Lagrange Formalism

-Introduction

In Newtonian mechanics, the motion of a system with N material points is obtained by solving N second-order vector differential equations or 6N scalar differential equations (3 for acceleration and 3 for velocity), projecting onto the 3 Cartesian coordinate axes and involving 6N integration constants.

However, there are circumstances where the application of Newtonian mechanics is delicate, particularly when the system has internal constraints due to binding forces that limit the movement of the system, thus reducing its degrees of freedom.

Newton's laws form what can be called vector mechanics, as most of the quantities involved in this description are of a vector nature (force, acceleration, velocity, position vector). In contrast to vector mechanics, analytical mechanics uses scalar functions (kinetic and potential energies).

II-Principle of Virtual Work

If $\vec{F_i}$ denotes the sum of all forces acting on the material point i of a system with N material points, the sum of the work done by the forces applied to the system in equilibrium is:

$$\delta W = \sum_{i=1}^{N} \vec{F}_i \cdot \delta\vec{r}_i = 0 \quad (2.1)$$

where $\overrightarrow{\delta r_i}$ is a virtual displacement of the system.

Some forces among the $\vec{F_i}$ have identically zero virtual work, and others have mutually zero virtual work. These forces are called binding forces, and the remaining forces are applied forces:

$$\vec{F}_i = \vec{F}_i^{(a)} + \vec{F}_i^{(l)}$$

By definition, for any infinitesimally small virtual displacement compatible with the constraints of the problem, we can write that the virtual work of the binding forces (this is what is called d'Alembert's principle (1743)) is:

$$\sum_{i=1}^{N} \vec{F}_i^{(l)} \cdot \delta\vec{r}_i = 0$$

In the case of equilibrium, we have:

$$\sum_{i=1}^{N} \vec{F}_i^{(a)} \cdot \delta\vec{r}_i = 0 \quad (2.2)$$

Example 1

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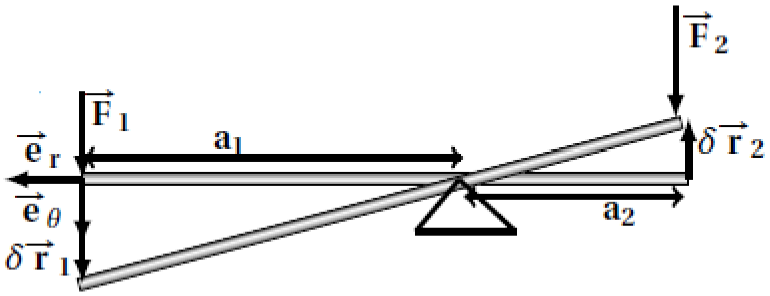
Consider a lever in equilibrium under the action of two forces $\vec{F}_1$ and $\vec{F}_2$. Determine the condition of equilibrium. The position vectors are written as:

$$\vec{r}_1 = a_1 \vec{e}_r$$
$$\vec{r}_2 = -a_2 \vec{e}_r$$

Giving the lever a virtual rotation $\delta\theta$, we get:

$$\delta \vec{e}_r = \delta\theta \vec{e}_\theta$$

$$\begin{cases} \delta \vec{r}_1 = a_1 \delta \vec{e}_r & \delta \vec{r}_1 = a_1 \delta\theta \vec{e}_\theta \\ \delta \vec{r}_2 = -a_2 \delta \vec{e}_r & \delta \vec{r}_2 = -a_2 \delta\theta \vec{e}_\theta \end{cases}$$



Since the reaction force $\vec{R}$ does not do work, the principle of virtual work can be written as:

$$\vec{F}_1 \cdot \delta \vec{r}_1 + \vec{F}_2 \cdot \delta \vec{r}_2 = 0$$

Given that:

$$\vec{F}_1 = F_1 \vec{e}_\theta$$
$$\vec{F}_2 = F_2 \vec{e}_\theta$$

It follows that:

$$(F_1 a_1 - F_2 a_2)\delta\theta = 0 \Rightarrow F_1 a_1 - F_2 a_2 = 0$$

Calculation of the reaction force $\vec{R}$:

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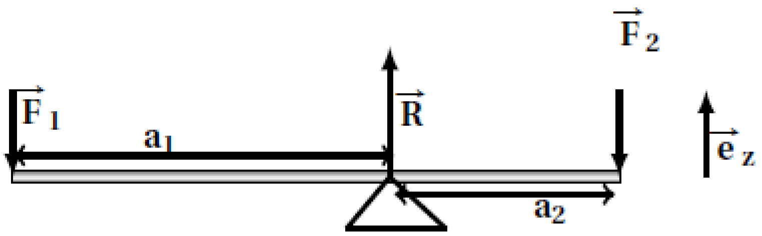


If the virtual displacement is incompatible with the binding force, we give the lever a virtual vertical translation $\delta\vec{r}=\delta z\vec{e_z}$. The principle of virtual work then becomes:

$$\vec{R}\cdot\delta\vec{r}+\vec{F}_1\cdot\delta\vec{r}+\vec{F}_2\cdot\delta\vec{r}=0$$

Therefore:

$$(R-F_1-F_2)\delta z=0\Rightarrow R=F_1+F_2$$

Example 2: Double Inclined Plane

Let two masses m_1 and m_2 placed on a double inclined plane and a lever in equilibrium under the action of two forces $\vec{F}_1$ et and $\vec{F}_2$. Determine the condition of equilibrium

The forces acting on the system include gravitational forces $\vec{p_1}$ and $\vec{p_2}$, tension forces $\vec{T}_1$ and $\vec{T_2}$,and reaction forces $\vec{R_1}$ and $\vec{R}_2$.

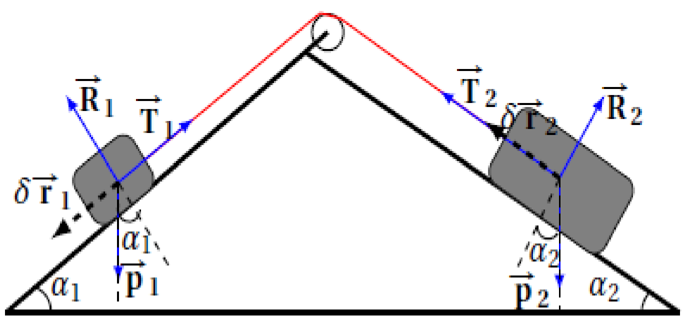
The principle of virtual work can be written as:

$$(\vec{p}_1+\vec{T}_1+\vec{R}_1)\cdot\delta\vec{r}_1+(\vec{p}_2+\vec{T}_2+\vec{R}_2)\cdot\delta\vec{r}_2=\vec{0}$$

Since $\vec{R}_1\cdot\delta\vec{r}_1=\vec{0}$ and $\vec{R}_2\cdot\delta\vec{r}_2=\vec{0}$ (as $\vec{R}_1$ and $\vec{R}_2$are perpendicular to the virtual displacements), the virtual work of $\vec{R}_1$and $\vec{R}_2$is identically zero.

The virtual work of $\vec{T_1}$ and $\vec{T}_2$ is mutually zero:

$$T_2\delta r_2-T_1\delta r_1=0$$



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Since $\delta r_2 = \delta r_1 = \delta r$, we have $(T_2 - T_1).\delta r = 0$, implying $T_1 = T_2$.

The virtual work of the weights is:

$$m_1 g \delta r \cos\left(\frac{\pi}{2} - \alpha_1\right) + m_2 g \delta r \cos\left(\frac{\pi}{2} + \alpha_2\right) = 0$$

Therefore:

$$g(m_1 \sin\alpha_1 - m_2 \sin\alpha_2)\delta r = 0$$

The equilibrium position is:

$$m_1 \sin\alpha_1 = m_2 \sin\alpha_2$$

Constraints

A constraint is associated with a binding force that restricts or limits the possible movements of the system. The equations of constraints can be expressed either by:

- Binding forces.
- Equations of constraints containing the coordinates, their derivatives, and time.

In the simplest case, these constraints can be expressed by a number l of independent equations among the n<N positions $\vec{r}_i$ and time t:

$$\begin{cases} f_1(\vec{r}_1, \ldots, \vec{r}_n, t) = 0 \\ \vdots \\ f_l(\vec{r}_1, \ldots, \vec{r}_n, t) = 0 \end{cases}$$

These constraints are called holonomic (from Greek: "whole law"). In these constraint relations, the velocity $\dot{\vec{r}}$ does not appear.

• If time is implicit in the equation, it is called holonomic scleronomic or holonomic stationary.
• If time is explicit in the equation, it is called holonomic rheonomic or holonomic non-stationary.

If these constraint relations do not involve the velocity \(\dot{\vec{r}}_i \): - If time

Degrees of Freedom

A free particle in a 3-dimensional configuration space has 3 independent coordinates to determine its motion. For a system of N free material points, 3N independent coordinates are needed to fully determine its motion. If the system is subject to constraints that limit its motion, the number of necessary coordinates to describe its motion reduces. In the simplest case, these constraints can be expressed by a number l of independent equations among the N positions $\vec{r}_i$ and time t : The minimum number of coordinates necessary to specify the position of a system, or multiple material points, is called the number of degrees of freedom (DOF): $n_{\{DOF\}} = 3N - l$

Generalized Coordinates

Generalized coordinates are the necessary and independent coordinates that allow tracking the evolution of a system over time. They can take the form of a coordinate or an angle. For nd degrees of freedom, we have nd generalized coordinates. It is assumed that the position vector $\vec{r}_i$ of material point iii is an arbitrary function of the generalized coordinates $\{q_j, j=1,\ldots,nd\}$ of the form:

$$\vec{r}_i(x_i, y_i, z_i) = \vec{r}_i(x_i(q_j), y_i(q_j), z_i(q_j))$$

From this equation, the differential form of Pfaff or Pfaffian is:

$$dx_i = \sum_{j=1}^{n} \frac{\partial x_i}{\partial q_j} dq_j + \frac{\partial x_i}{\partial t} dt$$
$$dy_i = \sum_{j=1}^{n} \frac{\partial y_i}{\partial q_j} dq_j + \frac{\partial y_i}{\partial t} dt$$
$$dz_i = \sum_{j=1}^{n} \frac{\partial z_i}{\partial q_j} dq_j + \frac{\partial z_i}{\partial t} dt$$

Generalization of the Principle of Virtual Work

To simplify, we consider holonomic constraints with n degrees of freedom. For any point in the system, we have:

$$\vec{r}_i = \vec{r}_i(q_1, \ldots, q_{nd}, t), \quad i = 1, \ldots, N$$

A compatible displacement from $\{q_1,\ldots,q_{nd},t\}$ is any virtual displacement that could be imposed on the system, taking into account the constraints existing at time t:

$$\delta\vec{r}_i = \sum_{j=1}^{nd} \frac{\partial \vec{r}_i}{\partial q_j} \delta q_j$$

D'Alembert's Equation

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Consider a system of material points of mass m_i, subjected to binding forces $\vec{F}_i^{\,l}$and applied forces $\vec{F}_i^{\,a}$According to the fundamental principle of dynamics, we have:

$$m_i\vec{\gamma}_i = \vec{F}_i^{(l)} + \vec{F}_i^{(a)}$$

According to the virtual work condition:

$$\sum_{i=1}^{N} \vec{F}_i^{(l)} \cdot \delta\vec{r}_i = 0$$

We obtain D'Alembert's equation (the sum of the virtual work of the applied forces and the inertia forces is zero for any compatible virtual displacement):

$$\sum_{i=1}^{N} (m_i\vec{\gamma}_i - \vec{F}_i^{(a)}) \cdot \delta\vec{r}_i = 0$$

Generalized Force

From D'Alembert's equation and equation (2.3), we have:

$$\sum_{i=1}^{N} (m_i\vec{\gamma}_i - \vec{F}_i^{(a)}) \cdot \sum_{j=1}^{nd} \frac{\partial\vec{r}_i}{\partial q_j}\delta q_j = 0$$

Introducing the generalized force Q_j:

$$Q_j = \sum_{i=1}^{N} \vec{F}_i^{(a)} \cdot \frac{\partial\vec{r}_i}{\partial q_j}$$

Lagrange's Equations of the First Kind

For a system with N material points subjected to holonomic rheonomic constraints $g_k(\vec{r}_1,\vec{r}_2,\ldots,\vec{r}_N,t)$ ($k=1,\ldots,\ell$), defining thus $nd=3N-\ell$ generalized coordinates $\{q_1,q_2,\ldots,q_{nd}\}$, we can invert the ndrelations $q_j=q_j(\vec{r}_1,\vec{r}_2,\ldots,\vec{r}_N,t)$ ($j=1,\ldots,nd_j$) to express the N relations:

$$\vec{r}_i=\vec{r}_i(q_1,q_2,\ldots,q_{nd},t)$$

The infinitesimal displacement is:

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$$d\vec{r}_i = \sum_{j=1}^{nd} \frac{\partial \vec{r}_i}{\partial q_j} dq_j + \frac{\partial \vec{r}_i}{\partial t} dt$$

The generalized velocity is defined as:

$$\dot{q}_j = \frac{dq_j}{dt}$$

Thus:

$$\vec{v}_i = \dot{\vec{r}}_i = \frac{d\vec{r}_i}{dt} = \sum_{j=1}^{nd} \frac{\partial \vec{r}_i}{\partial q_j} \dot{q}_j + \frac{\partial \vec{r}_i}{\partial t}$$

The left-hand side of D'Alembert's equation (2.5):

$$A_j = \sum_{i=1}^{N} m_i \vec{\gamma}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = \sum_{i=1}^{N} m_i \frac{d\dot{\vec{r}}_i}{dt} \cdot \frac{\partial \vec{r}_i}{\partial q_j}$$

Then:

$$A_j = \sum_{i=1}^{N} m_i \frac{d}{dt} \left(\dot{\vec{r}}_i \cdot \frac{\partial \dot{\vec{r}}_i}{\partial q_j} \right) - \sum_{i=1}^{N} m_i \dot{\vec{r}}_i \cdot \frac{d}{dt} \left(\frac{\partial \vec{r}_i}{\partial q_j} \right)$$

On the other hand:

$$\frac{d}{dt} \left(\frac{\partial \vec{r}_i}{\partial q_j} \right) = \frac{\partial}{\partial q_j} \left(\frac{d\vec{r}_i}{dt} \right) = \frac{\partial \vec{r}_i}{\partial q_j}$$

We have:

$$\frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_k} = \frac{\partial}{\partial \dot{q}_k} \left(\frac{d\vec{r}_i}{dt} \right) = \frac{\partial}{\partial \dot{q}_k} \left(\sum_{j=1}^{nd} \frac{\partial \vec{r}_i}{\partial q_j} \dot{q}_j + \frac{\partial \vec{r}_i}{\partial t} \right) = \sum_{j=1}^{nd} \frac{\partial \vec{r}_i}{\partial q_j} \frac{\partial \dot{q}_j}{\partial \dot{q}_k}$$

We have:

$$\frac{\partial \dot{q}_j}{\partial \dot{q}_k} = \delta_{jk} = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$$

is the Kronecker delta. Thus, we have:

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$$\frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_k} = \sum_{j=1}^{nd} \frac{\partial \vec{r}_i}{\partial q_j} \delta_{jk} = \frac{\partial \vec{r}_i}{\partial q_k}$$

Thus:

$$A_j = \sum_{i=1}^{N} \frac{d}{dt} \left(m_i \dot{\vec{r}}_i \cdot \frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_j} \right) - \sum_{i=1}^{N} m_i \dot{\vec{r}}_i \cdot \left(\frac{\partial \dot{\vec{r}}_i}{\partial q_j} \right)$$

Then:

$$A_j = \frac{d}{dt} \left(\frac{\partial}{\partial \dot{q}_j} \left(\sum_{i=1}^{N} \frac{1}{2} m_i \dot{\vec{r}}_i^2 \right) \right) - \frac{\partial}{\partial q_j} \left(\sum_{i=1}^{N} \frac{1}{2} m_i \dot{\vec{r}}_i^2 \right)$$

The kinetic energy of the system is defined as:

$$T = \sum_{i=1}^{N} \frac{1}{2} m_i \dot{\vec{r}}_i^2$$

Finally, for A_j, we have:

$$A_j = \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j}$$

Equation (2.5) becomes:

Lagrange's Equations of the First Kind

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} = Q_j \quad \text{for } j = 1, \ldots, nd$$

This is a system with nd equations called Lagrange equations.

Example: Simple Pendulum

It is a system composed of an object which is a material point of mass mmm. Thus, the number of coordinates is Nc=3. The constraints are the plane xoy with equation z=0 and since the string is inextensible, the coordinates of the mass mmm satisfy the relation $r=\sqrt{x^2+y^2}=\ell$, and the mass moves along an arc of a circle. The number of constraint equations is $\ell=2$, so the number of degrees of freedom is nd=1. The generalized coordinate is θ. The kinetic energy T is

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$T = \frac{1}{2}m\ell^2\dot{\theta}^2$

The generalized force is $Q_{\mathbf{p}}$

$Q_{\mathbf{p}} = \mathbf{p} \cdot \frac{\partial \mathbf{r}}{\partial \theta}$

$\mathbf{p} = mg\mathbf{j}$

$\mathbf{r} = \ell \sin\theta \mathbf{i} + \ell \cos\theta \mathbf{j}$

$\frac{\partial \mathbf{r}}{\partial \theta} = \ell \cos\theta \mathbf{i} - \ell \sin\theta \mathbf{j}$

$\mathbf{p} \cdot \frac{\partial \mathbf{r}}{\partial \theta} = -mg\ell \sin\theta$

Applying equation (2.7), we have

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = Q_{\mathbf{p}}$$

Thus, we find

$$m\ell^2\ddot{\theta} = -mg\ell \sin\theta$$

or

$$\ddot{\theta} + \omega_0^2 \sin\theta = 0$$

with $\omega_0 = \sqrt{\frac{g}{\ell}}$ being the natural frequency of the pendulum.

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Relation between Work and Generalized Force

$dW = \sum_{i=1}^{N} \mathbf{F}_i \cdot d\mathbf{r}_i$

as

$d\mathbf{r}_i = \sum_{j=1}^{nd} \frac{\partial \mathbf{r}_i}{\partial q_j} dq_j + \frac{\partial \mathbf{r}_i}{\partial t} dt$

we have

$dW = \sum_{j=1}^{nd} \left(\sum_{i=1}^{N} \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j}\right) dq_j$

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$$dU = \sum_{j=1}^{nd} \frac{\partial U}{\partial q_j} dq_j$$

as

$$dW = -dU = \sum_{j=1}^{nd} Q_j dq_j$$

therefore

$$Q_j = -\frac{\partial U}{\partial q_j}$$

then

$$dW = \sum_{j=1}^{nd} Q_j dq_j$$

On the other hand,

$$dW = \sum_{j=1}^{nd} \frac{\partial W}{\partial q_j} dq_j$$

thus

$$Q_j = \frac{\partial W}{\partial q_j}$$

Remark:

If the external forces are derivable from a potential

$$\mathbf{F}_i = -\frac{\partial U}{\partial \mathbf{r}_i}$$

(with $\frac{\partial}{\partial \mathbf{r}_i} = \nabla_{\mathbf{r}_i}$), we have

$$dW = -dU = -\sum_{j=1}^{nd} \left(\sum_{i=1}^{N} \frac{\partial U}{\partial \mathbf{r}_i} \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \right) dq_j$$

$$dU = \sum_{j=1}^{nd} \frac{\partial U}{\partial q_j} dq_j$$

as

$$dW = -dU = \sum_{j=1}^{nd} Q_j dq_j$$

therefore

$$Q_j = -\frac{\partial U}{\partial q_j}$$

Lagrange Equation in Presence of Conservative and Non-Conservative Forces

Suppose the material point i of the system is subject to conservative and non-conservative forces

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$$\mathbf{F}_i = \mathbf{F}_i^{(c)} + \mathbf{F}_i^{(nc)} = -\frac{\partial U}{\partial \mathbf{r}_i} + \mathbf{F}_i^{(nc)}$$

since $Q_j = -\frac{\partial U}{\partial q_j}$, then

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = -\frac{\partial U}{\partial q_j} + Q_j^{(nc)}$$

with

$$Q_j^{(nc)} = \sum_{i=1}^{N} \mathbf{F}_i^{(nc)} \cdot \frac{\partial \mathbf{r}_i}{\partial q_j}$$

If U does not depend on $\dot{q}_j$ and t, then we can write

$$\frac{d}{dt}\left(\frac{\partial (T-U)}{\partial \dot{q}_j}\right) - \frac{\partial (T-U)}{\partial q_j} = Q_j^{(nc)}$$

By defining the Lagrangian as

$$\mathbf{F}_i = \mathbf{F}_i^{(c)} + \mathbf{F}_i^{(nc)} = -\frac{\partial U}{\partial \mathbf{r}_i} + \mathbf{F}_i^{(nc)}$$

since $Q_j = -\frac{\partial U}{\partial q_j}$, then

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = -\frac{\partial U}{\partial q_j} + Q_j^{(nc)}$$

with

$$Q_j^{(nc)} = \sum_{i=1}^{N} \mathbf{F}_i^{(nc)} \cdot \frac{\partial \mathbf{r}_i}{\partial q_j}$$

If U does not depend on $\dot{q}_j$ and t, then we can write

$$\frac{d}{dt}\left(\frac{\partial (T-U)}{\partial \dot{q}_j}\right) - \frac{\partial (T-U)}{\partial q_j} = Q_j^{(nc)}$$

Example:

Let a mass m slide with friction on a plane inclined at an angle α relative to the horizontal. Determine its acceleration. This is a system with one degree of freedom and generalized coordinate x.

The kinetic energy T is

$$T=\frac{1}{2}m\dot{x}^2$$

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The potential energy of the system is gravitational in nature

$$Q_x^{(c)} = \frac{\partial W}{\partial x} = \frac{dW}{dx} = -\frac{dU}{dx}$$

but

$$Q_x^{(c)} = \mathbf{p} \cdot (dx\mathbf{e}_x)/dx = \mathbf{p} \cdot \mathbf{e}_x/dx = p_x = mg\sin\alpha$$

and

$$U_g = -mg\sin\alpha\int_0^x dx = -mg\sin\alpha x$$

The Lagrangian of the system is

$$L = T - U = \frac{1}{2}m\dot{x}^2 + mg\sin\alpha x$$

The non-conservative generalized force is

$$Q_x^{(nc)} = \frac{\partial W_c}{\partial x} = \frac{dW_c}{dx}$$

but

$$Q_x^{(nc)} = \mathbf{c} \cdot (dx\mathbf{e}_x)/dx = \mathbf{c} \cdot \mathbf{e}_x/dx = -cx = -f$$

and $f = \mu R = \mu mg\cos\alpha$ (μ is the coefficient of dynamic friction), thus

$$Q_x^{(nc)} = -\mu mg\cos\alpha$$

The Lagrange equation

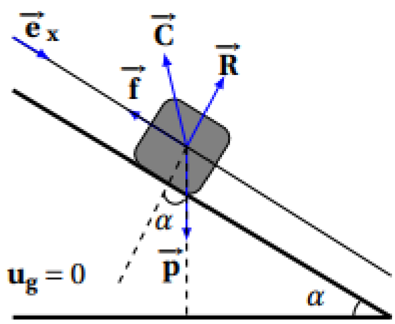
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = Q_x^{(nc)}$$

yields

$$\frac{d}{dt}(m\dot{x}) - mg\sin\alpha = -\mu mg\cos\alpha$$

finally resulting in

$$\ddot{x} = g(\sin\alpha - \mu\cos\alpha)$$



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Viscous Friction Force: Stokes Problem

Suppose some of the forces are not derived from the potential. This is the case with viscous friction forces

$$f_i = -k_i v_i^{\gamma}$$
$$\mathbf{F}_i = f_i \mathbf{u}$$

where **u** is a unit vector in the direction of the motion and $k_i>0$ is a proportionality coefficient, v_i *i*s the velocity of the material point i of the system with N material points. γ is a coefficient and depending on its value, we have three types of friction:

- γ=1 linear or dynamic friction
- γ=2 quadratic friction
- γ=0 Coulomb friction (solid)

The equation of motion is expressed as follows:

$m\ddot{x} + k_i\dot{x}^{\gamma}=F_I$

In the case of linear friction, the solution is

$$x = Ce^{-k_i t} + \frac{F_i}{k_i}t + D$$

Example:

Consider a material point m falling from a certain height and subject to a quadratic friction force. Determine the acceleration of the point. The system has one degree of freedom with generalized coordinate x.

The kinetic energy T is

$$T=\frac{1}{2}m\dot{x}^2$$

Lagrange Formalism

The potential energy of the system is gravitational in nature

$$Q_x^{(c)} = \frac{\partial W}{\partial x} = \frac{dW}{dx} = -\frac{dU}{dx}$$

with

$$Q_x^{(c)} = \mathbf{p} \cdot (dx\mathbf{e}_x)/dx = \mathbf{p} \cdot \mathbf{e}_x/dx = p_x = mg$$

and

$$U_g = -mg \int_0^x dx = -mgx$$

The Lagrangian of the system is

$$L = T - U = \frac{1}{2}m\dot{x}^2 + mgx$$

The non-conservative generalized force is

$$Q_x^{(nc)} = \frac{\partial W_c}{\partial x} = \frac{dW_c}{dx}$$

with $\mathbf{F}_i = -k_i v_i^2 \mathbf{u}$

$$Q_x^{(nc)} = -k_i v_i^2 \mathbf{u} \cdot \mathbf{e}_x/dx = -k_i \dot{x}^2$$

The Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = Q_x^{(nc)}$$

yields

$$m\ddot{x} + k_i \dot{x}^2 = mg$$

finally resulting in

$$\ddot{x} + \frac{k_i}{m}\dot{x}^2 = g$$

Viscous Friction

$$f_i = -k_i v_i^\gamma$$
$$\mathbf{F}_i = f_i \mathbf{u}$$

where $\vec{U}$ is a unit vector pointing in the direction of movement, and $k_i>0$ a proportionality coefficient. v_i is the velocity of the material point iii in a system of N material points. γ is a positive exponent defining the friction regime, and in general, $\gamma=\gamma(v)$; however, in the low-velocity regime, experiments show that $\gamma=1$. We are interested in the case where γ is a positive real number and independent of velocity v. From each vector, we can construct its unit vector, so

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$\vec{u}=\frac{\vec{V}_I}{V_I}$

then

$$\vec{F}_i = -k_i v_i^{\gamma-1}\vec{v}_i$$

In Cartesian coordinates, the components of the force are:

$$F_{xi} = -k_i v_i^{\gamma-1} v_{xi}, \quad F_{yi} = -k_i v_i^{\gamma-1} v_{yi}, \quad F_{zi} = -k_i v_i^{\gamma-1} v_{zi}$$

On the other hand, we have:

$$F_{xi} = -k_i \frac{\partial}{\partial v_{xi}}\left(\frac{1}{\gamma+1} v_i^{\gamma+1}\right), \quad F_{yi} = -k_i \frac{\partial}{\partial v_{yi}}\left(\frac{1}{\gamma+1} v_i^{\gamma+1}\right), \quad F_{zi} = -k_i \frac{\partial}{\partial v_{zi}}\left(\frac{1}{\gamma +}\right.$$

Therefore, we can write the dissipation function as:

$$\vec{F}_i = -\nabla_{\vec{v}_i} D_i \quad \text{with} \quad D_i = \frac{k_i}{\gamma+1} v_i^{\gamma+1}$$

$$D = \sum_{i=1} D_i$$

The associated generalized force is:

$$Q_j = \sum_{i=1}^{N} \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}$$

so

$$Q_j = -\sum_{i=1}^{N} \nabla_{\vec{v}_i} D_i \cdot \frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_j}$$

or

$$Q_j = -\sum_{i=1}^{N} \frac{\partial D_i}{\partial \vec{v}_i} \cdot \frac{\partial \vec{v}_i}{\partial \dot{q}_j}$$

since $\vec{v}_i = \dot{\vec{r}}_i$ and $\frac{\partial \vec{r}_i}{\partial q_j} = \delta_{ij}$, we obtain:

$$Q_j = -\frac{\partial}{\partial \dot{q}_j} \sum_{i=1}^{N} D_i$$

Generalized force associated with the dissipation function:

$$Q_j = -\frac{\partial D}{\partial \dot{q}_j} \quad \text{with} \quad D = \sum_{i=1}^{N} D_i$$

Note: In practice, D is calculated as a function of the generalized velocities. The Lagrange equations of the second kind in the presence of viscous dissipation forces are written as:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} + \frac{\partial D}{\partial \dot{q}_j} = Q_j^{nc}$$

Generalized Constraint Force

Some constraints, including holonomic ones, are sometimes too complex to allow for the emergence of new generalized coordinates in a reduced number. Other non-holonomic constraints, such as inequalities, do not offer a way to reduce the number of degrees of freedom. In these cases, the initial coordinates, which are too numerous, are retained, and the question arises about the impact of the constraints on the system's movement and their integration into the model. The answer lies in developing a general method of extending the Lagrangian formalism. It involves constructing a constrained Lagrangian and applying the principle of least action to derive modified Euler-Lagrange equations. The introduction of an additional unknown per constraint, called a Lagrange multiplier, allows the constraint to be adapted to the formalism, giving physical meaning to the constraint equation.

We use the set of $3N_C$ coordinates of the system, knowing that it has n_d independent coordinates and ℓ holonomic constraint equations, hence ℓ dependent coordinates. The first n_d coordinates are generalized and independent coordinates (no relations between them), so: $\frac{\partial f_i}{\partial q_j} = 0 \quad \forall i = 1, \ldots, n_d$

Only the coordinates linked by constraint equations of the form:

$$f_i(q_{n_d+1}, \ldots, q_{n_d+k}, t) \quad \forall i = n_d + 1, \ldots, n_d + k$$

Thus, we have:

$$\sum_{j=n_d+1}^{n_d+k} \frac{\partial f_i}{\partial q_j} \delta q_j = 0 \quad \forall i = 1, \ldots, k$$

Lagrange Formalism

By introducing k undetermined multipliers λ_i, also called Lagrange multipliers, into equation (2.13) and using the Lagrange equations of the first kind (2.7), we have:

$$\sum_{j=n_d+1}^{n_d+k}\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right)-\frac{\partial T}{\partial q_j}-Q_j\right]\delta q_j=\sum_{i=1}^{k}\lambda_i\left(\sum_{j=n_d+1}^{n_d+k}\frac{\partial f_i}{\partial q_j}\delta q_j\right)$$

Thus, we obtain:

$$\sum_{j=n_d+1}^{n_d+k}\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right)-\frac{\partial T}{\partial q_j}-Q_j-\sum_{i=1}^{k}\lambda_i\frac{\partial f_i}{\partial q_j}\right]\delta q_j=0$$

To make each of the k terms in equation (2.15) zero, the Lagrange multipliers λ_i_satisfy the equation of motion:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right)-\frac{\partial T}{\partial q_j}=Q_j+\sum_{i=1}^{k}\lambda_i\frac{\partial f_i}{\partial q_j}$$

This Lagrange equation of the first kind is only used for linked coordinates. Considering all the system's coordinates, the system's equations of motion are:

Lagrange equations of the first kind in the presence of generalized constraint forces

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right)-\frac{\partial T}{\partial q_j}=Q_j \quad \text{for } j=1,\ldots,n_d$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right)-\frac{\partial T}{\partial q_j}=Q_j+\sum_{i=1}^{k}\lambda_i\frac{\partial f_i}{\partial q_j} \quad \text{for } j=n_d+1,\ldots,n_d+k$$

The term $\lambda_i\frac{\partial f_i}{\partial q_j}$ homogeneous to a generalized force, it is a generalized constraint force. The use of equations (2.18) does not concern the number of degrees of freedom or the generalized coordinates to use. But, one must find the dependent coordinates and their constraint equations. The number of Lagrange multipliers λ_i equals the number of constraints. Each unknown is accompanied by a constraint equation.

For the N coordinates of the system with k non-holonomic constraints, we solve the system of equations given by (2.18) and the constraint equations f_i. Note that equations (2.18) include k more terms than the Lagrange equations of the second kind.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)-\frac{\partial L}{\partial q_j}=Q_j^{nc} \qquad \text{pour} \qquad j=1,\ldots,n_d$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)-\frac{\partial L}{\partial q_j}=Q_j^{nc}+\sum_{i=1}^{k}\lambda_i\frac{\partial f_i}{\partial q_j} \qquad \text{pour} \qquad j=n_d+1,\ldots,n_d+k$$

Lagrange Formalism

Example 1: Simple Pendulum

Consider a simple pendulum with mass mmm suspended by an inextensible string of length ℓ and negligible mass, fixed at point O. The pendulum is set in motion by displacing it from its equilibrium position.

The kinetic energy T of the system is given by:

$$T=\frac{1}{2}m\dot{r}^2+r^2\dot{\theta}^2$$

The potential energy U of the system, which is gravitational, is:

$$U=-mgr\cos\theta$$

The Lagrangian Lis:

$$L=T-U=\frac{1}{2}m(\dot{r}^2+r^2\dot{\theta})+mgr\cos\theta$$

The constraint equation is f(r,θ)=r− ℓ. The Lagrange multipliers are:

$$\lambda_r\frac{\partial f}{\partial r}=\lambda_r \quad \text{since} \quad \frac{\partial f}{\partial r}=1$$

$$\lambda_\theta\frac{\partial f}{\partial \theta}=0 \quad \text{since} \quad \frac{\partial f}{\partial \theta}=0$$

The Lagrange equations are:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right)-\frac{\partial L}{\partial r}=\lambda_r\frac{\partial f}{\partial r}$$

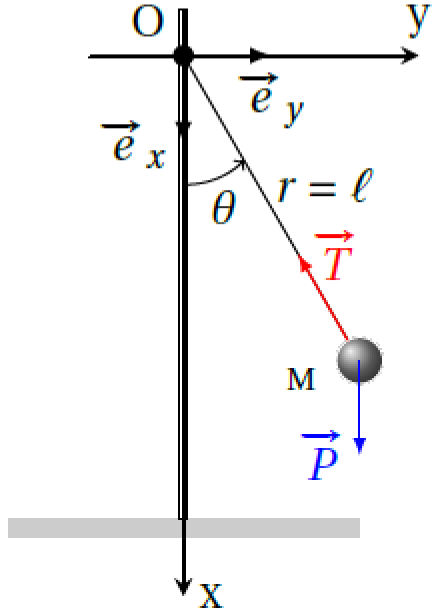
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right)-\frac{\partial L}{\partial \theta}=\lambda_\theta\frac{\partial f}{\partial \theta}$$

We obtain:

Lagrange Formalism

$$m\ddot{r} - mr\dot{\theta}^2 - mg\cos\theta = \lambda_r$$

$$m\frac{d}{dt}(r^2\dot{\theta}) + mgr\sin\theta = 0$$



Including the constraint, we get:

$$\lambda_r = -m\ell\dot{\theta}^2 - mg\cos\theta$$

$$\ddot{\theta} + \frac{g}{\ell}\sin\theta = 0$$

Here, λ_r represents the tension in the string, which balances the component of the weight along the radial direction $\vec{u}_r$ and the centripetal force.

Example 2: Particle Sliding on a Cylinder

Consider a material point with mass m sliding without friction on the external surface of a cylinder with radius a.

The kinetic energy T of the system is:

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta})$$

The potential energy U of the system, which is gravitational, is:

$$U = mgr\cos\theta$$

The Lagrangian L is:

$$L = T - U = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}) + mgr\cos\theta$$

Lagrange Formalism

The constraint equation is f(r,θ)=r−a. The Lagrange multipliers are:

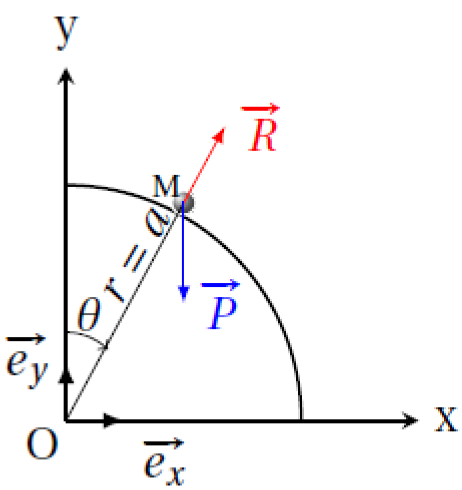
$$\lambda_r \frac{\partial f}{\partial r} = \lambda_r \quad \text{since} \quad \frac{\partial f}{\partial r} = 1$$

$$\lambda_\theta \frac{\partial f}{\partial \theta} = 0 \quad \text{since} \quad \frac{\partial f}{\partial \theta} = 0$$

The Lagrange equations are:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = \lambda_r \frac{\partial f}{\partial r}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = \lambda_\theta \frac{\partial f}{\partial \theta}$$



We obtain:

$$m\ddot{r} - mr\dot{\theta}^2 + mg\cos\theta = \lambda_r$$

$$\frac{d}{dt}(mr^2\dot{\theta}) - mgr\sin\theta = 0$$

Lagrange Formalism

$$\lambda_r = -ma\dot{\theta}^2 + mg\cos\theta$$

$$a\frac{d\dot{\theta}}{dt} - g\sin\theta = 0$$

λ_r is the radial component along $\vec{e}_r$ of the reaction force $\vec{R}=R\vec{e}_r$ of the lateral surface of the cylinder, balancing the component along $\vec{e}_r$ of the weight and the centripetal force.

If we consider the second differential equation:

$$a\frac{d\dot{\theta}}{dt} - g\sin\theta = 0$$

$$a\dot{\theta}\frac{d\dot{\theta}}{dt} - g\dot{\theta}\sin\theta = 0$$

$$\frac{d}{dt}\left(a^2\dot{\theta}^2 + g\cos\theta\right) = 0$$

$$a^2\dot{\theta}^2 + g\cos\theta = \text{constant}$$

Given that at t=0, θ=0 and $\dot{\theta}$=0, the constant is g.

$$a^2\dot{\theta}^2 = g(1-\cos\theta)$$

$$R = \lambda_r = -ma\dot{\theta}^2 + mg\cos\theta = -2mg(1-\cos\theta) + mg\cos\theta.$$

We have:

$$R = \lambda_r = mg(3\cos\theta - 2)$$

R>0 only if cosθ>2/3. Beyond θ=arccos(2/3), the particle will no longer be on the lateral surface of the cylinder.

Example 3: Cylinder Rolling Down an Incline

Consider a cylinder of mass M and radius R rolling without slipping on an inclined plane with an angle α\alphaα relative to the horizontal. Determine its acceleration.

The rolling without slipping condition constitutes the only constraint, so there is only one Lagrange multiplier. The constraint equation is f(x,θ)= x−Rθ=0, with:

$$\frac{\partial f}{\partial x}=1 \text{ and } \frac{\partial f}{\partial \theta}=-R$$

There is only one degree of freedom, but we use both coordinates xxx and θ\thetaθ. The kinetic energy TTT is:

$$T=\frac{1}{2}m\dot{x}^2+\frac{1}{2}J\dot{\theta}^2$$

The potential energy U, which is gravitational, is:

$$-\vec{p}\cdot d\vec{OM} = mg\cos\left(\frac{\pi}{2}-\alpha\right)dx = mg\sin\alpha dx$$

$$U_g = -mg\sin\alpha\int_0^x dx = -mg\sin\alpha x$$

The Lagrangian L of the system is:

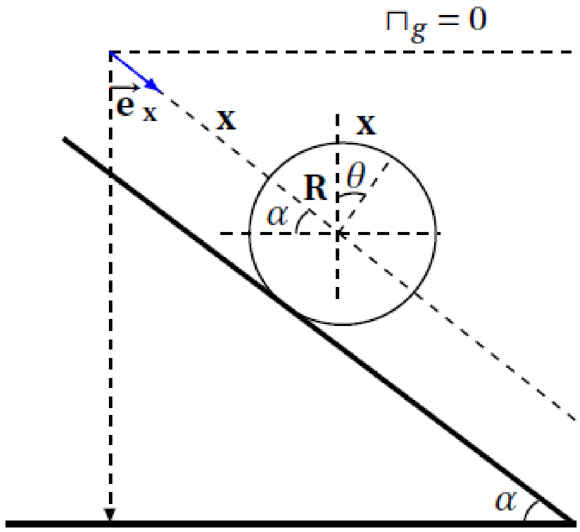
$$L = T - U = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}J\dot{\theta}^2 + mgx\sin\alpha$$

The Lagrange equations are:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = \lambda\frac{\partial f}{\partial x}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = \lambda\frac{\partial f}{\partial \theta}$$

Lagrange Formalism



We obtain:

$$m\ddot{x} - mg\sin\theta = \lambda$$

$$\downarrow$$

$$J\ddot{\theta} = -R\lambda$$

The Lagrange multiplier λ has the dimension of a force, and it can be expressed as:

$$\lambda = m\ddot{x} - mg\sin\theta = \frac{\mathcal{J}}{R}\ddot{\theta}$$

Soit on utilisant x ou θ

$$\left(\mathcal{J} + mR^2\right)\ddot{\theta} - mg\sin\alpha = 0$$