1.10 Propositional Formulas

Definition 1.10.1 *A propositional formula is said to be atomic if it cannot be written in terms of other propositional formulas using the fundamental connectives.*

Example 1.10.2 *The propositional formula p: "4 is a perfect square" is an atomic formula. The formula "4 is a perfect square then 4 is an even number" is not an atomic formula.*

Definition 1.10.3 *A propositional formula is defined by:*

1. *Every atomic formula is a propositional formula.*
2. *If P is a formula then P is also a formula.*
3. *If P and Q are formulas then $(P \wedge Q), (P \vee Q)$ and $(P \implies Q)$ are also formulas.*
4. *Nothing else is a formula.*

1.11 Predicates and Quantifiers

Definition 1.11.1 *A predicate or propositional function is a description of the property (or properties) a variable or subject may have. A proposition may be created from a propositional function by either assigning a value to the variable or by quantification.*

Definition 1.11.2 *The independent variable of a propositional function must have a* ***universe of discourse****, which is a set from which the variable can take values.*

Discussion

Recall from the introduction to logic that the sentence " $x+2=2x$ " is not a proposition, but if we assign a value for x then it becomes a proposition. The phrase " $x+2=2x$ " can be treated as a function for which the input is a value of x and the output is a proposition.

Another way we could turn this sentence into a proposition is to quantify its variable. For example, "for every real number x, $x+2=2x$ " is a proposition (which is, in fact, false, since it fails to be true for the number $x=0$).

This is the idea behind propositional functions or predicates. As stated above a predicate is a property or attribute assigned to elements of a particular set, called the universe of discourse. For example, the predicate " $x+2=2x$ ", where the universe for the variable x is the set of all real numbers, is a property that some, but not all, real numbers possess.

In general, the set of all x in the universe of discourse having the attribute $P(x)$ is called the truth set of P. That is, the truth set of P is

$$\{x \in U \mid P(x)\}$$

Examples 1.11.3 *The propositional function $P(x)$ is given by " $x>0$ " and the universe of discourse for x is the set of integers. To create a proposition from P, we may assign a value for x. For example,*

1. Setting $x=-3$, we get $P(-3)$: " $-3>0$ ", which is false.
2. Setting $x=2$, we get $P(2)$: " $2>0$ ", which is true.

Discussion: In this example we created propositions by choosing particular values for x. Here are two more examples:

Example 1.11.4 *Suppose $P(x)$ is the sentence " x has fur" and the universe of discourse for x is the set of all animals. In this example $P(x)$ is a true statement if x is a cat. It is false, though, if x is an alligator.*

Example 1.11.5 *Suppose $Q(y)$ is the predicate " y holds a world record," and the universe of discourse for y is the set of all competitive swimmers. Notice that the universe of discourse must be defined for predicates. This would be a different predicate if the universe of discourse is changed to the set of all competitive runners.*

1.11.1 Quantifiers

A quantifier turns a propositional function into a proposition without assigning specific values for the variable. There are primarily two quantifiers, the **universal quantifier** and the **existential quantifier**.

Definition 1.11.6 *The universal quantification of $P(x)$ is the proposition " $P(x)$ is true for all values x in the universe of discourse."*

Notation 1.11.7 *"For all $xP(x)$ " or "For every $xP(x)$ " is written*

$$\forall x P(x)$$

Remark 1.11.8 *There are another ways to express the existential quantifier: "For all element $xP(x)$", "For each $xP(x)$".*

Definition 1.11.9 *The existential quantification of $P(x)$ is the proposition*
"There exists an element x in the universe of discourse such that $P(x)$ is true."

Notation 1.11.10 *"There exists x such that $P(x)$ " or "There is at least one x such that $P(x)$ " is written*

$$\exists x P(x).$$

Remark 1.11.11 *There are another ways to express the existential quantifier: "There is ... ", "For some ... ",""For at least one ... ".*

Discussion

As an alternative to assigning particular values to the variable in a propositional function, we can turn it into a proposition by quantifying its variable. Here we see the two primary ways in which this can be done, the universal quantifier and the existential quantifier.
In each instance we have created a proposition from a propositional function by binding its variable.

Example 1.11.12 *Suppose $P(x)$ is the predicate "$x + 2 = 2x$", and the universe of discourse for x is the set $\{1, 2, 3\}$. Then*

1. $\forall x P(x)$ is the proposition "For every x in $\{1, 2, 3\}$ such that $x+2 = 2x$." This proposition is false.

2. $\exists x P(x)$ is the proposition "There exists x in $\{1, 2, 3\}$ such that $x + 2 = 2x$." This proposition is true.

Exercise 1.11.13 *Let $P(n, m)$ be the predicate "$mn > 0$", where the domain for m and n is the set of integers. Which of the following statements are true?*
(1) $P(-3, 2)$
(2) $\forall m P(0, m)$
(3) $\exists n P(n, -3)$

1.11.2 Converting from English.

Example 1.11.14 *Assume*
$F(x)$: *x is a fox.*
$S(x)$: *x is sly.*
$T(x)$: *x is trustworthy.*
and the universe of discourse for all three functions is the set of all animals.

1. Everything is a fox: $\forall x F(x)$.

2. All foxes are sly: $\forall x[F(x) \to S(x)]$.

3. If any fox is sly, then it is not trustworthy

$$\forall x[(F(x) \land S(x) \to \neg T(x)] \equiv \neg \exists x[F(x) \land S(x) \land T(x)].$$

Notice that in this example the last proposition may be written symbolically in the two ways given. Think about the how you could show they are the same using the logical equivalences in Module 2.2.

Additional Definitions

1. An assertion involving predicates is **valid** if it is true for every element in the universe of discourse.

2. An assertion involving predicates is **satisfiable** if there is a universe and an interpretation for which the assertion is true. Otherwise it is unsatisfiable.

3. The **scope** of a quantifier is the part of an assertion in which the variable is bound by the quantifier.

Discussion

You would not be asked to state the definitions of the terminology given, but you would be expected to know what is meant if you are asked a question like "Which of the following assertions are satisfiable?"

Example 1.11.15 *If the universe of discourse is* $U = \{1, 2, 3\}$, *then*
(1) $\forall x P(x) \Leftrightarrow P(1) \land P(2) \land P(3)$
(2) $\exists x P(x) \Leftrightarrow P(1) \lor P(2) \lor P(3)$

Suppose the universe of discourse U is the set of real numbers.
(1) If $P(x)$ *is the predicate* $x^2 > 0$, *then* $\forall x P(x)$ *is false, since* $P(0)$ *is false.*
(2) If $P(x)$ *is the predicate* $x^2 - 3x - 4 = 0$, *then* $\exists x P(x)$ *is true, since* $P(-1)$ *is true.*
(3) If $P(x)$ *is the predicate* $x^2 + x + 1 = 0$, *then* $\exists x P(x)$ *is false, since there are no real solutions to the equation* $x^2 + x + 1 = 0$.
(4) If $P(x)$ *is the predicate "If* $x \neq 0$, *then* $x^2 \geq 1$ *', then* $\forall x P(x)$ *is false, since* $P(0.5)$ *is false.*

Example 1.11.16 *Consider* $\forall x(\exists y(P(x, y)) \Longrightarrow Q(x))$.

1. The scope of the universal quantifier is $\exists y(P(x, y)) \Longrightarrow Q(x)$.

2. The scope of the existential quantifier is $(P(x, y)) \Longrightarrow Q(x))$.

Explanation: *1. The scope starts with the first (. To find the end, find the matching).*

2. The scope starts with the first (after ∃. To find the end, find the matching). So this statement says that, if there is some y such that P(x, y) is true, then Q is true of x. Often the scopes of all of the quantifiers end together at the end of the expression. But that isn't necessary, as we see here. Where the scope ends affects the meaning of the expression.

1.11.3 Multiple Quantifiers

Multiple quantifiers are read from left to right.

Suppose $P(x, y)$ is " $xy = 1$ ", the universe of discourse for x is the set of positive integers, and the universe of discourse for y is the set of real numbers.

Example 1.11.17 *1. $\forall x \forall y P(x, y)$ may be read "For every positive integer x and for every real number $y, xy = 1$. This proposition is false.*

2. $\forall x \exists y P(x, y)$ may be read "For every positive integer x there is a real number y such that $xy = 1$. This proposition is true.

3. $\exists y \forall x P(x, y)$ may be read "There exists a real number y such that, for every positive integer $x, xy = 1$. This proposition is false.

Discussion: Study the syntax used in these examples. It takes a little practice to make it come out right.

Ordering Quantifiers

The order of quantifiers is important, they may not commute. We state here some examples showing this importance.

Examples 1.11.18 *(1) $\forall x \forall y P(x, y) \equiv \forall y \forall x P(x, y)$, and*
(2) $\exists x \exists y P(x, y) \equiv \exists y \exists x P(x, y)$,
but
(3) $\forall x \exists y P(x, y)$ and $\exists y \forall x P(x, y)$ are not logically equivalent.

Discussion: The lesson here is that you have to pay careful attention to the order of the quantifiers. The only cases in which commutativity holds are the cases in which both quantifiers are the same. In the one case in which equivalence does not hold

$$\forall x \exists y P(x, y) \not\equiv \exists y \forall x P(x, y)$$

There is an implication in one direction. Notice that if $\exists y \forall x P(x, y)$ is true, then there is an element a in the universe of discourse for y such that $P(x, a)$ is true for all x in the universe of discourse for x. Thus, for all x there exists a y, namely a, such that $P(x, y)$. That is, $\forall x \exists y P(x, y)$. Thus,

$$\exists y \forall x P(x, y) \Rightarrow \forall x \exists y P(x, y)$$

Notice predicates use function notation and recall that the variable in function notation is really a place holder. The statement $\forall x \exists y P(x, y)$ means the same as $\forall s \exists t P(s, t)$. Now if this seems clear, go a step further and notice this will also mean the same as $\forall y \exists x P(y, x)$. When the domain of discourse for a variable is defined it is in fact defining the domain for the place that variable is holding at that time.

Example 1.11.19 *1. Here are some additional examples:*

$P(x, y)$ is " x is a citizen of y." $Q(x, y)$ is " x lives in y." The universe of discourse of x is the set of all people and the universe of discourse for y is the set of the states of Algeria.

2. All people who live in Ain Defla are citizens of Ain Defla

$$\forall x (Q(x, Ain\ Defla) \rightarrow P(x, Ain\ Defla))$$

3. Every state has a citizen that does not live in that state.

$$\forall y \exists x (P(x, y) \wedge \neg Q(x, y))$$

Example 1.11.20 *Suppose $R(x, y)$ is the predicate " x understands y," the universe of discourse for x is the set of students in your discrete class, and the universe of discourse for y is the set of examples in this course. Pay attention to the differences in the following propositions.*
1. $\exists x \forall y R(x, y)$ is the proposition "There exists a student in this class who understands every example in this course."
2. $\forall y \exists x R(x, y)$ is the proposition "For every example in this course there is a student in the class who understands that example."
3. $\forall x \exists y R(x, y)$ is the proposition "Every student in this class understands at least one example in this course."
4. $\exists y \forall x R(x, y)$ is the proposition "There is an example in these notes that every student in this class understands."

Exercise 1.11.21 *Let $P(x, y)$ be the predicate $2x + y = xy$, where the domain of discourse for x is $\{u \in \mathbb{Z} \mid u \neq 1\}$ and for y is $\{u \in \mathbb{Z} \mid u \neq 2\}$. Determine the truth value of each statement. Show work or briefly explain.*
1. $P(-1, 1)$
2. $\exists x P(x, 0)$
3. $\exists y P(4, y)$
4. $\forall y P(2, y)$
5. $\forall x \exists y P(x, y)$
6. $\exists y \forall x P(x, y)$
7. $\forall x \forall y [((P(x, y)) \wedge (x > 0)) \to (y > 1)]$

3.10. Unique Existential.

Definition 1.11.22 *The unique existential quantification of $P(x)$ is the proposition "There exists a unique element x in the universe of discourse such that $P(x)$ is true."*

Notation 1.11.23 *"There exists unique x such that $P(x)$ " or "There is exactly one $xP(x)$ " is written*

$$\exists! x P(x)$$

Discussion

Continuing with previous Example, the proposition $\forall x \exists! y R(x, y)$ is the proposition "Every student in this class understands exactly one example in this course (but not necessarily the same example for all students)."

Exercise 1.11.24 *Let $P(n, m)$ be the predicate $mn \geq 0$, where the domain for m and n is the set of integers. Which of the following statements are true?*
1. $\exists! n \forall m P(n, m)$
2. $\forall n \exists! m P(n, m)$
3. $\exists! m P(2, m)$

Remark 1.11.25 *A predicate is not a proposition until all variables have been bound either by quantification or by assignment of a value!*

De Morgan's Laws for Quantifiers.

The fundamental rules of negation of formulas are given by De Morgan's Laws of
1. $\neg \forall x P(x) \Leftrightarrow \exists x \neg P(x)$

- 2. $\neg \exists x P(x) \Leftrightarrow \forall x \neg P(x)$

Discussion

Example 1.11.26 *The negation of the proposition "Every fish in the sea has gills," is the proposition "there is at least one fish in the sea that does not have gills."*

Remark 1.11.27 *If there is more than one quantifier, then the negation operator should be passed from left to right across one quantifier at a time, using the appropriate De Morgan's Law at each step.*

Example 1.11.28 *Continuing further with previous example, suppose we wish to negate the proposition "Every student in this class understands at least one example in these notes." Apply De Morgan's Laws to negate the symbolic form of the proposition*

$$\begin{aligned} \neg(\forall x \exists y R(x, y)) &\Leftrightarrow \exists x(\neg \exists y R(x, y)) \\ &\Leftrightarrow \exists x \forall y \neg R(x, y) \end{aligned}$$

The first proposition could be read "It is not the case that every student in this class understands at least one example in these notes." The goal, however, is to find an expression for the negation in which the verb in each predicate in the scope of the quantifiers is negated, and this is the intent in any exercise, quiz, or test problem that asks you to "negate the proposition ... ." Thus, a correct response to the instruction to negate the proposition "Every student in this class understands at least one example in these notes" is the proposition "There is at least one student in this class that does not understand any of the examples in these notes."

Exercise 1.11.29 *Negate the rest of the statements in Example 3.9.2.*

It is easy to see why each of these rules of negation is just another form of De Morgan's Law, if you assume that the universe of discourse is finite: $U=\{x_1, x_2, \ldots, x_n\}$. For example,

$$\forall x P(x) \equiv P\left(x_1\right) \wedge P\left(x_2\right) \wedge \cdots \wedge P\left(x_n\right)$$

so that

$$\begin{aligned} \neg \forall x P(x) &\Leftrightarrow \neg\left[P\left(x_1\right) \wedge P\left(x_2\right) \wedge \cdots \wedge P\left(x_n\right)\right] \\ &\Leftrightarrow\left[\neg P\left(x_1\right) \vee \neg P\left(x_1\right) \vee \cdots \vee \neg P\left(x_1\right)\right] \\ &\Leftrightarrow \exists x \neg P(x) \end{aligned}$$

If U is an arbitrary universe of discouse, we must argue a little differently: Suppose $\neg \forall x P(x)$ is true. Then $\forall x P(x)$ is false. This is true if and only if there is some c in U such that $P(c)$ is false. This is true if and only if there is some c in U such that $\neg P(c)$ is true. But this is true if and only if $\exists x \neg P(x)$. The argument for the other equivalence is similar.

Exercise 1.11.30 *In the questions below suppose the variable x represents students and y represents courses, and:*

$U(y)$: *y is an upper-level course*

$M(y)$: *y is a math course*

$F(x)$: *x is a freshman*

$B(x)$: *x is a full-time student*

$T(x, y)$: *student x is taking course y.*

Write the statement using these predicates and any needed quantifiers.

1. Ahmed is taking Logic course: T(Ahmed, Logic course).

2. All students are freshmen: $\forall x F(x)$.

3. Every freshman is a full-time student: $\forall x(F(x) \Longrightarrow B(x))$.

4. No math course is upper-level: $\forall y(M(y) \Longrightarrow \neg U(y))$.

Exercise 1.11.31 *In the questions below suppose the variable x represents students and y represents courses, and:*

$U(y)$: *y is an upper-level course*

$M(y)$: *y is a math course*

$F(x)$: *x is a freshman*

$A(x)$: *x is a part-time student*

$T(x, y)$: *student x is taking course y.*

Write the statement using these predicates and any needed quantifiers.

1. Every student is taking at least one course: $\forall x \exists y$ T(x,y).

2. There is a part-time student who is not taking any math course: $\exists x \forall y[A(x) \wedge (M(y) \Longrightarrow \neg T(x, y))]$.

3. Every part-time freshman is taking some upper-level course: $\forall x \exists y[(F(x) \wedge A(x)) \Longrightarrow (U(y) \wedge T(x, y))]$.

Example 1.11.32 *Here is a formidable example from the calculus. Suppose a and L are fixed real numbers, and f is a real-valued function of the real variable x. Recall the rigorous definition of what it means to say "the limit of $f(x)$ as x tends to a is L ":*

$$\begin{aligned} \lim_{x \to a} f(x) &= L \Leftrightarrow \\ \text{for every } \epsilon &> 0 \text{ there exists } \delta > 0 \text{ such that, for every } x, \end{aligned}$$

$$\text{if } 0 < |x - a| < \delta, \text{ then } |f(x) - L| < \epsilon$$

Here, the universe of discourse for the variables ϵ, δ, and x is understood to be the set of all real numbers.

What does it mean to say that $\lim_{x \to a} f(x) \neq L$? In order to figure this out, it is useful to convert this proposition into a symbolic proposition. So, let $P(\epsilon, \delta, x)$ be the predicate " $0 < |x - a| < \delta$ " and let $Q(\epsilon, \delta, x)$ be the predicate " $|f(x) - L| < \epsilon$." (It is perfectly OK to list a variable in the argument of a predicate even though it doesn't actually appear!) We can simplify the proposition somewhat by restricting the universe of discourse for the variables ϵ and δ to be the set of positive real numbers. The definition then becomes

$$\forall \epsilon \exists \delta \forall x[P(\epsilon, \delta, x) \to Q(\epsilon, \delta, x)]$$

Use De Morgan's Law to negate:

$$\neg[\forall\epsilon\exists\delta\forall x[P(\epsilon,\delta,x)\to Q(\epsilon,\delta,x)]]\Leftrightarrow\exists\epsilon\forall\delta\exists x[P(\epsilon,\delta,x)\wedge\neg Q(\epsilon,\delta,x)],$$

and convert back into words:
There exists $\epsilon > 0$ such that, for every $\delta > 0$ there exists x such that,

$$0 < |x-a| < \delta \text{ and } |f(x)-L| \geq \epsilon$$

Distributing Quantifiers over Operators

Proposition 1.11.33 *Let P and Q two any predicates of one single variable, then we have*

1. $\forall x[P(x)\wedge Q(x)]\equiv\forall xP(x)\wedge\forall xQ(x)$, but
2. $\forall x[P(x)\vee Q(x)]\not\equiv\forall xP(x)\vee\forall xQ(x)$.
3. $\exists x[P(x)\vee Q(x)]\equiv\exists xP(x)\vee\exists xQ(x)$, but
4. $\exists x[P(x)\wedge Q(x)]\Longrightarrow\exists xP(x)\wedge\exists xQ(x)$.

Proof. 1. **Part1:** Show if $\forall x(P(x)\wedge Q(x))$ is true then $\forall xP(x)\wedge\forall xQ(x)$ is true. Assume $\forall x(P(x)\wedge Q(x))$ is true. If a is in the universe of P and Q, then $P(a)\wedge Q(a)$ is true. So, $P(a)$ is true and $Q(a)$ is true. Since $P(a)$ and $Q(a)$ are both true for every element in the universe of P and Q, $\forall xP(x)$ and $\forall xQ(x)$ are both true. So, $\forall xP(x)\wedge\forall xQ(x)$ is true.

Part2: Show if $\forall xP(x)\wedge\forall xQ(x)$ is true then $\forall x(P(x)\wedge Q(x))$ is true. Assume $\forall xP(x)\wedge\forall xQ(x)$ is true. So, $\forall xP(x)$ is true and $\forall xQ(x)$ is true. If a is in the universe of P and Q, then $P(a)$ is true and $Q(a)$ is true. If $P(a)$ is true and $Q(a)$ is true, then $P(a)\wedge Q(a)$ is true. Since $P(a)\wedge Q(a)$ is true for every element in the universe, $\forall x(P(x)\wedge Q(x))$ is true. So,

$$\forall x(P(x)\wedge Q(x))\equiv\forall xP(x)\wedge\forall xQ(x).$$

2. Give an example that $\forall x(P(x)\vee Q(x))$ and $\forall xP(x)\vee\forall xQ(x)$ have different truth values.
■

Proof. $P(x)$: x is odd. $Q(x)$: x is even. (in the domain of integers.).

For all element $(P(x)\vee Q(x))$ is true. (All x is odd or even.). So, $\forall x(P(x)\vee Q(x))$ is true. For all element $P(x)$ is false. (All x is not odd.). For all element $Q(x)$ is false. (All x is not even.). So, $\forall xP(x)\vee\forall xQ(x)$ is false. Thus, $\forall x(P(x)\vee Q(x))$ and $\forall xP(x)\vee\forall xQ(x)$ are not logically equivalent. ■

Discussion

Here we see that in only half of the four basic cases does a quantifier distribute over an operator, in the sense that doing so produces an equivalent proposition.

Exercise 1.11.34 *In each of the two cases in which the statements are not equivalent, there is an implication in one direction. Which direction? In order to help you analyze these two cases, consider the predicates $P(x)=[x\geq 0]$ and $Q(x)=[x<0]$, where the universe of discourse is the set of all real numbers.*

Exercise 1.11.35 *Write using predicates and quantifiers.*

1. *For every $m,n\in\mathbb{N}$ there exists $p\in\mathbb{N}$ such that $m<p$ and $p<n$.*
2. *For all nonnegative real numbers $a,b,$ and c, if $a^2+b^2=c^2$, then $a+b\geq c$.*
3. *There does not exist a positive real number a such that $a+\frac{1}{a}<2$.*
4. *Every student in this class likes mathematics.*
5. *No student in this class likes mathematics.*

Exercise 1.11.36 *Give the negation of each statement in previous exercice using predicates and quantifiers with the negation to the right of all quantifiers.*

Exercise 1.11.37 *Give the negation of each statement in previous exercice using an English sentence.*