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Series N° :03

Exercice 01 : Give a formal proof that the following argument is valid. Provide reasons.
1. $a \vee b$,
2. $\neg c \Longrightarrow \neg b$,
3. $\neg a$,
4. $\therefore c$.

Exercice 02 : Let p and q two propositions, we set

$$P : ((p \Longrightarrow q) \wedge \neg q) \Longrightarrow \neg p$$

1. Show that P is a tautology.
2. Modus tollens is a proof rule based on the tautology P. Apply this rule to show that for any odd natural number n the integer $3n + 7$ is even.

Exercice 03 : Using the principle of direct proof, show that for any strictly positive rational number, there exists an integer strictly greater than it.
1. Write the detailed (informal) demonstration using connectors, quantifiers, etc.
2. Give the full proof.

Exercice 04 : We recall that $\sqrt{2}$ is an irrational number.
1. Prove using proof by contradiction : If a and b are two integers such that $a + b\sqrt{2} = 0$, then $a = b = 0$.
a. Write the detailed (informal) demonstration using connectors, quantifiers, etc.
b. Give the full proof.
2. Deduce that if m, n, p and q are integers, then

$$m + n\sqrt{2} = p + q\sqrt{2} \Longleftrightarrow (m = p \text{ and } n = q).$$

Exercice 05 : Same questions with the proposition : $\sqrt{3}$ is an irrational number.

Exercice 06 : Let $n \in \mathbb{N}^*$. Prove that if n is the square of an integer, then $2n$ is not the square of an integer

Exercice 07 : If the integer $(n^2 - 1)$ is not divisible by 8, then the integer n is even.
1. Write the contrapositive of the previous proposition.
2. Noting that an odd integer is written in the form $n = 4k + r$ with $k \in \mathbb{N}$ and

$r \in \{1,3\}$ (to be justified), prove the contrapositive.
3. Have we demonstrated the property of the statement ?
<u>**Exercice 08 :**</u> Let $a \in \mathbb{R}$. Prove by contrapositive that

$$\forall \varepsilon > 0, \quad |a| \leq \varepsilon \Longrightarrow a = 0.$$

<u>**Exercice 09 :**</u> Using the proof by separation of cases, show
1. For all $(a,b) \in \mathbb{N}^2$, $ab(a^2 - b^2)$ est devisible par 3.
2. For all $(x,y) \in \mathbb{R}^2$, $\max(x,y) = \frac{1}{2}\left(x + y + |x - y|\right)$.
<u>**Exercice 10 : :**</u> Prove by using constructive and non-constructive proof the following proposition
"There exist two irrational numbers x and y such that x^y is rational".

Exercice 1 :1. We show that P is a tautology

$$\begin{gathered}((p \Longrightarrow q) \wedge \neg q) \Longrightarrow \neg p \\ \equiv ((\neg p \vee q) \wedge \neg q) \Longrightarrow \neg p \\ \equiv \neg ((\neg p \vee q) \wedge \neg q) \vee \neg p \\ \equiv ((p \wedge \neg q) \vee q) \vee \neg p \\ \equiv ((p \vee q) \wedge (q \vee \neg q)) \vee \neg p \\ \equiv ((p \vee q) \wedge V) \vee \neg p \\ p \vee \neg p \vee q \\ \equiv T \vee q \\ \equiv T.\end{gathered}$$

Then P is tautology.
2.a. Writing a proof following this rule takes the following form :
Proposition : If p then q
Preuve : Suppose $\neg q$....consequently $\neg p$

$$\neg q \text{ and } p \Longrightarrow q \text{ consequently } \neg p$$

b. Apply this rule to show that for any odd natural integer n the integer 3n+7 is even. We set :
q : n even natural number,
p : $3n+7$ is odd
Now, we show $p \Longrightarrow q$, that is if $3n+7$ est odd then n is even.
If $3n+7$ is odd $\Longrightarrow 3n$ is even $\Longrightarrow n$ est is even.
Then the proposition $p \Longrightarrow q$ is true, which implies that for any odd natural integer n the integer 3n+7 is even.
<u>Exercice 02 :</u> 1. Let $x \in \mathbb{Q}$ et suppose $x > 0$. There exist two intgers p and q with $q > 0$ such that $x = p/q$ (property of $\mathbb{Q}$).
Since q is an integer strictly positive, $q > 1$ (property of $\mathbb{N}$). Then $p = xq > x$ (rule).
In particular, $p > 0$ (rule). Hence $2p > p$ (rule). From this comes $2p > x$ (rule). Since $2p > 0$ (rule), we note that $2p \in \mathbb{N}$ (definition of $\mathbb{Z}$). We conclude that the double of the numerator $n = 2p$ is suitable.
2. We set **A :** $x \in \mathbb{Q}_*^+$, **B :** $\exists n \in \mathbb{N}$ such that $n > x$

$$\begin{aligned} A &\Longrightarrow \exists\,(p,q) \in \mathbb{N} \times \mathbb{N}^* : (p \text{ and } q \text{ coprime}) \wedge (x = p/q) \\ &\Longrightarrow \exists\,(p,q) \in \mathbb{N} \times \mathbb{N}^* : (p \text{ and } q \text{ coprime}) \wedge (p = xq > x) \\ &\Longrightarrow \exists p \in \mathbb{N} : (2p > x) \\ &\Longrightarrow \exists p \in \mathbb{N} : (2p > 0) \\ &\Longrightarrow \exists n \in \mathbb{N} : n\,(= 2p) > x \Longrightarrow B \end{aligned}$$

Exercice 03 : Let us reason by contradiction. Suppose that $a+b\sqrt{2}=0$ without $a=b=0$. Then, necessarily $b\neq 0$ because if $b=0$, then we should also have $a=0$, which is contrary to the hypothesis $(a,b)\neq(0,0)$. But, we have $\sqrt{2}=\frac{-a}{b}\in\mathbb{Q}$, which is false. The initial hypothesis is therefore false, and we have $a=b=0$.
2. We set

$$P:\forall\,(a,b)\in\mathbb{Z}:\left[\left(a+b\sqrt{2}=0\right)\wedge\left(\sqrt{2}\notin\mathbb{Q}\right)\Longrightarrow a=b=0\right].$$

Then

$$\bar{P}:\left(a+b\sqrt{2}=0\right)\wedge\left(\sqrt{2}\notin\mathbb{Q}\right)\wedge\left((a,b)\neq(0,0)\right).$$

We have

$$\begin{aligned}\bar{P}&\Longrightarrow(b\neq 0)\wedge\left(a+b\sqrt{2}=0\right)\wedge\left(\sqrt{2}\notin\mathbb{Q}\right)\\&\Longrightarrow(b\neq 0)\wedge\left(\sqrt{2}=\frac{-a}{b}\right)\wedge\left(\sqrt{2}\notin\mathbb{Q}\right)\\&\Longrightarrow\underbrace{\left(\sqrt{2}\in\mathbb{Q}\right)}_{Q}\wedge\underbrace{\left(\sqrt{2}\in\mathbb{Q}\right)}_{\bar{Q}}\Longrightarrow Q\wedge\bar{Q}.\end{aligned}$$

This constitutes a contradiction, therefore P is true.
3. On the other hand, if $m=p$ and $n=q$, then $m+n\sqrt{2}=p+q\sqrt{2}$. Conversely, let us suppose $m+n\sqrt{2}=p+q\sqrt{2}$. Then we have also

$$(m-p)+(n-q)\sqrt{2}=0.$$

The result of the previous question implies that $m-p=0$ and $n-q=0$, that is $m=p$ and $n=q$.
Exercice 03 : 1. Par l'absurde, si $\sqrt{3}\in\mathbb{Q}$, il existe $(p,q)\in\mathbb{N}^2$ tel que $\sqrt{3}=p/q$ avec (p,q) premiers ente eux. En élévant au carré, on obtient : (*) $p^2=3q^2$ et donc 3 divise p^2.

Comme 3 est premier, 3 divise p d'où l'existence de $p'\in\mathbb{N}$ tel que $p=3p'$. En reportant dans l'égalité (*), on arrive à $3p'^2=q^2$ donc 3 divise q, ce qui contredit (p,q) premiers ente eux. La contradiction assure que $\sqrt{3}$ est irrationnel.

2. On va revoir la preuve de l'exemple précédent plus en détail. On pose
P : $\sqrt{3}$est un nombre irrationel.

$$\begin{aligned}\bar{P}&\Longrightarrow\exists(p,q)\in\mathbb{N}\times\mathbb{N}^*:((p,q)\text{premiers entre eux})\wedge\left(\sqrt{3}=p/q\right)\\&\Longrightarrow(b\neq 0)\wedge\left(\sqrt{2}=\frac{-a}{b}\right)\wedge\left(\sqrt{2}\notin\mathbb{Q}\right)\\&\Longrightarrow\underbrace{\left(\sqrt{2}\in\mathbb{Q}\right)}_{Q}\wedge\underbrace{\left(\sqrt{2}\in\mathbb{Q}\right)}_{\bar{Q}}\Longrightarrow Q\wedge\bar{Q}\end{aligned}$$