Chapter 1

Logic and propositional calculus

1.1 Introduction

Propositional logic is a branch of mathematics that studies the logical relationships between propositions (or statements, sentences, assertions) taken as a whole, and connected via logical connectives. In this chapter, we will cover propositional logic and related topics in detail. Logic is the basis of all mathematical reasoning and all automated reasoning. The rules of logic specify the meaning of mathematical statements. These rules help us understand and reason with statements.

The rules of logic give precise meaning to mathematical statements. These rules are used to distinguish between valid and invalid mathematical arguments. Apart from its importance in understanding mathematical reasoning, logic has numerous applications in Computer Science, varying from the design of digital circuits to the construction of computer programs and verification of the correctness of programs.

1.2 Propositional calculus

1.2.1 Proposition

Definition 1.2.1 *A proposition (or statement) is a declarative statement that is either true or false (but not both).*

Notations 1.2.2

1. *If a statement is true, we assign it the truth value* ***T****. If a statement is false, we assign it the truth value* ***F****.*
2. *Propositions are generally denoted by the letters p, q, r... or p_1, p_2, ..*

Examples 1.2.3

1. *"4 is not a perfect square" is a false proposition.*
2. *"The 2nd year mathematics degree class at the University of Khemis Miliana has 60 students" is not a proposition.*
3. *""The 2nd year mathematics degree class at the University of Khemis Miliana for the academic year 2015-2016 includes 30 students" is a proposition.*
4. *"Riyad Mahrez is a good player" is not a proposition.*

1.3 Compound Propositions

Many propositions are composite, that is, composed of subpropositions and various connectives discussed subsequently. Such composite propositions are called compound propositions.

Definition 1.3.1 *A proposition is called to be primitive (or atomic) if it cannot be discomposed into simpler propositions.*

Examples 1.3.2

1. *The above proposition (3) is a primitive proposition. On the other hand, the proposition (1) is composite.*
2. *"Ahmed is smart or he studies every night". is composite proposition*
3. *"If it is raining, then I will stay inside". is composite proposition*
4. $\sqrt{2}$ *is an irrational number.*

The fundamental property of a compound proposition is that its truth value is completely determined by the truth values of its subpropositions together with the way in which they are connected to form the compound propositions. The next section studies some of these connectives.

1.4 Basic logical operations

This section discusses the three basic logical operations of conjunction, disjunction, and negation which correspond, respectively, to the English words "and," "or," and "not." These words are in logic called "**the connectives**" and are mathematically represented by "$\wedge$," "$\vee$," and "$\neg$.", respectively. Their role is to link propositions together to form new useful composed propositions.

1.4.1 Conjunction $p \wedge q$

Any two propositions can be combined by the word "and" to form a compound proposition called the conjunction of the original propositions.

Notation 1.4.1 *Symbolically,* $p \wedge q$ *read "p and q" denotes the conjunction of* p *and* q*.*

Since $p \wedge q$ is a proposition it has a truth value, and this truth value depends only on the truth values of p and q. Specifically:

Definition 1.4.2 *If* p *and* q *are true, then* $p \wedge q$ *is true; otherwise* $p \wedge q$ *is false.*

The truth value of the proposition $p \wedge q$ is given in the following truth table

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Example 1.4.3 *Let $p : 5$ be a rational number and $q : 15$ be a prime number. Is it a conjunction?*

Given that $p : 5$ is a rational number. This proposition is true. $q : 15$ is a prime number. This proposition is false as 15 is a composite number [1]*. Therefore, as per the truth table, p and q is a false statement. So, $p \wedge q = F$*

Example 1.4.4 *Let $p : x$ be greater than 9 and $q : x$ be a prime number. Is it a conjunction?*

Since x is a variable whose value we do not know. Let us define a range for p and q. To find the range let us take certain values for x:
When $x = 6$: p and q is false. Hence, $p \wedge q = F$.
When $x = 3$: p is false but q is true. But still, $p \wedge q = F$.
When $x = 10$: p is true but q is false. But still, $p \wedge q = F$.
When $x = 11$: p is true and q is true. Hence, $p \wedge q = T$.
Hence the conjunction p and q is only true when x is a prime number greater than 9.

1.4.2 Disjunction $p \vee q$

Any two propositions can be combined by the word "or" toform a compound proposition called the disjunction of the original propositions.

Notation 1.4.5 *Symbolically, $p \vee q$ read "p or q", denotes the disjunction of p and q. The truth value of $p \vee q$ depends only on the truth values of p and q as follows.*

Definition 1.4.6 *If p and q are false, then $p \vee q$ is false; otherwise $p \vee q$ is true.*

The truth value of $p \vee q$ may be defined equivalently by the following table

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Example 1.4.7 *Let $p : x$ is divisible by 2 or $q : x$ is divisible by 3.*

Let assume the different x values to prove the disjunction truth table.
For $x = 12$: p and q are true. Hence, $p \vee q = T$.
For $x = 4$: p is true but q is false. But still, $p \vee q = T$.
For $x = 9$: p is false but q is true. But still, $p \vee q = T$.
For $x = 7$: p is false and q is false. Hence, $p \vee q = F$.
Hence the disjunction p or q is only false when x is not divisible by 2 or 3.

1.4.3 Negation $\neg p$

Given any proposition p, another proposition, called the **negation** of p, can be formed by writing "It is not true that . . ." or "It is false that . . ." before p or, if possible, by inserting in p the word "not."

Notation 1.4.8 *Symbolically, the negation of p, read "not p," is denoted by $\neg p$ or $\bar{p}$*

[1] A composite number is a positive integer that can be formed by multiplying two smaller positive integers.

The truth value of $\neg p$ depends on the truth value of p as follows:

Definition 1.4.9 *If p is true, then $\neg p$ is false; and if p is false, then $\neg p$ is true.*

The truth value of $\neg p$ may be defined equivalently by the following table. Thus the truth value of the negation of p is always the opposite of the truth value of p.

p	$\bar{p}$
T	F
F	T

Example 1.4.10 *Find the negation of the given proposition " a number 6 is an even number"*

Let "p" be the given statement: p : "6 is an even number".
Therefore, the negation of the given statement is $\neg p$: "6 is not an even number".
Therefore, the negation of the proposition is " 6 is not an even number".

1.4.4 Conditional and biconditional statements

1. Conditional statement, $p \Longrightarrow q$

Many statements, particularly in mathematics, are of the form "If p then q." Such statements are called conditional statements and are denoted by

Notation 1.4.11 *Symbolically, $p \Longrightarrow q$ read "p implies q" or "p only if q", denotes the implication of p and q.*

The truth value of $p \Longrightarrow q$ depends only on the truth values of p and q as follows.

Definition 1.4.12 *The conditional $p \Longrightarrow q$ is false only when the first part p is true and the second part q is false.*

The truth value of $p \Longrightarrow q$ may be defined equivalently by the following table

p	q	$p \Longrightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Remark 1.4.13 *Here are another typical ways we can express a logical implication: "If p, then q" If p, q", "p is sufficient for q", "q if p", "q when p", "A necessary condition for q is p", "p only if q", "p is a sufficient condition for q", "q whenever p", "q is necessary for p", "q follows p", "p is a necessary condition for q".*

Example 1.4.14 *A teacher says to his students "If you answer correctly to the question I am going to ask, I will add an extra point to your exam score", the teacher asks his question and no student finds the correct answer, the teacher adds a point to each student and they do not understand."*

Example 1.4.15 *If we write p : "answer correctly to the question" and q : "Add a point to exam score". The logical implication $p \Longrightarrow q$ is true in both situations (whether the teacher adds points or not) because p is false.*

Definition 1.4.16 *Given an implication $p \Longrightarrow q$, we define three related implications:*

1. Its converse is defined as: $q \Longrightarrow p$.
2. Its inverse is defined as: $\neg p \Longrightarrow \neg q$
3. Its contrapositive is defined as: $\neg q \Longrightarrow \neg p$.

Among them, the contrapositive $\neg q \Longrightarrow \neg p$ is the most important one.

Remark 1.4.17 *An implication and its contrapositive always have the same truth value, but this is not true for the converse.*

p	q	$p \Longrightarrow q$	$q \Longrightarrow p$	$\neg q \Longrightarrow \neg p$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	T
F	F	T	T	T

Example 1.4.18

$$p \; : \; \textit{The sky is overcast.}$$
$$q \; : \; \textit{The sun is not visible.}$$

In this example, $p \Longrightarrow q$ is a true statement, assuming we are at the surface of the earth, below the cloud layer. However, the statement $q \Longrightarrow p$ is not necessarily true because it might be a clear night. In this case, logical implication does not work both ways. However, the sense of logical implication is reversed if the negation of both statements is used. That is, $p \Longrightarrow q \equiv \neg q \Longrightarrow \neg p$.

Using the above sentences as examples, we can say that if the sun is visible, then the sky is not overcast. This conditional statement is always true. In fact, there is logical equivalence between the two statements $p \Longrightarrow q$ and $\neg q \Longrightarrow \neg p$.

2. Biconditional statement, $p \Longleftrightarrow q$

Another common statement is of the form "p if and only if q". Such statements are called biconditional statements.

Notation 1.4.19 *Symbolically, $p \Longleftrightarrow q$ read "p if and only if q" or "p is equivalent to q", denotes the implication of p and q.*

The truth value of $p \Longrightarrow q$ depends only on the truth values of p and q as follows:

Definition 1.4.20 *The biconditional $p \Longleftrightarrow q$ is true whenever p and q have the same truth values and false otherwise.*

The truth value of $p \Longleftrightarrow q$ may be defined equivalently by the following table

p	q	$p \Longleftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Remark 1.4.21 *Here are another typical ways we can express a biconditional $p \Longleftrightarrow q$: "p is necessary and sufficient for q", "if p then q, and conversel".*

Remark 1.4.22 *A biconditional statement can also be defined as the compound statement $(p \Longrightarrow q) \wedge (q \Longrightarrow p)$. This explains why we call it a biconditional statement. A biconditional statement is often used to define a new concept.*

1.4.5 Logical equivalence

Definition 1.4.23 *Logical Equivalence. We say two propositions are logically equivalent if* $p \iff q$ *is a tautology. We denote this by* $p \equiv q$.

Examples 1.4.24 *Prove the following are equivalent*

1. $p \vee (p \wedge q) \equiv p$.
   We shall prove that $p \vee (p \wedge q) \iff p$ is au tautology, that is the implication is true in the two senses for all propositions values. We have

$$\begin{aligned} 1.p \vee (p \wedge q)) \implies p &\equiv \neg (p \vee (p \wedge q)) \vee p \implies \\ &\equiv (\neg p \wedge \neg(p \wedge q)) \vee p \\ &\equiv (\neg p \wedge (\neg p \vee \neg q)) \vee p \\ &\equiv (\neg p \vee p) \wedge (\neg p \vee \neg q \vee p) \\ &\equiv T \wedge (\neg p \vee p \vee \neg q) \\ &\equiv T \wedge (T \vee \neg q) \equiv T. \end{aligned}$$

For the second implication, we have

$$\begin{aligned} 2.p \implies (p \vee (p \wedge q)) &\equiv \neg p \vee (p \vee (p \wedge q)) \\ &\equiv (\neg p \vee p) \vee (p \wedge q) \\ &\equiv T \vee (p \wedge q) \equiv T \end{aligned}$$

2. $\neg p \implies (q \wedge \neg q) \equiv p$.

1.5 Propositions and truth tables

Let $P(p, q, ...)$ denote a compound proposition constructed from logical variables $p, q, ...$, which take on the value TRUE (T) or FALSE (F), and the logical connectives $\wedge$, $\vee$, and $\neg$ (and others discussed subsequently). Such an expression $P(p, q, ...)$ will be called a proposition.

A **truth table** shows how the truth or falsity of a compound proposition depends on the truth or falsity of the simple statements from which it is constructed.

Remark 1.5.1 *When we construct a truth table, we have to consider all possible assignments of True (T) and False (F) to the component proposition. For example, suppose the component propositions are p, q, and r. Each of these statements can be either true or false, so there are* $2^3 = 8$ *possibilities. Generally, this can be determined by the formula:* 2^n *where n is the number of component propositions.*

Example 1.5.2 *Construct a truth table for the proposition:* $\bar{p} \wedge (\bar{p} \vee q)$.

p	q	$\bar{p}$	$\bar{p} \vee q$	$\bar{p} \wedge (\bar{p} \vee q)$
T	T	F	T	F
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

1.6 Tautology and Contradiction

Definition 1.6.1 *A tautology* [1] *is a proposition that is always true, regardless of the truth values of the component propositions it contains.*

Definition 1.6.2 *A proposition that is always false is called a contradiction.*

Notation 1.6.3 *To indicate that a formula is a tautology we note:*$\vdash P$.

Remark 1.6.4 *A proposition that is neither a tautology nor a contradiction is called a contingency. The term contingency is not as widely used as the terms tautology and contradiction.*

Example 1.6.5 *From the following truth table*

p	$\bar{p}$	$p \wedge \bar{p}$	$p \vee \bar{p}$
T	F	F	T
F	T	F	T

We gather that $p \vee \bar{p}$ is a tautology, and $p \wedge \bar{p}$ is a contradiction. In words, $p \vee \bar{p}$ says that either the statement p is true, or the statement $\bar{p}$ is true (that is, p is false). This claim is always true. The compound statement $p \wedge \bar{p}$ claims that p is true, and at the same time, $\bar{p}$ is also true (which means p is false). This is clearly impossible. Hence, $p \wedge \bar{p}$ must be false.

1.7 Propositional laws

Propositions satisfy various laws which we list below. We state this result formally (T and F design truth values "True" and "False", respectively.).

Theorem 1.7.1 *Propositions satisfy the following rules:*
Idempotent laws: *1.* $p \vee p \equiv p$, *2.* $p \wedge p \equiv p$.
Associative laws:

1. $(p \vee q) \vee r \equiv p \vee (q \vee r)$,
2. $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$.

Commutative laws: *1.* $p \vee q \equiv q \vee p$, *2.* $p \wedge q \equiv q \wedge p$.
Distributive laws:

1. $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$,
2. $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$.

[1] The notion of tautology in the propositional calculus was first developed in the early 20th century by the American philosopher Charles Sanders Peirce, the founder of the school of pragmatism and a major logician. The term itself, however, was introduced by the Austrian-born British philosopher Ludwig Wittgenstein, who argued in the Logisch-philosophische Abhandlung (1921; Tractatus Logico-Philosophicus, 1922) that all necessary propositions are tautologies and that there is, therefore, a sense in which all necessary propositions say the same thing ”viz, nothing at all.

Identity laws:

1. $p \vee F \equiv p$, *2.* $p \wedge T \equiv p$, *3.* $p \vee T \equiv T$, *4.* $p \wedge F \equiv F$.
Involution law: $\neg\neg p \equiv p$.
Complement laws:
1. $p \vee \neg p \equiv T$, *2.* $p \wedge \neg p \equiv T$, *3.* $\neg T \equiv F$ *4.* $\neg F \equiv T$.
De Morgan laws [1]

1. $\neg(p \vee q) \equiv \neg p \wedge \neg q$,

2. $\neg(p \wedge q) \equiv \neg p \vee \neg q$.

Implication identity: $p \Longrightarrow q \equiv \neg p \vee q$

1.8 Principle of Substitution

Let $P(p, q, ...)$ be a tautology, and let $Q_1(p, q, ...), Q_2(p, q, ...), \ . \ . \ .$ be any propositions.

Theorem 1.8.1 *(Principle of Substitution): If $P(p, q, ...)$ is a tautology, then $P(Q_1, Q_2, ...)$ is a tautology for any propositions P_1, P_2, ...*

Interpretation: Since $P(p, q, ...)$ does not depend upon the particular truth values of its variables $p, q, ...$, we can substitute P_1 for p, P_2 for q, . . . in the tautology $P(p, q, ...)$ and still have a tautology.

Example 1.8.2 *Since $\neg p \vee p$ is a tautology, then by the principle of substitution the propositions $\neg (p \wedge q) \vee p$ and $\neg (p \vee q) \vee p$ are also tautologies.*

1.9 Completeness for a set of connectives

Definition 1.9.1 *A set C of connectives is said to be complete if any propositional formula is equivalent to a formula using only the connectives of C.*

Proposition 1.9.2 *The set $\{\neg, \vee\}$ is a complete system of connectives.*

Proof. It suffices to show that the propositions $(p \wedge q)$, $(p \Longrightarrow q)$ and $(p \Longleftrightarrow q)$ can be written using only the connectives $\neg$ and $\vee$.
1. $p \wedge q \equiv \neg (\neg p \vee \neg q)$,
2. $p \Longrightarrow q \equiv \neg p \vee q$,
3. $p \Longleftrightarrow q \equiv (\neg p \vee q) \wedge (\neg q \vee p)$. ■

[1] De Morgan's laws are identities between logical propositions. They were formulated by the British mathematician Augustus De Morgan (1806-1871. De Morgan's Laws describe how mathematical statements and concepts are related through their opposites). In set theory, De Morgan's Laws relate the intersection and union of sets through complements. In propositional logic, De Morgan's Laws relate conjunctions and disjunctions of propositions through negation. De Morgan's Laws are also applicable in computer engineering for developing logic gates. Interestingly, regardless of whether De Morgan's Laws apply to sets, propositions, or logic gates, the structure is always the same.