1.1 Vector space

In which follows $\Bbbk$ designates the field $(\mathbb{R}, +, \times)$ or $(\mathbb{C}, +, \times)$. We shall give the axioms which defines a vector space.

Definition 1.1 ($\Bbbk$-vector space) *Let $(\Bbbk, +, \times)$ be a field. We call vector space on a field $\Bbbk$ any set E equipped of a internal law $+$ (addition)*

$$\circledast : \begin{cases} E \times E & \rightarrow & E \\ (x, y) & \mapsto & x \circledast y \end{cases}$$

and a external law $\cdot$ (multiplication by a scalar)

$$. : \begin{cases} \Bbbk \times E & \rightarrow & E \\ (\lambda, y) & \mapsto & \lambda . x \end{cases}$$

Such that :

1. *$(E, \circledast)$ is a commutative group. We notice by 0_E its neutral element.*
2. *For all $(\alpha, \beta) \in \Bbbk^2$, and for all $(x, y) \in E^2$, we have*

— $(\alpha + \beta).x = \alpha.x \circledast \beta.x$ *Axiom 1*
— $(\alpha \times \beta).x = \alpha.(\beta.x)$ *Axiom 2*
— $\alpha.(x \circledast y) = \alpha.x \circledast \alpha.y$ *Axiom 3*
— $1_{\Bbbk}.x = x$ *Axiom 4*

We say then that $(E, \circledast, \cdot)$ is a $\Bbbk$-vector space. The elements of $\Bbbk$ are called scalars, and of E, vectors. neutral element of $(E, \circledast)$, 0_E is called null vector.

Example 1.1 *The commutative field $\Bbbk$, is a vector space on itself , rof the addition and the product existing on $\Bbbk$.*

Example 1.2 *The set $\mathbb{R}_n[x]$ of polynômial functions with coefficients in $\mathbb{R}$ of degree $\leq n$, i.e :*

$$\mathbb{R}_n[x] = \{P : \mathbb{R} \rightarrow \mathbb{R} |\ P(x) = a_0 + a_1 x + ... + a_n x^n,\ a_i \in \mathbb{R}\}$$

is a vector space on $\mathbb{R}$ for the laws :

$$\begin{aligned} (a_0 + a_1 x + ... + a_n x^n) \circledast (b_0 + b_1 x + ... + b_n x^n) &: = (a_0 + b_0) + ... + (a_n + b_n) x^n \\ \lambda.(a_0 + a_1 x + ... + a_n x^n) &: = \lambda a_0 + \lambda a_1 x + ... + \lambda a_n x^n. \end{aligned}$$

Example 1.3 *Let A be a non-empty set and a vector space E on the field $\mathbb{k}$. The set E^A of the maps from A to E is equipped with vector structure on $\mathbb{k}$ as follows.*

Let f and g be two functions from A to E, and λ a scalar of $\mathbb{k}$ we dfine the function $f+g$ by

$$\forall x \in A,\ (f+g)(x) = f(x) + g(x),$$

and the function $\lambda.f$ by

$$\forall x \in A, (\lambda.f)(x) = \lambda.f(x),$$

Then, $(E^{A,}+,.)$ is a vector space on $\mathbb{k}$. Its null vector is the null function from A with values in E,

$$\begin{array}{rccc} 0_{E^A}: & A & \to & E \\ & x & \mapsto & 0_E \end{array}$$

Example 1.4 *Let $E_1, ..., E_n$ be n vector spaces on the commutative field $\mathbb{k}$, we define the cartesian product $E = E_1 \times E_2 \times ... \times E_n$ denoted by $E = \prod_{i=1}^n E_i$, which its elements are $X = (X_1, ..., X_n)$, with $\forall i = 1...n$, $X_i \in E_i\cdot$*

It is easy to verify that for the addition $+$ fefined by

$$(X_1, ..., X_n) + (Y_1, ..., Y_n) = (X_1 + Y_1, ..., X_i + Y_i, ..., X_n + Y_n)$$

E is a commutative group, and with the product

$$\lambda.X = (\lambda X_1, \lambda X_2, ..., \lambda X_n),$$

we equip E with a vector structure.

1.2 Calculation rules

Proposition 1.1 *Let $(E, +, \cdot)$ be a $\mathbb{k}$-vector space. For all scalars $\alpha, \beta, \lambda \in \mathbb{k}$ and for all vectors $x, y \in E$, we have*

1. $0_{\mathbb{k}} \cdot x = 0_E$
2. $(-1) \cdot x = -x$
3. $(-\lambda) \cdot x = -(\lambda \cdot x) = \lambda \cdot (-x)$
4. $(\alpha - \beta).x = \alpha.x - \beta.x$
5. $\lambda.(x - y) = \lambda.x - \lambda.y$
6. $\lambda \cdot 0_E = 0_E$
7. $\lambda \cdot x = 0_E \Longleftrightarrow (\lambda = 0_{\mathbb{k}} \quad ou \quad x = 0_E)$.

Proof. 1. We have

$$\begin{aligned} 0_{\mathbb{k}}\cdot x + 0_E &= 0_{\mathbb{k}}\cdot x \text{ since } (E,+) \text{ is a group} \\ &= (0_{\mathbb{k}} + 0_{\mathbb{k}})\cdot x \text{ since } \mathbb{k} \text{ is a field} \\ &= 0_{\mathbb{k}}\cdot x + 0_{\mathbb{k}}\cdot x. \end{aligned}$$

Then $0_{\mathbb{k}}\cdot x = 0_E$ by subtracting $0_{\mathbb{k}}\cdot x$ to the leftside of the two members of this equality.
2. We have :

$$\begin{aligned} x + (-1)\cdot x &= 1.x + (-1)\cdot x \text{ after the axiom 4} \\ &= (1 + (-1)).x \text{ after the axiom 1} \\ &= 0_{\mathbb{k}}\cdot x \text{ since } \mathbb{k} \text{ is a field} \\ &= 0_E \text{ after 1.} \end{aligned}$$

so, $(-1).x$ is the inverse of x. We can write then : $-x = (-1).x$.
3. We have :

$$\begin{aligned} (-\lambda)x &= (-1.\lambda)x \text{ since } \mathbb{k} \text{ is a field} \\ &= (-1).(\lambda.x) \text{ after the axiom 2} \\ &= -\lambda.x \text{ after 2.} \end{aligned}$$

4. We have :

$$\begin{aligned} (\alpha - \beta).x &= (\alpha + (-\beta)).x \text{ since } \mathbb{k} \text{ is a field} \\ &= \alpha.x + (-\beta).x \text{ after the axiom 1} \\ &= \alpha.x - \beta.x \text{ after 3.} \end{aligned}$$

5. We have :

$$\begin{aligned} \lambda.(x - y) &= \lambda.(x + (-y)) \text{ since } (E,+) \text{ is a group} \\ &= \lambda.x + \lambda.(-y) \text{ after the axiom 3} \\ &= \lambda.x + \lambda.(-1y) \text{ after 2} \\ &= \lambda.x + (\lambda.(-1)).y \text{ after the axiom 2} \\ &= \lambda.x + (-\lambda).y \text{ since } \mathbb{k} \text{ is a field} \\ &= \lambda.x - \lambda.y \text{ after 3.} \end{aligned}$$

6. We have :

$$\begin{aligned} \lambda\cdot 0_E &= \lambda\cdot(x - x) \text{ since } (E,+) \text{ is a group} \\ &= \lambda\cdot x + \lambda\cdot(-x) \text{ after the axiom 1} \\ &= \lambda\cdot x - \lambda\cdot x \text{ after 3.} \\ &= 0_E \text{ since } (E,+) \text{ is a group.} \end{aligned}$$

7. Suppose that $\lambda \cdot x = 0_E$, If $\lambda = 0_{\mathbb{k}}$ then after 1, $\lambda \cdot x = 0_E$. otherwise, if $\lambda \neq 0_{\mathbb{k}}$ then since $\mathbb{k}$ is a field, λ^{-1} exists and

$$x = 1.x = \left(\lambda.\lambda^{-1}\right).x = \lambda^{-1}.\left(\lambda.x\right) = \lambda^{-1}.0_E = 0_E,$$

and thus $x = 0_E$. The converse is obvious. □

1.3 Subspaces

Definition 1.2 (Subspaces) *We call Subspaces of a vector space E on the field $\mathbb{k}$, any part F of E which is an additive sub-group of E such that $\forall \lambda \in \mathbb{k}$, $\forall x \in F$, $\lambda.x \in F$.*

Remark 1.1 *It is easy to show that F is a vector space on $\mathbb{k}$, the conditions of definition 1.1 are verified.*

Example 1.5 $\{0_E\}$ *and E are sub-spaces of E.*

Definition 1.3 (Linear combination) ∘ *Let $x_1, x_2, ..., x_n$ be n vectors of E vector space E on the field $\mathbb{k}$. We call linear combination of these n vectors every vector $x \in E$ of the form*

$$x = \lambda_1 x_1 + ... + \lambda_p x_n = \sum_{k=1}^{n} \lambda_k x_k$$

where $(\lambda_1, ..., \lambda_n) \in \mathbb{k}^n$.

∘ *If A is a part of E, we call linear combination of elements of A every linear combination of a finite number of elements of A.*

In principle, to show that F is a vector subspace, one would have to verify the eight axioms of the Definition 1.1 In fact, it is enough to verify the "stability" of the laws of composition as asserted by the following proposition :

Proposition 1.2 *A part F of a vector space E on $\mathbb{k}$ is a sub-space of E if and only if :*

1. $F \neq \varnothing$
2. $\forall (\alpha, \beta) \in \mathbb{k}^2$, $\forall (x, y) \in F^2$, $\alpha.x + \beta.y \in F$.

Proof. $\Longrightarrow$) First if F is a subspace of E, as an additive subgroup F is non-empty because it contains 0, the null element of E, then if x and y are in F and α and β in the field $\mathbb{k}$, $\alpha.x$ et $\beta.y \in F$ (cf. définition 1.2) which is stable for the addition, therefore $\alpha.x + \beta.y \in F$.

$\Longleftarrow$) if 1. and 2. are verified, with $\alpha = 1$ and $\beta = -1$ and x and y in F we have :

$$(F \neq \varnothing) \text{ and } \left(\forall (x,y) \in F^2, x - y \in F\right),$$

which already justifies that F is an additive subgroup of E, then 2. with $\beta = 0$ gives

$$\forall \lambda \in \mathbb{k}, \forall x \in F, \lambda . x \in F.$$

We therefore have F subspace of E. □

Remark 1.2 *As we saw during the proof, if F is a vector subspace, then F necessarily contains the null vector 0_E.*

1.3.1 Fundamental Examples of Subspaces

1. Vector line :

Let $v \in E$, $v \neq 0$, then :

$$F = \{y \in E |\ \exists \lambda \in \mathbb{k} : y = \lambda v\}$$

is a vector subspace of E called vector line generated (spanned) by v.

In fact, $F \neq \emptyset$, since $v \in F$. In addition, F is stable for the laws of E, since if $x, y \in F$ (i.e: $x = \lambda v$, $y = \mu v$), we have :

$$x + y = \lambda v + \mu v = (\lambda + \mu)v \in F$$

Likewise, if $x \in F$ (i.e $x = \lambda v$), we have : $\mu x = \mu(\lambda v) = (\mu\lambda)v \in F$.

2. Vector plane :

Let $x_1, x_2 \in E$ then :

$$F = \{y \in E |\ \exists \lambda_1, \lambda_2 \in \mathbb{k} : y = \lambda_1 x_1 + \lambda_2 x_2\}$$

F is a subspace of E, called the subspace generated by x_1, x_2. If x_1 and x_2 are not null and x_2 does not belong to the vector line generated by x_1, F is said to be a vector plane generated by x_1 and x_2.

3. Generated subspace :

More generally, if $x_1, x_2, ..., x_p \in E$ then :

$$F = \{y \in E |\ \exists \lambda_1, ..., \lambda_p \in \mathbb{k} : y = \lambda_1 x_1 + ... + \lambda_p x_p\}$$

is a vector subspace of E denoted $Vect\{x_1, x_2, ..., x_p\}$, said subspace generated by $x_1, ..., x_p$, or also space of linear combinations of $x_1, x_2, ..., x_p$. We will see subsequently that, basically, all subspaces are of this type, that is to say obtained by "linear combinations" of a family of elements of E.

Remark 1.3 *Let E be the vector space of vectors of origin O. A vector line is a line passing through O. Similarly, a vector plane is a plane passing through O. More generally, a vector subspace of $\mathbb{R}^n$ can be visualized as a "plane of dimension p" passing through O. We could give a precise meaning to the notion of "plane of dimension p", but this is not necessary. Let us remember for the moment the fact that it must pass through O, because any vector subspace must contain the zero vector. So, for example, a line not passing through O is not a vector subspace : the points of the line are the ends of the vectors coming from O and the null vector is not among them.*

Example 1.6 *Let*

$$F = \{(x, y, z) \in \mathbb{R}^3 |\ 3x + y + 2z = 0\}.$$

F is a subspace of $\mathbb{R}^3$. Indeed, Let $v_1 = (x_1, y_1, z_1)$ and $v_2 = (x_2, y_2, z_2) \in F$; we have :

$$3x_1 + y_1 + 2z_1 = 0 \text{ and } 3x_2 + y_2 + 2z_2 = 0$$

so, by summing : $3(x_1 + x_2) + (y_1 + y_2) + 2(z_1 + z_2) = 0$, i.e

$$v_1 + v_2 = (x_1 + x_2, y_1 + y_2, z_1 + z_2) \in F.$$

Likewise, we see that if $\lambda \in \Bbbk$ and $v \in F$ we have $\lambda v \in F$.

Example 1.7 *We have :*

$$G = \{(x, y, z) \in \mathbb{R}^3 |\ x + 4y + z = 1\},$$

is not a subspace of $\mathbb{R}^3$ because $0_{\mathbb{R}^3} = (0, 0, 0) \notin G$ $(0 + 4.0 + 0 \neq 1)$.

Proposition 1.3 *Let F and G be two subspaces of E.*

1. *$F \cap G$ is a subspace of E.*
2. *$F \cup G$ is not in general a subspace of E.*
3. *The complementary $E \backslash F$ of a subspace F is not a subspace of E.*

Proof. **1.** First, we have $F \cap G \neq \varnothing$, because $0_E \in F \cap G$.

Let $x, y \in F \cap G$, we have : $x, y \in F$ then $x + y \in F$. Likewise, if $x, y \in G$, $x + y \in G$ Consequently $x + y \in F \cap G$.

If $\lambda \in \Bbbk$ and $x \in F \cap G$, we have : $x \in F$, then $\lambda x \in F$, and $x \in G$, thus $\lambda x \in G$, so : $\lambda x \in F \cap G$.

2. This is due to the fact that in general $F \cup G$ is not stable by the sum. For example, let $E = \mathbb{R}^2$, F the vector line generated by $(1, 0)$ and G the vector line generated by $(0, 1)$. We have : $(1, 0) \in F$ then $(1, 0) \in F \cup G$. $(0, 1) \in G$ then $(0, 1) \in F \cup G$ but : $w = (1, 0) + (0, 1) = (1, 1) \notin F \cup G$.

3. $E \backslash F$ does not contain 0_E, therefore it is not a subspace of E(Remarque 1.2). □

1.4 Bases (in finite dimension)

Definition 1.4 (spanning set) *A family of vectors $\{v_1, ..., v_p\}$ of a vector space E is called spanning set, if $E = Vect\{v_1, ..., v_p\}$, we write also, $E = Vect\{v_1, ..., v_p\}$. Which means that $\forall x \in E$, x decomposed on vectors v_i, or every $x \in E$ is a linear combination of vectors v_i,*

$$x = \lambda_1 v_1 + ... + \lambda_p v_p = \sum_{k=1}^{p} \lambda_k v_k.$$

Example 1.8 *In $\mathbb{R}^2$, let $v_1 = (1, 1)$ and $v_2 = (1, -1)$. Let's show that $\{v_1, v_2\}$ is a spanning set. Let $x = (a, b) \in \mathbb{R}^2$ wit a, b arbitrary : it is an issue of showing that there exists $x_1, x_2 \in \mathbb{R}$ such that $x = x_1 v_1 + x_2 v_2$, i.e :*

$$x = (a, b) = (x_1, x_1) + (x_2, -x_2) = (x_1 + x_2, x_1 - x_2)$$

This means that $\forall (a, b) \in \mathbb{R}^2$, $\exists\ x_1, x_2 \in \mathbb{R}$ satisfies the system :

$$\begin{cases} x_1 + x_2 = & a \\ x_1 - x_2 = & b, \end{cases}$$

Solving, we find :

$$x_1 = \frac{a+b}{2} \text{ and } x_2 = \frac{a-b}{2},$$

solution defined for arbitrary a,b. So $\{v_1, v_2\}$ spans $\mathbb{R}^2$.

Definition 1.5 *A vector space is said to be of finite dimension, if there exists a finite spanning (generating) family, otherwise, it is said to be of infinite dimension.*

Definition 1.6 (Linearly independent family) *Let $\{v_1, ..., v_p\}$, a finite family of elements of E. It is said to be linearly independent if :*

$$\lambda_1 v_1 + ... + \lambda_p v_p = 0 \Longrightarrow \lambda_1 = \lambda_2 = ... = \lambda_p = 0.$$

A family which is not linearly independent is said linearly dependent.

Example 1.9 *In $\mathbb{R}^3$, the vectors $v_1 = (1, 1, -1)$, $v_2 = (0, 2, 1)$ and $v_3 = (0, 0, 5)$ are linearly independent. Indeed, suppose that there exist reals λ_1, λ_2, λ_3 so that $\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 = 0_{\mathbb{R}^3}$, i.e :*

$$\lambda_1(1, 1, -1) + \lambda_2(0, 2, 1) + \lambda_3(0, 0, 5) = 0_{\mathbb{R}^3}$$

We obtain :

$$(\lambda_1, \lambda_2, -\lambda_1 + \lambda_2 + 5\lambda_3) = 0_{\mathbb{R}^3},$$

so

$$\begin{cases} \lambda_1 = & 0 \\ \lambda_2 = & 0 \\ -\lambda_1 + \lambda_2 + 5\lambda_3 = & 0 \end{cases}$$

which immediately gives $\lambda_1 = \lambda_2 = \lambda_3 = 0$.

Example 1.10 *In* $\mathcal{F}(\mathbb{R}, \mathbb{R})$ *family* $\{\sin, \cos, \exp\}$ *is linearly independent. Let* $\alpha, \beta, \gamma \in \mathbb{R}$ *so that* $\alpha \sin + \beta \cos + \gamma \exp = 0$. *Then for* $x \in \mathbb{R}$, $\alpha \sin(x) + \beta \cos(x) + \gamma \exp(x) = 0$, *which is also written*

$$\alpha \frac{\sin(x)}{\exp(x)} + \beta \frac{\cos(x)}{\exp(x)} + \gamma = 0.$$

It is easily shown that

$$lim_{+\infty} \frac{\sin(x)}{\exp(x)} = lim_{+\infty} \frac{\cos(x)}{\exp(x)} = 0.$$

we deduce that $\gamma = 0$. *We have then* $\forall x \in \mathbb{R}$, $\alpha \sin(x) + \beta \cos(x) + \gamma \exp(x) = 0$. *If we make* $x = 0$ *we obtain* $\beta = 0$ *and if we make* $x = \pi/2$, *we get* $\alpha = 0$. *We then clearly showed that* $\alpha = \beta = \gamma = 0$.

Proposition 1.4 *A family* $\{v_1, ..., v_p\}$ *is linearly dependent if and only if at least one of the vectors* v_i *is written as a linear combination of the other vectors of the family.*

Proof. $\Longrightarrow)$: If $\{v_1, ..., v_p\}$ is linearly dependent, there exists λ_1,..., λ_p not all zero such that $\lambda_1 v_1 + ... + \lambda_p v_p = 0$. If, for example $\lambda_1 \neq 0$, we can write :

$$v_1 = -\frac{\lambda_2}{\lambda_1} v_2 + ... + \frac{-\lambda_p}{\lambda_1} v_p$$

$\Longleftarrow)$: Suppose for example that $v_1$is a linear combination of vectors $\lambda_2 v_2, ..., \lambda_p v_p$, then there exists $\mu_2, ..., \mu_p \in \mathbb{k}$ such that $v_1 = \lambda_2 v_2 + ... + \lambda_p v_p$, i.e. :

$$v_1 - \lambda_2 v_2 - ... - \lambda_p v_p = 0.$$

There therefore exists a linear combination of vectors $\{v_1, ..., v_p\}$ which is zero, without the coefficients being all zero. So the family is linearly dependent. □

Proposition 1.5 *Let* $\{v_1, ..., v_p\}$ *be a family linearly independent and* x *any vector in the space generated by the vectors* v_i *(i.e.* x *is linear combination of* v_i*). Then the decomposition of* x *on* v_i *is unique.*

Proof. Let

$$\begin{aligned} x &= \lambda_1 v_1 + ... + \lambda_p v_p, \\ x &= \beta_1 v_1 + ... + \beta_p v_p, \end{aligned}$$

two decompositions of x. By making the difference we find :

$$(\lambda_1 - \beta_1)\, v_1 + ... + (\lambda_p - \beta_p)\, v_p = 0,$$

Since the family is linearly independent, we have $(\lambda_1 - \beta_1) = ... = (\lambda_p - \beta_p) = 0$, i.e. : $\lambda_1 = \beta_1, ..., \lambda_p = \beta_p$. □

Definition 1.7 (Base) *A family that is both generative and linearly independent is called a basis.*

Proposition 1.6 *A family $\{v_1, ..., v_p\}$ is a base of E if and only if all $x \in E$ decomposes in a unique way on the v_i. i.e. :*
$\forall x \in E$ there is a unique $(\lambda_1, ..., \lambda_n) \in \mathbb{k}^n$ so that :

$$x = \lambda_1 v_1 + ... + \lambda_n v_n.$$

Proof. The existence of decomposition for all $x \in E$ is equivalent to the fact that the family is generative ; uniqueness to the fact that the family is linearly independent. □

Example 1.11 (Canonical basis of $\mathbb{R}^n$) *Let the vectors :*

$$e_1 = (1, 0, ..., 0), ..., e_i = (0, ..., \underbrace{1}_{rank\ i}, ..., 0), ..., e_n = (0, 0, ..., 1).$$

We already know that they form a generative family. Let's show that it is linearly independent. We have :

$$\lambda_1 e_1 + ... + \lambda_i e_i + ... + e_n v_n = 0_{\mathbb{R}^n},$$

i.e. :

$$\lambda_1(1, 0, ..., 0) + ... + \lambda_i(0, ..., \underbrace{1}_{i^e rang}, ..., 0) + ... + \lambda_n(0, 0, ..., 1) = 0_{\mathbb{R}^n},$$

then :

$$(\lambda_1, \lambda_2, ..., \lambda_n) = 0_{\mathbb{R}^n}.$$

Therefore $\{e_1, ..., e_n\}$ is a basis of $\mathbb{R}^n$, called the canonical basis.

Example 1.12 (***Canonical basis of*** $\mathbb{R}_n[x]$) *The family $\mathcal{B} = \{1, x, ..., x^n\}$ is a basis of $\mathbb{R}_n[x]$, In fact, every $P(x) = a_0 + a_1 x + ... + a_n x^n$, $a_i \in \mathbb{R}$; B is therefore generative. Moreover :*

$$\lambda_0 1 + \lambda_1 x + ... + \lambda_n x^n = 0 \Longrightarrow \lambda_0 = \lambda_1 = ... = \lambda_n = 0.$$

Example 1.13 *Let $F = \{(x, y, z) \in \mathbb{R}^3 | \ 2x + y + 2z = 0\}$. Find a basis of F. We have seen F is a subspace of $\mathbb{R}^3$, We have : $(x, y, z) \in F \Leftrightarrow y = -2x - 2z$ then :*

$$u \in F \Leftrightarrow u = (x, -2x - 2z, z) \Leftrightarrow u = x(1, -2, 0) + z(0, -2, 1).$$

Therefore the vectors $v_1 = (1, 2, 0), v_2 = (0, -3, 1)$ form a family generating F. On the other hand :

$$\lambda_1 v_1 + \lambda_2 v_2 = 0 \Leftrightarrow \lambda_1(1, -2, 0) + \lambda_2(0, -2, 1) = (0, 0, 0),$$

which is equivalent to $\lambda_1 = \lambda_2 = 0$. Then $\{v_1, v_2\}$ is linearly independent and therefore it is a basis of F.

Proposition 1.7 *We have :*

1. *$\{x\}$ is a linearly independent family $\Longleftrightarrow x \neq 0_E$.*
2. *Any family containing a generating family is generating.*
3. *Any subfamily of a linearly independent family is linearly independent.*
4. *Any family containing a linearly dependent family is linearly dependent.*
5. *Any family $\{v_1, ..., v_p\}$ of which one of the vectors v_i is null is linearly dependent.*

Proof.

1. $\Longleftarrow$) D'après Proposition 1.1 (7), $\lambda \cdot x = 0_E \Longleftrightarrow (\lambda = 0_{\mathbb{k}}$ ou $x = 0_E)$. Donc, si $x \neq 0$, $\lambda x = 0$ implique $\lambda = 0$, ce qui signifie que $\{x\}$ est une famille libre.

   $\Longrightarrow$) Supposons que $\{x\}$ libre. Alors, d'après la définition de famille libre, si $\lambda x = 0$ on a nécessairement $\lambda = 0$, ce qui signifie, toujours d'après la proposition 1.1 (7), que $x \neq 0$.
2. Soit $\{v_1, ..., v_p\}$ une famille génératrice et $x = \lambda_1 v_1 + ... + \lambda_p v_p$ un élément arbitraire de E. On peut aussi écrire :

   $$x = \lambda_1 v_1 + ... + \lambda_p v_p + 0w_1 + ... + 0w_q, \quad w_1, ..., w_q \in E.$$

   Donc tout $x \in E$ est combinaison linéaire de $v_1, ..., v_p, w_1, ..., w_q$.
3. Soit $\mathcal{T} = \{v_1, ..., v_p\}$ une famille libre et $\mathcal{T}'$ une sous-famille de $\mathcal{T}$. Quitte à changer la numérotation, on peut supposer que $\mathcal{T}\prime = \{v_1, ..., v_k\}$ (avec $k < p$). Si $\mathcal{T}\prime$ était liée, l'un des vecteurs $v_1, ..., v_k$ serait combinaison linéaire des autres. Il existerait donc un élément de $\mathcal{T}$ qui s'écrirait comme combinaison linéaire de certains éléments de $\mathcal{T}$. Or, cela est impossible car $\mathcal{T}$ est libre (voir Proposition 1.4).
4. Soit $\mathcal{F} = \{v_1, ..., v_p\}$ une famille liée et $\mathcal{G} = \{v_1, ..., v_p, w_1, ..., w_q\}$. D'après proposition 1.4, l'un des v_i est combinaison linéaire des autres. Or, les vecteurs v_i appartiennent à $\mathcal{G}$; donc l'un des éléments de $\mathcal{G}$ est combinaison linéaire des autres, et par conséquent $\mathcal{G}$ est liée.
5. Évident d'après 4., car il s'agit d'une famille contenant $\{0\}$, et $\{0\}$ est liée, d'après 1.

$\square$

1.5 Dimension of a vector space

Definition 1.8 (Dimension of a vector space) *If $E = \{0\}$, we say that E is of dimension 0 and we write $\dim E = 0$. Otherwise, if E is a vector space on $\Bbbk$ of finite dimension not reduced to $\{0\}$, we call dimension of E the cardinal of a base of E and we write $\dim_{\Bbbk} E$.*

Example 1.14 $\dim \mathbb{R}^n = n$, $\dim \mathbb{R}_n[x] = n + 1$.

Remark 1.4 *Une famille f d'au moins $n+1$ vecteurs dans un espace E de dimension n est toujours liée. En effet, si elle était libre alors on aurait une famille libre f de cardinal plus grand que celui de n'importe quelle base $\mathcal{B}$ de E. Or $\mathcal{B}$ est une famille génératrice de E et le cardinal d'une famille libre est toujours plus petit que celui d'une famille génératrice.*

1.6 Dimension of subspace

Proposition 1.8 *Let E be a vector space of finite dimension n and F a subspace of E. We have :*

1. *F is of finite dimension and $\dim F \leq \dim E$.*
2. *$(\dim F = \dim E) \Leftrightarrow F = E$*

Remark 1.5 *To check that two vector subspaces F and G are equal*
- *We show that $F \subset G$.*
- *We show that $\dim F = \dim G$.*

Example 1.15 *Let $E = \mathbb{R}^4$ and*

$$\begin{aligned} F &= Vect((1,1,\alpha,3),(0,1,1,2)), \\ G &= \{(x,y,z,t) \in \mathbb{R}^4 |\ x - y + z = 0, x + 2y - t = 0\}. \end{aligned}$$

Let's look at what condition on $\alpha \in \mathbb{R}$ so that $F = G$?

Let us first assume that $F = G$. Then $(1,1,\alpha,3) \in G$ and must satisfy in particular the equation $x - y + z = 0$. We then find that $\alpha = 0$.

Let us show that if $\alpha = 0$ then $F = G$. We know that $F = Vect((1,1,0,3),(0,1,1,2))$. The vectors $(1,1,0,3),(0,1,1,2)$ generate F and they are non-collinear so they form a linearly independent family. Consequently, it is a base of F and $\dim F = 2$. Moreover

$$\begin{aligned} G &= \{(x,y,z,t) \in E |\ x - y + z = 0, x + 2y - t = 0\} \\ &= \{(x,y,y-x,x+2y) |\ x,y \in \mathbb{R}\} \\ &= Vect((1,0,-1,1),(0,1,1,2)), \end{aligned}$$

we then show in the same way as before that $\dim G = 2$. *In addition* $(1, 1, 0, 3)$ *and* $(0, 1, 1, 2)$ *satisfy the system*

$$\begin{cases} x - y + z = 0 \\ x + 2y - t = 0 \end{cases}$$

then $(1, 1, 0, 3), (0, 1, 1, 2) \in G$ *and since* G *is a subspace,*

$$F = Vect((1, 1, \alpha, 3), (0, 1, 1, 2) \subset G.$$

Finally, as $\dim F = \dim G$ *we obtain* $F = G$.

1.7 Sum of subspaces

Definition 1.9 (Sum of two subspaces) *Let* F *and* G *be two subspaces of a vector space* E *on* $\Bbbk$. *We call the sum of* F *and* G *and we write* $F + G$ *the subspace of* E *given by*

$$F + G = \{x + y \mid (x, y) \in F \times G\}.$$

Remark 1.6 *The subset* $F+G$ *is indeed a subspace of* E. *Indeed,* $F+G \subset E$ *because* E *is stable for addition. Furthermore,* $F+G$ *is non-empty because* F *and* G *are. Finally if* $u = x+y \in F+G$ *and* $u' = x' + y' \in F + G$ *with* $x, x' \in F$ *and* $y, y' \in G$ *then, for* $\alpha, \beta \in \Bbbk$.

$$\alpha u + \beta u' = \underbrace{\alpha x + \beta x'}_{\in F} + \underbrace{\alpha y + \beta y'}_{\in G} \in F + G$$

because F *and* G *are vector subspaces of* E.

Proposition 1.9 *Let* F *and* G *be two subspaces of a vector space* E *on* $\Bbbk$. *Then* $F + G$ *is the smallest subspace of* E *containing* $F \cup G$.

Proof. We proved in the previous remark that $F + G$ is a subspace of E. It contains F and G because 0_E is an element of F and G and therefore $F = F + 0_E \subset F + G$ and $G = 0_E + G \subset F + G$. Moreover, if we consider a subspace H of E which contains $F \cup G$ then let's show that $F + G \subset H$. Let $x + y \in F + G$ with $x \in F$ et $y \in G$. Since $F \cup G \subset H$, we have also $x, y \in H$ and since H is a subspace, it follows that $x + y \in H$. Then $F + G \subset H$ and $F + G$ is the smallest vector subspace of E containing F and G. □

Example 1.16 *In* $\mathcal{F}(\mathbb{R}, \mathbb{R})$ *let* $F = Vect(\sin)$ *and* $G = Vect(\exp)$ *then :*

$$F + G = Vect\,(\sin, \exp) = \{x \mapsto \alpha \sin(x) + \beta \exp(x) \mid |\, |_{|\alpha, \beta \in \mathbb{R}\}}$$

Proposition 1.10 *Let A and B be two parts of a vector space E on $\mathbb{k}$ then*

$$Vect(A) + Vect(B) = Vect(A \cup B).$$

Example 1.17 *Dans l'espace $\mathbb{R}^3$, on considère les parties $F = \{(x, 0, 0) \mid x \in \mathbb{R}\}$ et $G = \{(x, x, 0) \mid x \in \mathbb{R}\}$. Montrons que ce sont des sous-espaces vectoriels de $\mathbb{R}^3$ et déterminons le sous-espace $F + G$. On a $F = Vect(1, 0, 0)$ et $G = Vect(1, 1, 0)$ donc F et G sont des sous-espaces vectoriels de $\mathbb{R}^3$. De plus $F + G = Vect((1, 0, 0), (1, 1, 0))$ et on reconnait que $F + G$ est le plan vectoriel de $\mathbb{R}^3$ engendré par $(1, 0, 0)$ et $(1, 1, 0)$.*

1.7.1 Direct sum, supplementary subspaces

Definition 1.10 (Direct sum) *We say that two vector subspaces F and G of E are in direct sum if $F \cap G = \{0_E\}$. We denote then by $F \oplus G$ their sum.*

In other words :

$$\zeta = F \oplus G \Leftrightarrow \begin{cases} \zeta = F + G \\ et \\ F \cap G = \{0_E\} \end{cases}$$

Example 1.18 *In $\mathbb{C}$, the subspaces $F = \mathbb{R}$ and $G = i\mathbb{R}$ are in direct sum :*

Let $x \in F \cap G$, so, $x \in F$ thus x is real, and $x \in G$ then x is pure imaginary. x is a complex both real and pure imaginary therefore $x = 0_E$.

Example 1.19 *In $E = \mathcal{F}(\mathbb{R}, \mathbb{R})$, We consider*

$$F = \{f \in E \mid f(0) = 0\}\} \text{ et } G = Vect(x \mapsto 1).$$

It is clear that F and G are subspaces of E. G is the set of constant maps of $\mathbb{R}$ in $\mathbb{R}$.

Let $f \in F \cap G$. $f \in F$ then $f(0) = 0$. Likewise $f \in G$ so there exists $a \in \mathbb{R}$ so that $f(x) = a$. But $a = f(0) = 0$ then $f = 0_E$. Thus F and G are in direct sum.

Proposition 1.11 *Let F and G be two subspaces of the vector space E. F and G are in direct sum if and only if $\forall x \in F + G, \exists!(x_1, x_2) \in F \times G : x = x_1 + x_2$ (that is, the decomposition of x is unique).*

Proof. $\Longrightarrow$) Suppose that F and G are in direct sum and let $x \in F + G$. By definition, there exists $x_1 \in F$ and $x_2 \in G$ so that $x = x_1 + x_2$. Suppose there exists $x_1' \in F$ and $x_2' \in G$ such that we still have $x = x_1' + x_2'$. Since $x = x_1 + x_2 = x_1' + x_2'$, we have the equality : $x_1 - x_1' = x_2 - x_2'$. Let us denote this vector by y. Since F and G are subspaces of E, $y = x_1 - x_1' \in F$ and $y = x_2 - x_2' \in G$. Therefore, $y \in F \cap G$. But F and G being in direct sum, we have : $F \cap G = \{0_E\}$ then $y = 0$. Therefore, $x_1 = x_1'$ and $x_2 = x_2'$ and then uniqueness.

$\Longleftarrow$) Let $x \in F \cap G$. There are then two pairs of $F \times G$ allowing us to decompose x into a vector of F and a vector of G : $(x, 0)$ and $(0, x)$. By hypothesis, they are equal : $(x, 0) = (0, x)$. Consequently $x = 0$ and the subspaces F and G are in direct sum. $\square$

Definition 1.11 (Supplementary subspaces) *Let E be a vector space and F, G be two subspaces of E. We say that F and G are supplementary, if $E = F \oplus G$.*

Proposition 1.12 *Let E be a vector space and F, G be two subspaces of E. Then $E = F \oplus G$ (F et G are supplementary) if and only if for any basis $\mathcal{B}_1$ of F and for any basis $\mathcal{B}_2$ of G, $\{\mathcal{B}_1, \mathcal{B}_2\}$ is a basis of E.*

Proof. $\Longleftarrow$) Let $\mathcal{B}_1 = \{v_\alpha\}_{\alpha \in A}$ and $\mathcal{B}_2 = \{w_\beta\}_{\beta \in B}$ basis of F and G respectively and suppose that $\{v_\alpha, w_\beta\}_{(\alpha,\beta) \in A \times B}$ is a basis of E. Then tout $x \in E$ is written by unique way :

$$x = \lambda_1 v_{\alpha_1} + ... + \lambda_p v_{\alpha_p} + \mu_1 w_{\beta_1} + ... + \mu_q w_{\beta_q},$$

i.e. any $x \in E$ is written by unique way $x = x_1 + x_2$ with $x_1 \in F$ and $x_2 \in G$, then $E = F \oplus G$.
$\Longrightarrow$) If $E = F \oplus G$, any $x \in E$ is decomposed by unique way in $F + G$ and, therefore, in the family $\mathcal{B} = \{\mathcal{B}_1, \mathcal{B}_2\}$. We deduce that $\mathcal{B}$ is a basis of E. $\square$

Corollaire 1.1 *Let E be a vector space. For any subspace F of E, there exists always a supplementary. The supplementary of F is not unique, but if E is of finite dimension, all the supplementaries of F have the same dimension.*

Let E be a vector spaceof finite dimension and F, G be two subspaces of E. Then

$$\dim (F + G) \Leftrightarrow \dim F + \dim G - \dim F \cap G.$$

Proof. Supposons que $\dim F = p$, $\dim G = q$ et $\dim F \cap G = r$. Notons que, puisque $F \cap G$ est un sous-espace vectoriel de F et de G, on a $r < p$ et $r < q$. Considérons une base $\{a_1, ..., a_r\}$ de $F \cap G$. Puisque la famille $\{a_1, ..., a_r\}$ est libre, on peut la compléter en une base de F et aussi en une base de G. On peut donc construire : une base de F du type $\{a_1, ..., a_r, e_{r+1}, ..., e_p\}$ et une base de G du type $\{a_1, ..., a_r, f_{r+1}, ..., f_q\}$.

On sait que tout vecteur de $F + G$ s'écrit comme somme d'un vecteur de F, et d'un vecteur de G et donc il est de la forme :

$$\begin{aligned} x &= \lambda_1 a_1 + ... + \lambda_r a_r + \lambda_{r+1} e_{r+1} + ... + \lambda_p e_p + \\ &\quad \mu_1 a_1, ... + \mu_r a_r + \mu_{r+1} f_{r+1}, ... + \mu_q f_q, \end{aligned}$$

c'est-à-dire, en posant $\tau_i = \lambda_i + \mu_i$, pour $i = 1, ..., r$:

$$x = \tau_1 a_1 + ... + \tau_r a_r + \lambda_{r+1} e_{r+1} + ... + \lambda_p e_p + \mu_{r+1} f_{r+1}, ... + \mu_q f_q. \quad (1.1)$$

Par conséquent, la famille $\{a_1, ..., a_r, e_{r+1}, ..., e_p, f_{r+1}, ..., f_q\}$ engendre $F+G$. Montrons qu'elle est libre. Soit une combinaison linéaire nulle :

$$\underbrace{\tau_1 a_1 + ... + \tau_r a_r}_{\alpha \in F\cap G} + \underbrace{\lambda_{r+1} e_{r+1} + ... + \lambda_p e_p}_{\beta \in F} + \underbrace{\mu_{r+1} f_{r+1}, ... + \mu_q f_q}_{\gamma \in G} = 0.$$

On a $\alpha + \beta + \gamma = 0$, donc $\gamma = -(\alpha + \beta)$. Alors $\gamma \in g$ et $\alpha + \beta \in F$, donc $\gamma \in F \cap G$.
Par conséquent, γ peut s'écrire comme combinaison linéaire des a_i :

$$\mu_{r+1} f_{r+1}, ... + \mu_q f_q = \delta_1 a_1 + ... + \delta_r a_r.$$

Mais $\{a_1, ..., a_r, \mu_{r+1} f_{r+1}, ... + \mu_q f_q\}$ est une base de G donc tous les coefficients de cette combinaison linéaire doivent être nuls. En particulier, $\mu_{r+1} = 0, ..., \mu_q = 0$. De même $\lambda_{r+1} = 0, ..., \lambda_q = 0$. D'après (1.1) on déduit alors que :

$$\tau_1 a_1 + ... + \tau_r a_r = 0$$

Or la famille $\{a_1, ..., a_r\}$ est libre, donc $\tau_1 = 0, ..., \tau_r = 0$. Ainsi la famille

$$\{a_1, ..., a_r, e_{r+1}, ..., e_p, f_{r+1}, ..., f_q\}$$

est libre et donc elle est une base de $F+G$. On en déduit :

$$\begin{aligned} \dim(F+G) &= r + (p-r) + (q-r) = p + q - r \\ &= \dim E + \dim G + \dim F \cap G. \end{aligned}$$

□

Corollaire 1.2 *Let E be a vector space of finite dimensionand F, G two subspaces of E. Then*

$$E = F \oplus G \Leftrightarrow \begin{cases} F \cap G = \{0_E\} \\ \dim E = \dim F + \dim G \end{cases}$$

Example 1.20 *In $E = \mathbb{C}$, $F = Vect(\{1\})$ et $G = Vect(\{i\})$. We know that F and G are subspaces of E. If $x \in F \cap G$ then x is both real and pure imaginary, so, $x = 0$ and $F \cap G = \{0_E\}$. Morever*

$$F + G = Vect(1) + Vect(i) = Vect(1, i) = \mathbb{C}$$

thus, $E = F + G$. We have proved that F and G are supplementary.

Example 1.21 *In $\mathbb{R}^3$, let π a vector plan and v a vecteur not contained in this plan. We have*

$$\mathbb{R}^3 = \pi \oplus Vect(v)$$

because if $\{e_1, e_2\}$ is a basis of π, then $\{e_1, e_2, v\}$ is a basis of $\mathbb{R}^3$.