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Level : 1^{st} year LMD
Year : 2024– 2025
Algebra 1

Series 4 : Algebraic structures

Exercise 1 :

1. We equip $\mathbb{R}$ with the internal composition law $*$ defined by :

$$\forall x, y \in \mathbb{R}, \quad x * y = xy + \left(x^2 - 1\right)\left(y^2 - 1\right)$$

Prove that $*$ is commutative, not associative, and that 1 is the neutral element.

2 We equip $\mathbb{R}^+$ with the internal composition law $*$ defined by :

$$\forall x, y \in \mathbb{R}^+, \quad x * y = \sqrt{x^2 + y^2}$$

Prove that $*$ is commutative, associative, and that 0 is the neutral element.
Prove that for each element x of $\mathbb{R}^+$, x hasn't an inverse by $*$.

3 We equip $\mathbb{R}$ with the internal composition law $*$ defined by :

$$\forall x, y \in \mathbb{R}, \quad x * y = \sqrt[3]{x^3 + y^3}$$

Prove that the map $x \mapsto x^3$ is an isomorphism of $(\mathbb{R}, *)$ to $(\mathbb{R}, +)$. deduce that $(\mathbb{R}, *)$ is a group commutative.

Exercise 2 : Let $G = \mathbb{R}^* \times \mathbb{R}$ and $*$ the law on G defined by

$$(x, y) * (x', y') = (xx', xy' + y)$$

1. Prove that $(G, *)$ is a group not commutative.
2. Prove that $(]0, +\infty[\times \mathbb{R}, *)$ is a sub-group of $(G, *)$.

Exercise 3 :

We equip $A = \mathbb{R} \times \mathbb{R}$ with two laws defined by :

$$(x, y) + (x', y') = (x + x', y + y') \text{ and } (x, y) * (x', y') = (xx', xy' + x'y)$$

1. Prove that $(A, +)$ is a group commutative.
2.

a) Prove that $*$ is commutative.
b) Prove that $*$ is associative
c) Determine the neutral element of A for the law $*$.
d) Prove that $(A, +, *)$ is a ring commutative.