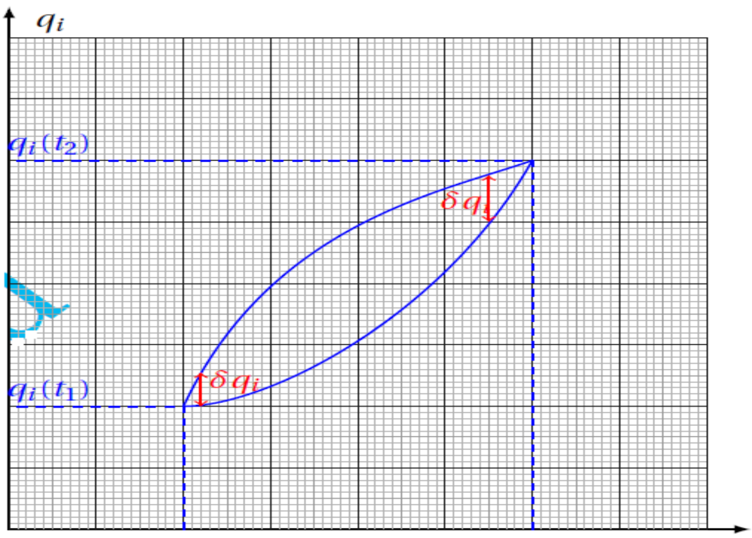
# Action and Principle of Least Action

## Principle of Least Action

The state of a physical system with n degrees of freedom is described by the generalized coordinate $q_i(t)$ at each instant t. At any given moment t, the system's state can be represented by a point in an n-dimensional Cartesian space, called the "configuration space." Each axis corresponds to a generalized coordinate $q_i$. As a mechanical system evolves between two instants $t_1$ and $t_2$, it traces a curve in the configuration space between two points $q_i(t_1)$ and $q_i(t_2)$ which we (for lack of a better term) call a "trajectory". Time is considered as a parameter of the curve. The real trajectory corresponds to the actual dynamics followed by the system, while the "varied trajectory" or "fictitious trajectory," which is infinitely close, corresponds at each instant to the positions $q_i+\delta q_i$, where $\delta q$ is an infinitesimal increment of the position. These two trajectories must satisfy the same initial and final conditions:

The fact that $q_i$ represents the actual trajectory of the system is obtained by solving the Lagrange equation. This equation is derived by minimizing the functional S defined by

$$\delta q_i(t_1) = \delta q_i(t_2) = 0$$

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$$S[q_i] = \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t)$$

The physical trajectory is the path that gives a minimal value to the action S (it is generally a minimum; hence the principle of least action, but in some cases, the extremum is a maximum). The motion of a system from instant $t_1$ to $t_2$ is such that the action S is stationary (minimal), where

$$\delta S=0$$

The variation of $\delta S$ is written as:

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$$\delta S = \int_{t_1}^{t_2} \left[ L(q_i + \delta q_i, \dot{q}_i + \delta \dot{q}_i, t) - L(q_i, \dot{q}_i, t) \right] dt = 0$$

Knowing that $\delta \dot{q}_i = \delta(dq_i/dt) = d/dt(\delta q_i)$, and after a first-order expansion of $\delta q_i$, we have:

$$L(q_i + \delta q_i, \dot{q}_i + \delta \dot{q}_i, t) = L(q_i, \dot{q}_i, t) + \sum_{i=1}^{nd} \frac{\partial L}{\partial q_i} \delta q_i + \sum_{i=1}^{nd} \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i + \frac{\partial L}{\partial t} \delta t$$

The last term $\frac{\partial L}{\partial t} \delta t$ does not contribute since the variation is taken at constant time $\delta t=0$, so:

$$\delta S = \sum_{i=1}^{nd} \left[ \int_{t_1}^{t_2} \frac{\partial L}{\partial q_i} \delta q_i dt + \int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i dt \right].$$

Since $\delta \dot{q}_i$=d($\delta q_i$)/dt, the second integral can be rewritten as:

$$\delta S = \sum_{i=1}^{nd} \int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{q}_i} \frac{d\delta q_i}{dt} dt$$

By performing an integration by parts on this integral, we obtain:

$$\delta S = \sum_{i=1}^{nd} \left[ \frac{\partial L}{\partial \dot{q}_i} \delta q_i \right]_{t_1}^{t_2} - \sum_{i=1}^{nd} \int_{t_1}^{t_2} \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i dt$$

The first term in brackets is zero since $\delta q_i(t_1)=\delta q_i(t_2)=0$. Therefore, for all expressions of $\partial L/\partial \dot{q}_i$ , we obtain:

$$\delta S = \sum_{i=1}^{nd} \int_{t_1}^{t_2} \left[ \frac{\partial L}{\partial q_i} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) \right] \delta q_i dt = 0$$

The $q_i$ are generalized coordinates and thus independent. The sum can only vanish for arbitrary and independent $\delta q_i$ if the n differential equations:

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$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \quad \text{for } 1 \leq i \leq nd$$ are simultaneously satisfied.

## 2.2 Conjugate Momentum

The conjugate momentum associated with the generalized coordinate qiq_iqi is defined by:

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

$p_i$ sometimes called generalized momentum.

**Example:**

For a material point of mass m immersed in a potential U, we have:

$$L(q, \dot{q}, t) = \frac{m}{2}\dot{q}^2 - U \Rightarrow p = \frac{\partial L}{\partial \dot{q}} \Rightarrow p = m\dot{q}$$

**Example:**

For a material point of mass m rotating around a force center c at a distance r, we have:

$$L(\theta, \dot{\theta}, t) = \frac{m}{2}r^2\dot{\theta}^2 - U(r) \Rightarrow p_\theta = \frac{\partial L}{\partial \dot{\theta}} \Rightarrow p = mr\dot{\theta}$$

$P_\theta$ is the component of the angular momentum perpendicular to the plane of rotation.

**Example:**

For a generalized potential:

$$L(q, \dot{q}, t) = \frac{m}{2}\dot{q}^2 + e\phi - e\vec{A} \cdot \dot{\vec{q}} \Rightarrow p_i = \frac{\partial L}{\partial \dot{q}_i} \Rightarrow p = m\dot{q}_i + eA_i \text{ with } i = x, y, z$$

### 2.3 Conservation Laws

In the context of solving motion equations, conservation laws play a fundamental role in physics:

- They reflect certain fundamental physical properties.
- They provide important information about the system's motion.

### A. Cyclic Variable:

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If the Lagrangian L is independent of the generalized coordinate q, then:

$$\frac{\partial L}{\partial q}=0$$

In this case, q is called a cyclic or hidden coordinate, and we have:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{dp}{dt} = 0 \Rightarrow p = \text{constant}$$

The conjugate momentum p associated with the generalized coordinate q is a constant of motion (time-independent) or first integral.

## B. Time Homogeneity:

If the Lagrangian $L(q_i;\dot{q}_i)$ does not explicitly depend on time t (i.e., ∂L/∂t=0), then the quantity:

$$\epsilon = \sum_{i=1}^{nd} \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L(q_i, \dot{q}_i)$$

is a first integral, where:

$$\frac{d\epsilon}{dt}=0$$

By hypothesis, we have $\frac{\partial L}{\partial t}=0$ but $\frac{dL}{dt}\neq 0$, so we have:

$$\frac{d\epsilon}{dt} = \sum_{i=1}^{i=n_d}\left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\dot{q}_i + \frac{\partial L}{\partial \dot{q}_i}\frac{d\dot{q}_i}{dt}\right] - \frac{dL}{dt}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i}$$

$$L(q_i, \dot{q}_i) \Rightarrow \frac{dL}{dt} = \sum_{i=1}^{i=n_d}\left[\frac{dL}{dq_i}\frac{dq_i}{dt} + \frac{\partial L}{\partial \dot{q}_i}\frac{d\dot{q}_i}{dt}\right] \Rightarrow \frac{dL}{dt} = \sum_{i=1}^{i=n_d}\left[\frac{dL}{dq_i}\dot{q}_i + \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i\right]$$

$$\frac{d\epsilon}{dt} = \sum_{i=1}^{i=n_d}\left[\frac{\partial L}{\partial q_i}\dot{q}_i + \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i\right] - \sum_{i=1}^{i=n_d}\left[\frac{dL}{dq_i}\dot{q}_i + \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i\right]$$

$$\frac{d\epsilon}{dt} = 0$$

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Nous supposons que le potentiel $U(q_i)$ ne dépend pas de la vitesse généralisée $\dot{q}_i$ et que les liaisons sont holonomes scléronomes (c'est-à-dire indépendantes du temps t). L'expression de l'énergie cinétique T est alors de la forme quadratique par rapport aux vitesses généralisées $q_i$.

$$T = \sum_{i=1}^{N} \frac{1}{2} m_i \dot{\vec{r}}_i^2$$

where $m_i$ is the mass of particle ii, and $\overrightarrow{\dot{r}_i}$ i is the velocity of that particle.

Since:

$$\dot{\vec{r}}_i = \frac{d\vec{r}_i}{dt} = \sum_{j=1}^{nd} \frac{\partial \vec{r}_i}{\partial q_j} \dot{q}_j$$

..

Then

$$\dot{\vec{r}}_i^2 = \sum_{j=1}^{nd} \frac{\partial \vec{r}_i}{\partial q_j} \dot{q}_j \cdot \sum k = 1^{nd} \frac{\partial \vec{r}_i}{\partial q_k} \dot{q}_k$$

Expanding this, we obtain:

$$T = \frac{1}{2} \sum_{i=1}^{N} m_i \left( \sum_{j=1}^{nd} \frac{\partial \vec{r}_i}{\partial q_j} \dot{q}_j \cdot \sum_{k=1}^{nd} \frac{\partial \vec{r}_i}{\partial q_k} \dot{q}_k \right)$$

This can be rewritten as:

$$T = \frac{1}{2} \sum_{j,k=1}^{nd} \left( \sum_{i=1}^{N} m_i \frac{\partial \vec{r}_i}{\partial q_j} \cdot \frac{\partial \vec{r}_i}{\partial q_k} \right) \dot{q}_j \dot{q}_k$$

Hence:

$$T = \frac{1}{2} \sum_{j=1}^{nd} \sum_{k=1}^{nd} \alpha_{jk} \dot{q}_j \dot{q}_k$$

with:

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$$\alpha_{jk} = \sum_{i=1}^{N} m_i \frac{\partial \vec{r}_i}{\partial q_j} \cdot \frac{\partial \vec{r}_i}{\partial q_k}$$

Thus:

$$\sum_{i=1}^{nd} \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i = \sum_{i=1}^{nd} \frac{\partial T}{\partial \dot{q}_i} \dot{q}_i = \sum_{i=1}^{nd} \dot{q}_i \frac{\partial}{\partial \dot{q}_i} \left[ \frac{1}{2} \sum_{j=1}^{nd} \sum_{k=1}^{nd} \alpha_{jk} \dot{q}_j \dot{q}_k \right] \dot{q}_i$$

Knowing that:

$$\frac{\partial \dot{q}_j}{\partial \dot{q}_i} = \delta_{ji} = \delta_{ij} \quad \text{and} \quad \frac{\partial \dot{q}_k}{\partial \dot{q}_i} = \delta_{ki} = \delta_{ik}$$

then

$$\sum_{i=1}^{nd} \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i = \frac{1}{2} \sum_{i=1}^{nd} \dot{q}_i \sum_{j=1}^{nd} \sum_{k=1}^{nd} \alpha_{jk} \left( \delta_{ij} \dot{q}_k + \delta_{ik} \dot{q}_j \right)$$

or

$$\sum_{i=1}^{nd} \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i = \frac{1}{2} \sum_{i=1}^{nd} \sum_{k=1}^{nd} \alpha_{ik} \dot{q}_i \dot{q}_k + \frac{1}{2} \sum_{j=1}^{nd} \sum_{k=1}^{nd} \alpha_{jk} \dot{q}_j \dot{q}_k$$

Finally, we obtain:

$$\sum_{i=1}^{nd} \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i = T + T = 2T \quad \text{and} \quad L = T - U$$

Thus, the expression for ϵ\epsilonϵ is:

$$\epsilon = \sum_{i=1}^{nd} \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L = 2T - (T - U) = T + U$$

This shows that the total energy $\epsilon$ is the sum of the kinetic energy T and the potential energy U.

## Non-uniqueness of the Lagrangian

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Consider a physical system described by a Lagrangian L($q_i$,$\dot{q}$,t), which satisfies the Lagrange equation 2.21. The following Lagrangians:

$$\tilde{L}=\lambda L$$

$$\tilde{L} = L + \frac{df}{dt}$$

describe the same evolution equation, where f($q_i$,t) is a real and differentiable scalar function.

For the first Lagrangian, this follows directly from the linear form of L in the Lagrange equation.

For the second Lagrangian, we have:

$$\frac{d}{dt}\left(\frac{\partial \tilde{L}}{\partial \dot{q}_i}\right) - \frac{\partial \tilde{L}}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_i}\left(L + \frac{df}{dt}\right)\right) - \frac{\partial}{\partial q_i}\left(L + \frac{df}{dt}\right)$$

$$= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} + \frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_i}\left(\frac{df}{dt}\right)\right) - \frac{\partial}{\partial q_i}\left(\frac{df}{dt}\right)$$

On the other hand, we have:

$$f(q_i, t) \Rightarrow df = \sum_{i=1}^{nd} \frac{\partial f}{\partial q_i} dq_i + \frac{\partial f}{\partial t} dt \Rightarrow \frac{df}{dt} = \sum_{i=1}^{nd} \frac{\partial f}{\partial q_i} \dot{q}_i + \frac{\partial f}{\partial t}$$

Thus:

$$\frac{\partial}{\partial \dot{q}_j}\left(\frac{df}{dt}\right) = \frac{\partial f}{\partial q_j}$$

since:

$$\frac{\partial}{\partial \dot{q}_j}\left(\frac{\partial f}{\partial t}\right) = 0$$

Then:

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_i}\left(\frac{df}{dt}\right)\right) - \frac{\partial}{\partial q_i}\left(\frac{df}{dt}\right) = \frac{d}{dt}\left(\frac{\partial f}{\partial q_i}\right) - \frac{\partial}{\partial q_i}\left(\frac{df}{dt}\right) = 0$$

## 2.5 Principle of Least Action and Holonomic Constraints

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Suppose that the physical system is subjected to ℓ constraints or holonomic constraints:

$$f_j(\vec{r}_1,\ldots,\vec{r}_N)=0 \qquad \forall 1\leq j\leq \ell$$

or, in terms of the 3N coordinates $q_i$, for $i=1,\ldots,3N$

$$f_j(q_1,\ldots,q_{3N})=0 \qquad \forall 1\leq j\leq \ell$$

For a virtual displacement, we have:

$$\delta f_j = \sum_{i=1}^{3N} \frac{\partial f_j}{\partial q_i}\delta q_i = 0$$

Thus:

$$\sum_{i=1}^{3N} \int_{t_1}^{t_2} \left( \sum_{j=1}^{\ell} \lambda_j \frac{\partial f_j}{\partial q_i} \right) \delta q_i dt = 0 \quad (2.24)$$

Moreover, according to the calculus of variations, if we add a virtual displacement $\delta q_i$ such that $\delta q_i(t_1)=\delta q_i(t_2)=0$ to a real trajectory $q_i$, the variation of the action is:

$$\delta S = \sum_{i=1}^{3N} \int_{t_1}^{t_2} \left[ \frac{\partial L}{\partial q_i} - \frac{d}{dt}\left( \frac{\partial L}{\partial \dot{q}_i} \right) \right] \delta q_i dt = 0 \quad (2.25)$$

Given equations (2.24) and (2.25), we have:

$$\sum_{i=1}^{3N} \int_{t_1}^{t_2} \left[ \frac{\partial L}{\partial q_i} - \frac{d}{dt}\left( \frac{\partial L}{\partial \dot{q}_i} \right) + \sum_{j=1}^{\ell} \lambda_j \frac{\partial f_j}{\partial q_i} \right] \delta q_i \, dt = 0 \quad (2.26)$$

The 3N unknowns $q_i$ are given by the Lagrange equations:

$$\begin{aligned} &\frac{d}{dt}\left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = \sum_{j=1}^{\ell} \lambda_j \frac{\partial f_j}{\partial q_i} \quad \text{for } i = 1,\ldots,3N \\ &f_j(q_1,\ldots,q_{3N},t) = 0 \quad \text{for } j = 1,\ldots,\ell \end{aligned} \quad (2.27)$$

Applications

A. **Two-Body Problem:** Two particles of masses $m_1$ and $m_2$ are represented by position vectors $\vec{r_1}$ and $\vec{r_2}$. Assuming the system formed by $m_1$ and $m_2$ is closed, the total external force on the system is zero. Additionally, the particles interact via the potential energy $U(\vec{r_1}, \vec{r_2}.)=U(\vec{r_1}, \vec{r_2})$.

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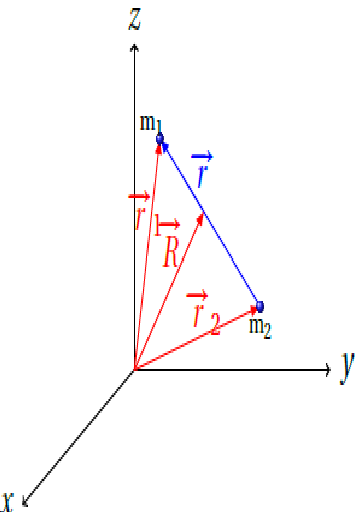
Let $\vec{R}$ be the position vector of the center of mass (CM):




$$\vec{R} = \frac{m_1\vec{r}_1+m_2\vec{r}_2}{m_1+m_2}$$

Let the new vector $\vec{r}$ be defined as:

$$\vec{r}=\vec{r}_1-\vec{r}_2$$

The kinetic energy of the system is:

$$T=\frac{1}{2}m\dot{r}^2$$

After substitution and simplification:

$$T=\frac{1}{2}(m_1+m_2)\dot{R}^2+\frac{1}{2}\frac{m_1 m_2}{m_{1+}+m_2}\dot{r}^2$$

Finally, the kinetic energy can be written as:

$$T=\frac{1}{2}M\dot{R}^2+\frac{1}{2}\mu\dot{r}^2$$

where $M=m_1+m_2$ is the total mass of the system, and $\mu=\frac{m_1 m_2}{m_{1+}+m_2}$ is the reduced mass.

In the case where $m_1 \ll m_2$ , such as in the Earth-Sun system, $M=m_2$ and $\mu=\frac{m_1 m_2}{m_{1+}+m_2}$.
The Lagrangian of the system is:

$$L=T-U(\vec{r}_1-\vec{r}_2)=\frac{1}{2}M\dot{\vec{R}}^2+\frac{1}{2}\mu\dot{\vec{r}}^2-U(\vec{r})$$

This Lagrangian can be written as:

$$L=L_{CM}+L_{\mu}$$

where:

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$$L_{CM}=\frac{1}{2}M\dot{\vec{R}}^2$$

$\vec{R}$ is a Hidden or Cyclic Variable:

$$\text{Since} \qquad \frac{\partial L}{\partial \vec{R}}=\frac{\partial L_{CM}}{\partial \dot{\vec{R}}}=\vec{0},$$

then:

$$\frac{d}{dt}\left(\frac{\partial L_{CM}}{\partial \dot{\vec{R}}}\right)=\vec{0} \Rightarrow \frac{\partial L_{CM}}{\partial \dot{\vec{R}}}=M\dot{\vec{R}}$$

$M\dot{\vec{R}}=\overrightarrow{cte}$ the center of mass (CM) moves at a constant velocity.

The second part of the Lagrangian is:

$$L\mu=\frac{1}{2}\mu\,\dot{\vec{r}}^2-U(\vec{r})$$

This appears as the Lagrangian of a particle with mass μ\muμ and position vector $\vec{r}$. The Lagrangian L is the Lagrangian of the relative motion between two particles, reducing it to a problem of a single fictitious particle with reduced mass μ\muμ and position vector $\vec{r}$.

The two-body problem can be reduced to a one-body problem. However, the N-body problem (N > 2) has no analytical solution.

**Central Potential:**

We are interested in cases where the interaction depends only on the distance between two bodies (gravitational interaction, electrostatic interaction) or between the body and the origin of the force (central force) (mass-spring, sun-earth). In this case, $U(\vec{r})=U(r)$.

Since the force here is purely radial, $\vec{F}= f(r)\overrightarrow{e_r}$ then:

$$\vec{M}_{\vec{F}/O}=\overrightarrow{OM}\wedge\vec{F}=\vec{r}\overrightarrow{e_r}\wedge f(r)\,\overrightarrow{e_r}=r\,f(r)\,\overrightarrow{e_r}\wedge\overrightarrow{e_r}=\vec{0}$$

The moment of the force applied to the mass mmm is zero. The angular momentum is:

$$\overrightarrow{L}_{M/O}=\overrightarrow{OM}\wedge m\vec{v}=r\vec{e}_r\wedge m\left(\dot{r}\vec{e}_r+r\dot{\theta}\vec{e}_\theta\right)=mr\dot{r}\vec{e}_r\wedge\vec{e}_r+mr^2\dot{\theta}(\vec{e}_r\wedge\vec{e}_\theta)$$

Since $\overrightarrow{e_r}\wedge\overrightarrow{e_\theta}=\overrightarrow{e_z}$, the angular momentum with respect to O is:

$$\overrightarrow{L}_{M/O}=mr^2\dot{\theta}\vec{e}_z$$

According to the angular momentum theorem:

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$$\frac{d\overrightarrow{L}_{M/O}}{dt} = \overrightarrow{\mathscr{M}}_{\overrightarrow{F}/O} = \overrightarrow{0}$$

$\vec{L}_{M/O}$ is constant in magnitude and direction, so the trajectory of the material point lies in the XOY plane (the plane is a constraint, reducing one degree of freedom). We can also write:

$$\frac{d^{\cdot}}{dt}(mr^2\dot{\theta})=0$$

The particle of mass mmm has two degrees of freedom, so two generalized coordinates rrr and θ\thetaθ. The Lagrangian is:

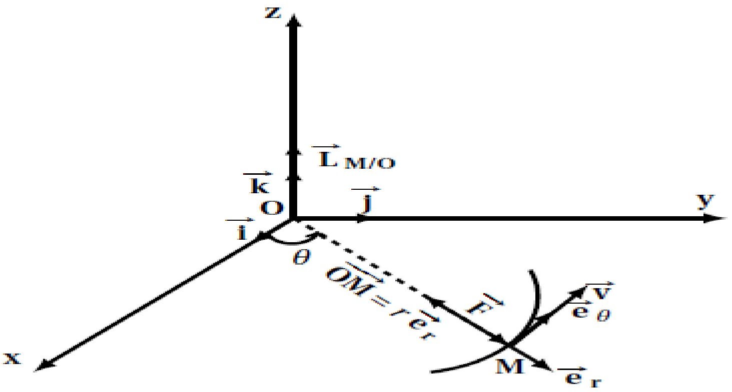
$$L=\frac{1}{2}m(r^{\cdot 2}+r^2\theta^{\cdot 2})-U(r)$$

Immediately, we notice that θ is a cyclic or "hidden" coordinate, and therefore:

$$\frac{\partial L}{\partial \theta} = 0 \quad \Rightarrow \quad \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \quad \Rightarrow \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{d}{dt}(mr^2\dot{\theta}) = 0 \quad \Rightarrow \quad mr^2\dot{\theta} = \sigma_0$$

This is the law of conservation of angular momentum. With respect to the generalized coordinate r, we have:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = m\ddot{r} - mr\dot{\theta}^2 + \frac{\partial U}{\partial r} = 0 \quad \Rightarrow \quad m\ddot{r} = mr\dot{\theta}^2 - \frac{\partial U}{\partial r}$$




The particle of mass mmm is subjected to two forces:

Central force

$$-\frac{\partial U}{\partial r}\mathbf{e}_r$$

Centrifugal force:

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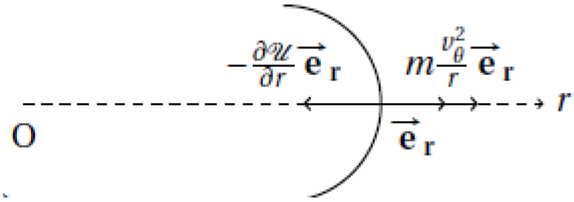
$$mr\dot{\theta}^2\,\mathbf{e}_r = m\frac{v_\theta^2}{r}\,\mathbf{e}_r = \frac{\sigma_0^2}{mr^3}\,\mathbf{e}_r$$




$m\frac{v_\theta^2}{r}\vec{\mathbf{e}}_{\mathbf{r}}$ is a repulsive force, meaning that a body moving along a curvilinear trajectory with respect to O is effectively pushed outward.

From the conservation of angular momentum, we have:

$$\dot{\theta} = \frac{\sigma_0}{mr^2} \quad \Rightarrow \quad m\ddot{r} = \frac{\sigma_0^2}{mr^3} - \frac{\partial U}{\partial r}$$

This simplifies to:

$$m\ddot{r} = -\frac{\partial}{\partial r}\left(U + \frac{\sigma_0^2}{2mr^2}\right) \quad \Rightarrow \quad m\ddot{r} = -\frac{\partial U_{\text{eff}}}{\partial r}$$

where we have defined $U_{eff}$ as:

$$U_{\text{eff}}(r) = U(r) + \frac{\sigma_0^2}{2mr^2}$$

The term $\frac{{\sigma_0}^2}{2mr^2}$ represents a potential barrier. The problem is thus reduced to studying the motion of a particle of mass m with one degree of freedom in the generalized coordinate r, subjected to an effective force:

$$F_{\text{eff}} = -\frac{\partial U_{\text{eff}}}{\partial r} = -\frac{\partial U}{\partial r} + \frac{\sigma_0^2}{mr^3}$$

and governed by the fundamental principle of dynamics:

$$m\ddot{r} = F_{eff}$$

## B - Potential in $\frac{1}{r}$:

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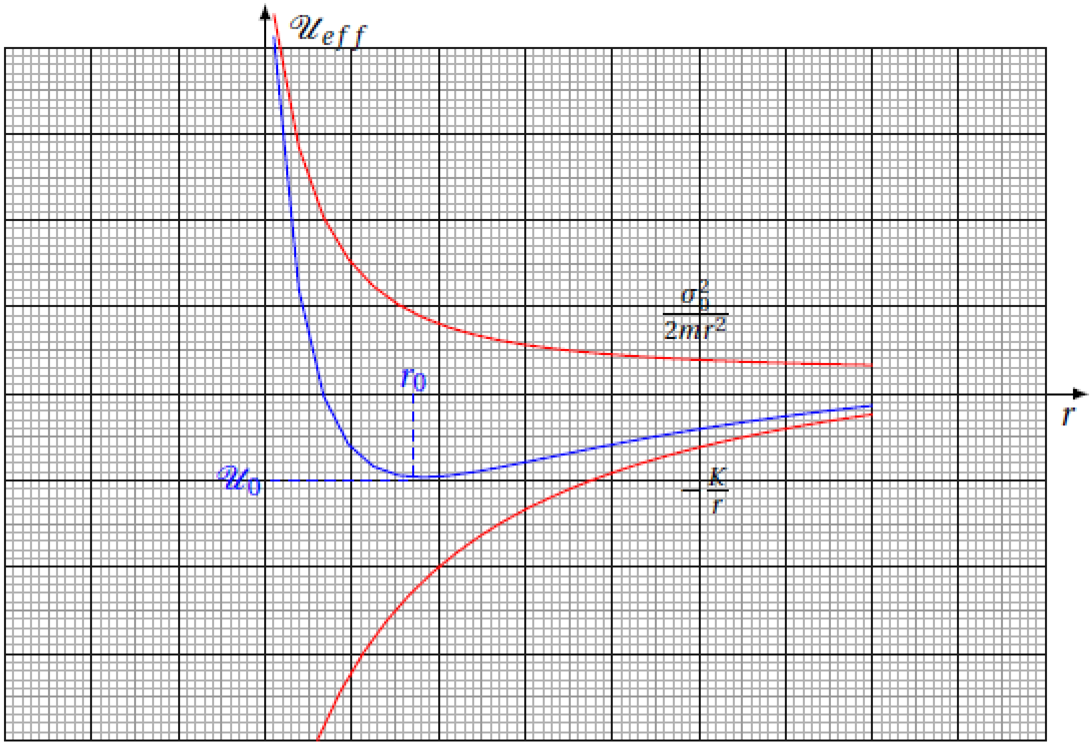
In the case of gravitational or electrostatic interaction, the Newtonian potential is written as:

$$U(r) = -\frac{K}{r}$$




Le potentiel effective à un minimum $\mathscr{U}_0$ pour

with $K=Gm_1m_2$ the gravitational case and $K=\frac{|q_1q_2|}{4\pi\epsilon 0}$ in the electrostatic case. Thus, the effective potential is written as:

$$U_{\text{eff}}(r) = -\frac{K}{r} + \frac{\sigma_0^2}{2mr^2}$$

The effective potential has a minimum $U_0$ at:

$$\frac{dU_{\text{eff}}}{dr} = \frac{K}{r^2} - \frac{\sigma_0^2}{mr^3} = 0$$

We obtain:

$$r_0 = \frac{\sigma_0^2}{Km} \quad \Rightarrow \quad U_0 = -\frac{K}{2m}$$

Thus, for $r=r_0$, we have $m\ddot{r}=0$ and the effective force acting on the particle is zero. This corresponds to a circular trajectory (where only θ\thetaθ varies). According to the conservation of angular momentum, the angular velocity is:

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$$\dot{\theta} = \dot{\theta}_0 = \frac{\sigma_0}{m r_0^2} = \text{constant}$$

The equation of motion for θ(t) is:

$$\theta(t) = \dot{\theta}_0 t + \theta_0$$

For each time interval, the period τ\tauτ, during which θ\thetaθ increases by 2π (to complete one orbit), is given by: