = \sqrt{|x-4|}$;

(xv) $f(x) = \dfrac{1}{1+e^x}$;

(xvi) $f(x) = \dfrac{e^{2x}}{e^x - 1}$;

(xvii) $f(x) = e^{-x}\sin x$;

(xviii) $f(x) = x^2 \sin\dfrac{1}{x}$.

Problem 8.31 Draw the graph of the following functions:

(i) $f(x) = \min\{\log|x^3 - 3|, \log|x+3|\}$;

(ii) $f(x) = \dfrac{1}{|x|-1} - \dfrac{1}{|x-1|}$;

(iii) $f(x) = \dfrac{1}{1+|x|} - \dfrac{1}{1+|x-a|}$, $(a > 0)$;

(iv) $f(x) = x\sqrt{x^2-1}$;

(v) $f(x) = \arctan\log|x^2 - 1|$;

(vi) $f(x) = 2\arctan x + \arcsin\left(\dfrac{2x}{1+x^2}\right)$.

Problem 8.32 Plot the function

$$f(x) = \begin{cases} \dfrac{e^{1/x}}{1+x}, & x \neq 0, \\ 0 & x = 0, \end{cases}$$

and discuss how many real solutions has the equation $\dfrac{e^{1/x}}{1+x} = x^3$.

Problem 8.33 Given the function $f(x) = \dfrac{1+x}{3+x^2}$ plot the functions $g(x) = \sup\limits_{y>x} f(y)$ and $h(x) = \inf\limits_{y>x} f(y)$.

Problem 8.34 Determine the equations of the tangents to $f(x) = \log(1+x^2)$ at its inflection points and plot them along with the graph of $f(x)$.