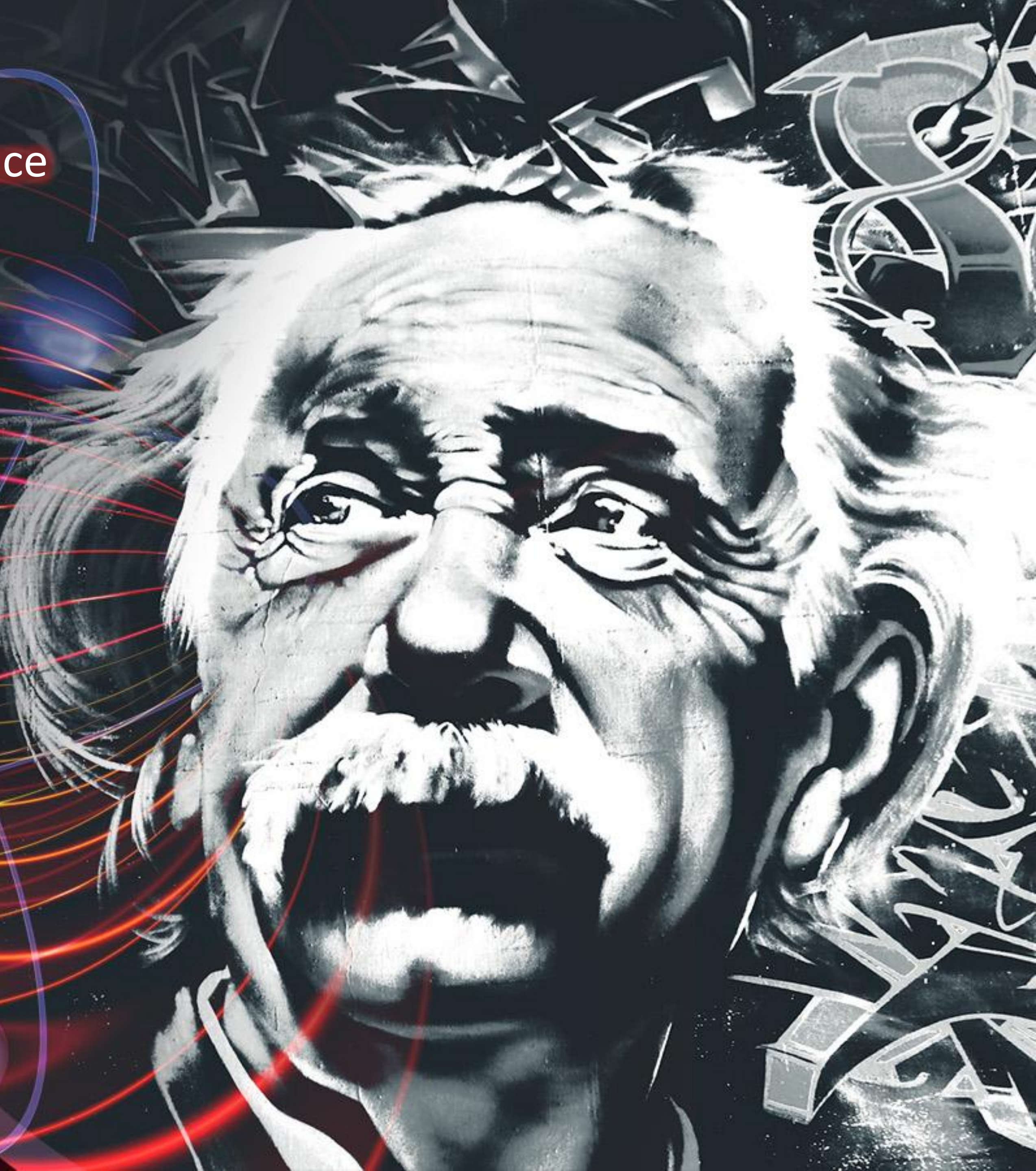
Khemis Miliana University
Faculty of Material Sciences and Computer Science
Dept. of Physics

Special Relativity

Chapter04: Special Relativity and Electromagnetism

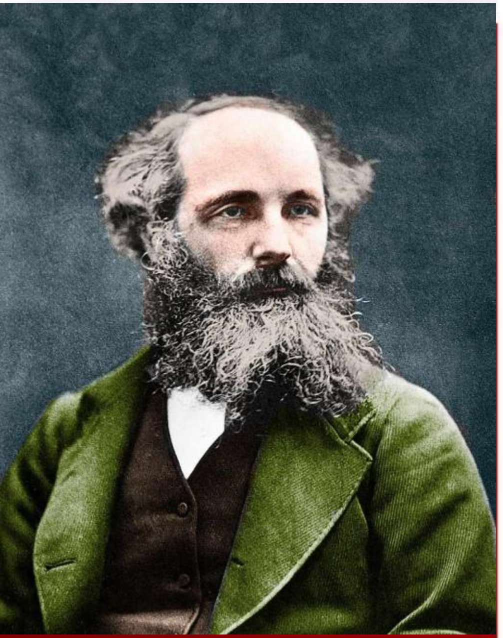
L3 Fundamental Physics



Dr. S.E. Bentridi

Chapiter 04: Special Relativity and Electromagnetism

- Reminder: Maxwell’s equations and Galilean relativity
- Invariance of the wave equation
- Implications of the invariance of Maxwell’s equations under Lorentz Transformations
- Lorentz Transformation of Electromagnetic field
- Four-vector charge-current



J.C. Maxwell
(1831-1879, UK)

Reminder: Maxwell's equations and Galilean relativity

- **The Maxwell's equations:**

(1) $\vec{\nabla}.\vec{E} = \frac{\rho}{\varepsilon_0}$ (2) $\vec{\nabla} \wedge \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$

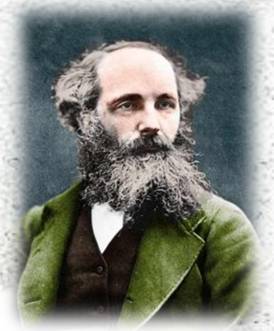
(3) $\vec{\nabla}.\vec{B} = 0$ (4) $\vec{\nabla} \wedge \vec{B} - \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J}$

With the continuity equation: $\frac{\partial \rho(t)}{\partial t} + \vec{\nabla}.\vec{J} = 0$

Equations (2) and (3) are structural equations

Equations (1) and (4) link fields to the sources

EM and Newton relativity



Maxwell's equations under Galilean transformations:

- **Now, lets rewrite the Maxwell's equations in the new frame (R'), since we know that in (R) we have:**

(1) $\vec{\nabla}.\vec{E} = \frac{\rho}{\varepsilon_0}$ (2) $\vec{\nabla} \wedge \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$ in addition of the continuity equation: $\frac{\partial \rho(t)}{\partial t} + \vec{\nabla}.\vec{J} = 0$

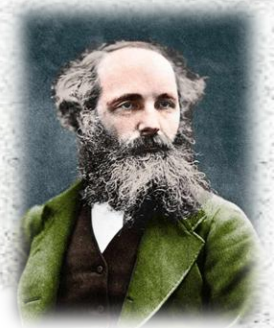
(3) $\vec{\nabla}.\vec{B} = 0$ (4) $\vec{\nabla} \wedge \vec{B} - \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J}$

Let's verify if these equations are invariant under Galilean transformations $(\rho' = \rho, \vec{J'} = 0)$**:**

(1) $\vec{\nabla'}.\vec{E'} = \frac{\rho}{\varepsilon_0}$ (2) $\vec{\nabla'} \wedge \vec{E'} + \frac{\partial \vec{B'}}{\partial t'} = 0$

(3) $\vec{\nabla'}.\vec{B'} = 0$ (4) $\vec{\nabla'} \wedge \vec{B'} - \frac{1}{c^2}\frac{\partial \vec{E'}}{\partial t'} = \mu_0 \vec{J}'$

EM et relativité Newtonienne



Maxwell’s equations under Galilean transformations:

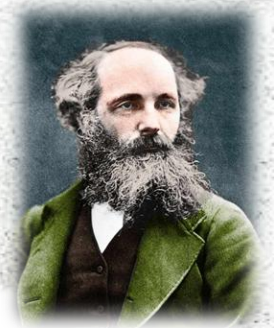
- By replacing with: $\vec{E'} = \vec{E} + \vec{u} \wedge \vec{B}$ and $\vec{B}' = \vec{B}$, and by using: $\vec{\nabla}' = \vec{\nabla}$, $\frac{\partial}{\partial t'} = \frac{\partial}{\partial t} + \vec{u}.\vec{\nabla}$, we get for the 1st equation:

$$\vec{\nabla}.\left(\vec{E} + \vec{u} \wedge \vec{B}\right) = \frac{\rho}{\varepsilon_0} \rightarrow \vec{\nabla}.\vec{E} + \vec{\nabla}.(\vec{u} \wedge \vec{B}) = \underbrace{\vec{\nabla}.\vec{E}}_{\frac{\rho}{\varepsilon_0}} + \underbrace{\vec{B}.\vec{\nabla} \wedge \vec{u}}_{=0} - \vec{u}(\vec{\nabla} \wedge \vec{B}) = \frac{\rho}{\varepsilon_0} - \frac{1}{c^2}\vec{u}.\frac{\partial \vec{E}}{\partial t} = \frac{\rho}{\varepsilon_0}$$

- By considering the vector identity: $\vec{\nabla} \wedge \left(\vec{A} \wedge \vec{B}\right) = \vec{A}\left(\vec{\nabla}.\vec{B}\right) - \vec{B}\left(\vec{\nabla}.\vec{A}\right) + \left(\vec{B}.\vec{\nabla}\right)\vec{A} - \left(\vec{A}.\vec{\nabla}\right)\vec{B}$, the 2nd equation will verify the same one as in (R) :

$$\vec{\nabla} \wedge \left(\vec{E} + \vec{u} \wedge \vec{B}\right) + \frac{\partial \vec{B}}{\partial t'} = \underbrace{\vec{\nabla} \wedge \vec{E} + \frac{\partial \vec{B}}{\partial t}}_{=0} + \underbrace{\vec{\nabla} \wedge \left(\vec{u} \wedge \vec{B}\right) + \left(\vec{u}.\vec{\nabla}\right)\vec{B}}_{=0} = 0$$

EM et relativité Newtonienne



Maxwell's equations under Galilean transformations:

By replacing with: $\vec{E'} = \vec{E} + \vec{u} \wedge \vec{B}$ and $\vec{B}' = \vec{B}$, and by using: $\vec{\nabla'} = \vec{\nabla}$, $\frac{\partial}{\partial t'} = \frac{\partial}{\partial t} + \vec{u}.\vec{\nabla}$, we get for the 3rd equation:

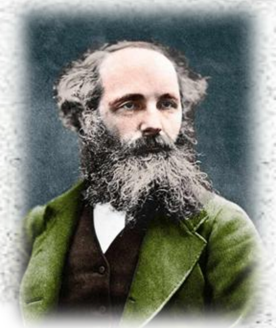
$$\vec{\nabla'}.\vec{B'} = \vec{\nabla}.\vec{B} = 0$$

And for the 4th equation, we have:

$$\vec{\nabla} \wedge \vec{B} - \frac{1}{c^2}\frac{\partial\left(\vec{E} + \vec{u} \wedge \vec{B}\right)}{\partial t'} = \underbrace{\vec{\nabla} \wedge \vec{B} - \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}}_{\mu_0 \vec{J}} - \frac{1}{c^2}(\vec{u}.\vec{\nabla})\vec{E} - \frac{1}{c^2}\frac{\partial}{\partial t}(\vec{u} \wedge \vec{B}) - \frac{1}{c^2}(\vec{u}.\vec{\nabla})\vec{u} \wedge \vec{B} = 0$$

The Galilean transformation did not preserve the Maxwell's equations !!!

EM et relativité Newtonienne



Maxwell’s equations under Galilean transformations:

In the same way, we could get similar results for the wave equation of E.M fields when we try to write it in a moving inertial frame (R’), where we get non-invariant equation under Galilean transformations:

(S): $\Delta\vec{E} - \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} = 0 \; et \; \Delta\vec{B} - \frac{1}{c^2}\frac{\partial^2\vec{B}}{\partial t^2} = 0$

(S’): $\Delta\vec{E} - \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t'^2} + \frac{1}{c^2}\left(2u\frac{\partial^2\vec{E}}{\partial x'\partial t'} - u^2\frac{\partial^2\vec{E}}{\partial x'^2}\right) = 0 \; et \; \Delta\vec{B} - \frac{1}{c^2}\frac{\partial^2\vec{B}}{\partial t'^2} + \frac{1}{c^2}\left(2u\frac{\partial^2\vec{B}}{\partial x'\partial t'} - u^2\frac{\partial^2\vec{B}}{\partial x'^2}\right) = 0$ ✗

The Galilean transformation did not preserve the EM wave equation !!!

Exercise 08

Show that the electromagnetic wave equation:

$$\Delta\varphi - \frac{1}{c^2}\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} - \frac{1}{c^2}\varphi = 0$$

Is not invariant under the Galilean transformations.

$$x = x' + ut'; y = y'; z = z'; t = t'$$

Use the following rules of the differential derivation to pass from a frame to another.

$$\frac{\partial\varphi}{\partial x_i} = \frac{\partial\varphi}{\partial x}{}'; \frac{\partial x'}{\partial x_i} + \frac{\partial\varphi}{\partial y}{}'; \frac{\partial y'}{\partial x_i} + \frac{\partial\varphi}{\partial z}{}'; \frac{\partial z'}{\partial x_i} + \frac{\partial\varphi}{\partial t}{}'; \frac{\partial t'}{\partial x_i}$$

Solution:

$$\frac{\partial x'}{\partial x} = 1; \frac{\partial x'}{\partial t} = -u; \frac{\partial x'}{\partial y} = \frac{\partial x'}{\partial z} = 0, \qquad \frac{\partial y'}{\partial y} = 1; \frac{\partial y'}{\partial t} = \frac{\partial y'}{\partial x} = \frac{\partial y'}{\partial z} = 0$$

$$\frac{\partial t'}{\partial x} = \frac{\partial t'}{\partial y} = \frac{\partial t'}{\partial z} = 0; \frac{\partial t'}{\partial t} = 1, \qquad \frac{\partial z'}{\partial z} = 1; \frac{\partial z'}{\partial t} = \frac{\partial z'}{\partial x} = \frac{\partial z'}{\partial z} = 0$$

Exercise 08

Show that the electromagnetic wave equation:

$$\Delta\varphi - \frac{1}{c^2}\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} - \frac{1}{c^2}\varphi = 0$$

Is not invariant under the Galilean transformations.

$$x = x' + ut'; y = y'; z = z'; t = t'$$

Use the following rules of the differential derivation to pass from a frame to another.

$$\frac{\partial\varphi}{\partial x_i} = \frac{\partial\varphi}{\partial x'}; \frac{\partial x'}{\partial x_i} + \frac{\partial\varphi}{\partial y'}; \frac{\partial y'}{\partial x_i} + \frac{\partial\varphi}{\partial z'}; \frac{\partial z'}{\partial x_i} + \frac{\partial\varphi}{\partial t'}; \frac{\partial t'}{\partial x_i}$$

Solution:

This will give:

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x'}; \frac{\partial}{\partial y} = \frac{\partial}{\partial y'}; \frac{\partial}{\partial z} = \frac{\partial}{\partial z'}; \frac{\partial}{\partial t} = \frac{\partial}{\partial t'} - u\frac{\partial}{\partial x'}$$

Implying:

$$\frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial x'^2}; \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial y'^2}; \frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial z'^2}; \frac{\partial^2}{\partial t^2} = \frac{\partial^2}{\partial t'^2} - 2u\frac{\partial}{\partial t'}\frac{\partial}{\partial x'} + u^2\frac{\partial^2}{\partial x'^2}$$

Exercise 08

Show that the electromagnetic wave equation:

$$\Delta\varphi-\frac{1}{c^2}\varphi=\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}+\frac{\partial^2\varphi}{\partial z^2}-\frac{1}{c^2}\varphi=0$$

Is not invariant under the Galilean transformations.

$$x=x'+ut';y=y';z=z';t=t'$$

Use the following rules of the differential derivation to pass from a frame to another.

$$\frac{\partial\varphi}{\partial x_i}=\frac{\partial\varphi}{\partial x}{}';\frac{\partial x'}{\partial x_i}+\frac{\partial\varphi}{\partial y}{}';\frac{\partial y'}{\partial x_i}+\frac{\partial\varphi}{\partial z}{}';\frac{\partial z'}{\partial x_i}+\frac{\partial\varphi}{\partial t}{}';\frac{\partial t'}{\partial x_i}$$

Solution:

By replacing this in the wave equation written in (R'):

$$\frac{\partial^2\varphi}{\partial x'^2}+\frac{\partial^2\varphi}{\partial y'^2}+\frac{\partial^2\varphi}{\partial z'^2}-\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t'^2}+\frac{1}{c^2}\left(2u\frac{\partial^2\varphi}{\partial t'\partial x'}-u^2\frac{\partial^2\varphi}{\partial x'^2}\right)=0$$

The wave equation is no more invariant under the Galilean transformations !!!

Invariance of the wave equation

Let's examine the transformation of the wave equation under L.T:

$$\Delta'\varphi - \frac{1}{c^2}\frac{\partial^2}{\partial t'^2}\varphi = \frac{\partial^2\varphi}{\partial x'^2} + \frac{\partial^2\varphi}{\partial y'^2} + \frac{\partial^2\varphi}{\partial z'^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t'^2}\varphi = 0$$

By using the partial derivation rule:

$$\frac{\partial\varphi}{\partial x_i} = \frac{\partial\varphi}{\partial x'}\frac{\partial x'}{\partial x_i} + \frac{\partial\varphi}{\partial y'}\frac{\partial y'}{\partial x_i} + \frac{\partial\varphi}{\partial z'}\frac{\partial z'}{\partial x_i} + \frac{\partial\varphi}{\partial t'}\frac{\partial t'}{\partial x_i}$$

With $T.L(R \to R'):x' = \gamma(x - ut);\ y = y';\ z = z';t' = \gamma\left(t - \frac{u}{c^2}x\right)$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial}{\partial y'}\frac{\partial y'}{\partial x} + \frac{\partial}{\partial z'}\frac{\partial z'}{\partial x} + \frac{\partial}{\partial t'}\frac{\partial t'}{\partial x} = \gamma\frac{\partial}{\partial x'} - \gamma\frac{u}{c^2}\frac{\partial}{\partial t'} = \gamma\left(\frac{\partial}{\partial x'} - \frac{u}{c^2}\frac{\partial}{\partial t'}\right)$$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial x'}\frac{\partial x'}{\partial t} + \frac{\partial}{\partial y'}\frac{\partial y'}{\partial t} + \frac{\partial}{\partial z'}\frac{\partial z'}{\partial t} + \frac{\partial}{\partial t'}\frac{\partial t'}{\partial t} = -\gamma u\frac{\partial}{\partial x'} + \gamma\frac{\partial}{\partial t'} = \gamma\left(-u\frac{\partial}{\partial x'} + \frac{\partial}{\partial t'}\right)$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial x'}\frac{\partial x'}{\partial y} + \frac{\partial}{\partial y'}\frac{\partial y'}{\partial y} + \frac{\partial}{\partial z'}\frac{\partial z'}{\partial y} + \frac{\partial}{\partial t'}\frac{\partial t'}{\partial y} = \frac{\partial}{\partial y'};\frac{\partial}{\partial z} = \frac{\partial}{\partial z'}$$

Invariance of the wave equation

Thus, we could also find the square of the derivations:

$$\frac{\partial^2}{\partial x^2} = \gamma^2 \left(\frac{\partial}{\partial x'} - \frac{u}{c^2}\frac{\partial}{\partial t'}\right)\left(\frac{\partial}{\partial x'} - \frac{u}{c^2}\frac{\partial}{\partial t'}\right) = \cdots$$

$$\frac{\partial^2}{\partial t^2} = \gamma^2 \left(-u\frac{\partial}{\partial x'} + \frac{\partial}{\partial t'}\right)\left(-u\frac{\partial}{\partial x'} + \frac{\partial}{\partial t'}\right) = \cdots$$

$$\frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial y'^2}; \frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial z'^2}$$

Invariance of the wave equation

Thus, we could also find the square of the derivations:

$$\frac{\partial^2}{\partial x^2} = \gamma^2 \left(\frac{\partial}{\partial x'} - \frac{u}{c^2}\frac{\partial}{\partial t'}\right)\left(\frac{\partial}{\partial x'} - \frac{u}{c^2}\frac{\partial}{\partial t'}\right) = \gamma^2 \left[\frac{\partial^2}{\partial x'^2} - 2\frac{u}{c^2}\frac{\partial}{\partial x'}\frac{\partial}{\partial t'} + \frac{u^2}{c^4}\frac{\partial^2}{\partial t'^2}\right]$$

$$\frac{\partial^2}{\partial t^2} = \gamma^2 \left(-u\frac{\partial}{\partial x'} + \frac{\partial}{\partial t'}\right)\left(-u\frac{\partial}{\partial x'} + \frac{\partial}{\partial t'}\right) = \gamma^2 \left[u^2\frac{\partial^2}{\partial x'^2} - 2u\frac{\partial}{\partial x'}\frac{\partial}{\partial t'} + \frac{\partial^2}{\partial t'^2}\right]$$

$$\frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial y'^2}; \frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial z'^2}$$

Invariance of the wave equation

By replacing now into the wave equation:

$$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}+\frac{\partial^2\varphi}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\varphi=0$$

We get:

$$\boldsymbol{\gamma^2\left[\frac{\partial^2}{\partial x'^2}-2\frac{u}{c^2}\frac{\partial}{\partial x'}\frac{\partial}{\partial t'}+\frac{u^2}{c^4}\frac{\partial^2}{\partial t'^2}\right]\varphi+\frac{\partial^2}{\partial y'^2}\varphi+\frac{\partial^2}{\partial z'^2}\varphi-\frac{1}{c^2}\gamma^2\left[u^2\frac{\partial^2}{\partial x'^2}-2u\frac{\partial}{\partial x'}\frac{\partial}{\partial t'}+\frac{\partial^2}{\partial t'^2}\right]\varphi=0}$$

$$\boldsymbol{\underbrace{\gamma^2\left[1-\frac{u^2}{c^2}\right]}_{=1}\frac{\partial^2}{\partial x'^2}\varphi+\frac{\partial^2}{\partial y'^2}\varphi+\frac{\partial^2}{\partial z'^2}\varphi-\frac{1}{c^2}\underbrace{\gamma^2\left[1-\frac{u^2}{c^2}\right]}_{=1}\frac{\partial^2}{\partial t'^2}\varphi=0}$$

Invariance of the wave equation

We obtain finally:

$$\frac{\partial^2\varphi}{\partial x'^2}+\frac{\partial^2\varphi}{\partial y'^2}+\frac{\partial^2\varphi}{\partial z'^2}-\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t'^2}=\Delta'\varphi-\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t'^2}=0$$

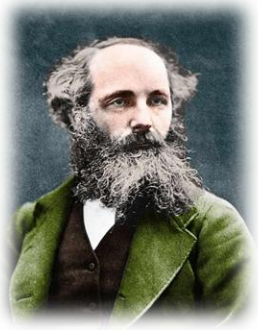
Which is equivalent to the wave equation in the frame (R):

$$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}+\frac{\partial^2\varphi}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\varphi=0$$

This result implies that the wave equation is invariant under L.T and preserve the same formulation when we pass from an inertial frame to another.

What about the Maxwell's equations ?

Implications of the invariance of Maxwell's equations under L.T



(1) $\vec{\nabla}.\vec{E} = \frac{\rho}{\varepsilon_0}$ (2) $\vec{\nabla} \wedge \vec{E} + \frac{\partial \vec{B}}{\partial t} = \mathbf{0}$

(3) $\vec{\nabla}.\vec{B} = \mathbf{0}$ (4) $\vec{\nabla} \wedge \vec{B} - \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J}$

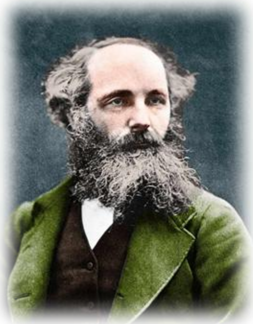
To do, we will focus on both structural equations (2) and (3), due to their relative simplicity, and let's see what their invariance will imply on both fields $\vec{E}$ et $\vec{B}$

$$(R): \begin{cases} \vec{\nabla} \wedge \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \\ \vec{\nabla}.\vec{B} = 0 \end{cases} \rightarrow (R'): \begin{cases} \vec{\nabla'} \wedge \vec{E'} + \frac{\partial \vec{B'}}{\partial t'} = 0 \\ \vec{\nabla'}.\vec{B'} = 0 \end{cases}$$

We recall, that partial derivations between the two frames (R) et (R'):

$$\frac{\partial}{\partial x} = \gamma\left(\frac{\partial}{\partial x'} - \frac{u}{c^2}\frac{\partial}{\partial t'}\right); \frac{\partial}{\partial t} = \gamma\left(-u\frac{\partial}{\partial x'} + \frac{\partial}{\partial t'}\right); \frac{\partial}{\partial y} = \frac{\partial}{\partial y'}; \frac{\partial}{\partial z} = \frac{\partial}{\partial z'}$$

Implications of the invariance of Maxwell's equations under L.T



First, we rewrite the two equations (2 & 3):

$$\vec{\nabla}\wedge\vec{E} = -\frac{\partial \vec{B}}{\partial t} \rightarrow \begin{cases} \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\frac{\partial B_x}{\partial t} \\ \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\frac{\partial B_y}{\partial t} \\ \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\frac{\partial B_z}{\partial t} \end{cases} \leftrightarrow \begin{cases} \frac{\partial E_z}{\partial y'} - \frac{\partial E_y}{\partial z'} = -\gamma\left(-u\frac{\partial}{\partial x'} + \frac{\partial}{\partial t'}\right)B_x \\ \frac{\partial E_x}{\partial z'} - \gamma\left(\frac{\partial}{\partial x'} - \frac{u}{c^2}\frac{\partial}{\partial t'}\right)E_z = -\gamma\left(-u\frac{\partial}{\partial x'} + \frac{\partial}{\partial t'}\right)B_y \\ \gamma\left(\frac{\partial}{\partial x'} - \frac{u}{c^2}\frac{\partial}{\partial t'}\right)E_y - \frac{\partial E_x}{\partial y'} = -\gamma\left(-u\frac{\partial}{\partial x'} + \frac{\partial}{\partial t'}\right)B_z \end{cases}$$

$$\vec{\nabla}.\vec{B} = 0 \rightarrow \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = \gamma\left(\frac{\partial}{\partial x'} - \frac{u}{c^2}\frac{\partial}{\partial t'}\right)B_x + \frac{\partial B_y}{\partial y'} + \frac{\partial B_z}{\partial z'} = 0$$

After a rearrangement ($\partial_{x,y,z}$ on the left, ∂_t on the right):

$$\begin{cases} \frac{\partial E_x}{\partial z'} - \gamma\frac{\partial}{\partial x'}(E_z + uB_y) = -\gamma\frac{\partial}{\partial t'}\left(B_y + \frac{u}{c^2}E_z\right) \\ \gamma\frac{\partial}{\partial x'}(E_y - uB_z) - \frac{\partial E_x}{\partial y'} = -\gamma\frac{\partial}{\partial t'}\left(B_z - \frac{u}{c^2}E_y\right) \end{cases} \leftrightarrow \begin{cases} \frac{\partial E'_x}{\partial z'} - \frac{\partial E'_z}{\partial x'} = -\frac{\partial B'_y}{\partial t'} \\ \frac{\partial E'_y}{\partial x'} - \frac{\partial E'_x}{\partial y'} = -\frac{\partial B'_z}{\partial t'} \end{cases} \rightarrow \begin{cases} E'_x = E_x; E'_z = \gamma(E_z + uB_y) \\ E'_y = \gamma(E_y - uB_z) \\ B'_y = \gamma\left(B_y + \frac{u}{c^2}E_z\right) \\ B'_z = \gamma\left(B_z - \frac{u}{c^2}E_y\right) \end{cases}$$

Implications of the invariance of Maxwell's equations under L.T

Let's arrange the terms in remain equations ($\partial_{x,y,z}$ on the left, ∂_t on the right):

$$\begin{cases} \dfrac{\partial E_z}{\partial y'} - \dfrac{\partial E_y}{\partial z'} - \gamma u \dfrac{\partial B_x}{\partial x'} = -\gamma \dfrac{\partial B_x}{\partial t'} \\ \gamma \dfrac{\partial B_x}{\partial x'} + \dfrac{\partial B_y}{\partial y'} + \dfrac{\partial B_z}{\partial z'} = \gamma \dfrac{u}{c^2} \dfrac{\partial B_x}{\partial t'} \end{cases} \leftrightarrow \begin{cases} \dfrac{\partial E_z}{\partial y'} - \dfrac{\partial E_y}{\partial z'} - \gamma u \dfrac{\partial B_x}{\partial x'} = -\gamma \dfrac{\partial B_x}{\partial t'} \\ \gamma u \dfrac{\partial B_x}{\partial x'} + u \dfrac{\partial B_y}{\partial y'} + u \dfrac{\partial B_z}{\partial z'} = \gamma \dfrac{u^2}{c^2} \dfrac{\partial B_x}{\partial t'} \end{cases}$$

By summing both equations from part to another:

$$\rightarrow \frac{\partial E_z}{\partial y'} - \frac{\partial E_y}{\partial z'} + u \frac{\partial B_y}{\partial y'} + u \frac{\partial B_z}{\partial z'} = -\gamma \left(1 - \frac{u^2}{c^2}\right) \frac{\partial B_x}{\partial t'} = \frac{-1}{\gamma} \frac{\partial B_x}{\partial t'}$$

$$-\frac{\partial B_x}{\partial t'} = \gamma \frac{\partial E_z}{\partial y'} - \gamma \frac{\partial E_y}{\partial z'} + \gamma u \frac{\partial B_y}{\partial y'} + \gamma u \frac{\partial B_z}{\partial z'} = \frac{\partial \gamma (E_z + u B_y)}{\partial y'} - \frac{\partial \gamma (E_y - u B_z)}{\partial z'} = \frac{\partial E'_z}{\partial y'} - \frac{\partial E'_y}{\partial z'} = -\frac{\partial B'_x}{\partial t'}$$

This implies that the missing equation concerns the component x: $B'_x = B_x$

Implications of the invariance of Maxwell's equations under L.T

Therfore, the invariance of the Maxwell's equations, under L.T implies the following L.T of electric and magnetic fields:

$T.L: S \rightarrow S'$		$T.L: S' \rightarrow S$	
Champ E'	*Champ B'*	*Champ E*	*Champ B*
$E'_x = E_x$	$B'_x = B_x$	$E_x = E'_x$	$B_x = B'_x$
$E'_y = \gamma(E_y - uB_z)$	$B'_y = \gamma\left(B_y + \frac{u}{c^2}E_z\right)$	$E_y = \gamma(E'_y + uB'_z)$	$B_y = \gamma\left(B'_y - \frac{u}{c^2}E'_z\right)$
$E'_z = \gamma(E_z + uB_y)$	$B'_z = \gamma\left(B_z - \frac{u}{c^2}E_y\right)$	$E_z = \gamma(E'_z - uB'_y)$	$B_z = \gamma\left(B'_z + \frac{u}{c^2}E'_y\right)$

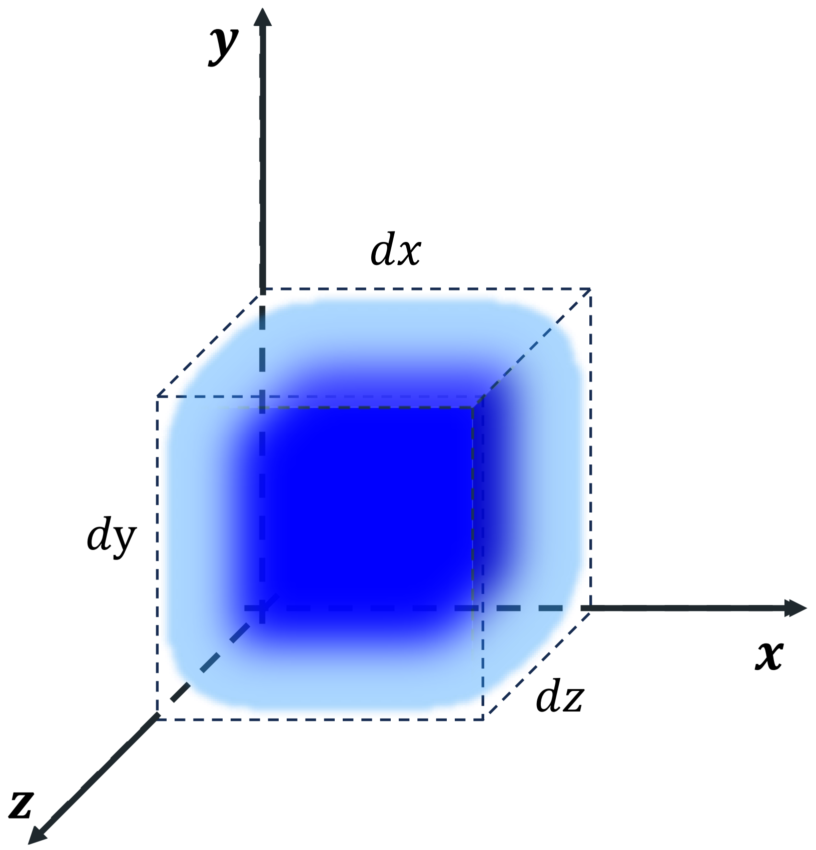
These transformations could also be rewritten under vector form

$T.L: S \rightarrow S'$		$T.L: S' \rightarrow S$	
Champ E'	*Champ B'*	*Champ E*	*Champ B*
$\vec{E'}_\parallel = \vec{E}_\parallel$	$\vec{B'}_\parallel = \vec{B}_\parallel$	$\vec{E}_\parallel = \vec{E'}_\parallel$	$\vec{B}_\parallel = \vec{B'}_\parallel$
$\vec{E'}_\perp = \gamma(\vec{E}_\perp + \vec{u} \wedge \vec{B}_\perp)$	$\vec{B'}_\perp = \gamma\left(\vec{B}_\perp - \frac{1}{c^2}\vec{u} \wedge \vec{E}_\perp\right)$	$\vec{E}_\perp = \gamma\left(\vec{E'}_\perp - \vec{u} \wedge \vec{B'}_\perp\right)$	$\vec{B}_\perp = \gamma\left(\vec{B'}_\perp + \frac{1}{c^2}\vec{u} \wedge \vec{E'}_\perp\right)$

Four-vector charge-current

We define the density of a group of point charges Q as the quotient charge/volume:

$$\rho = \frac{dQ}{dV}$$



Now, if we consider the volume of a cube with edge dl. Thus, the previous expression becomes ($V = dx \times dy \times dz = l^3$):

$$\rho = \frac{dQ}{dx.dy.dz} = \frac{dQ}{dl^3}$$

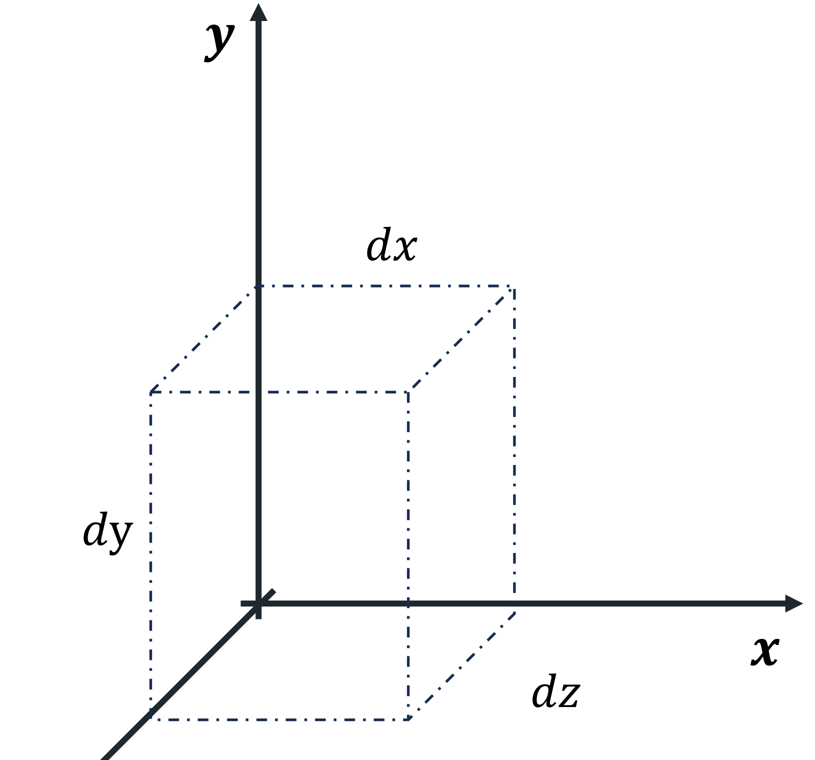
If this group of charge will move with a given velocity according the direction OX, then we could define an electrical current, even a current density:

$$i = \frac{dQ}{dt} = \frac{dV}{dV}\frac{dQ}{dt} = \frac{dQ}{dV}\frac{dV}{dt} = \rho\frac{dx}{dt}dy.dz = \rho dSv \rightarrow \frac{di}{dS} = j = \rho.v \rightarrow \vec{j} = \rho\vec{v}$$

Four-vector charge-current

How both quantities ρ et $\vec{j}$ will transform in special relativity?

To assess, we consider a volume distribution of charge defined in rest within a frame (R') moving with respect to another stationary frame (R) with a velocity $\vec{u}$ along $x - axis$.

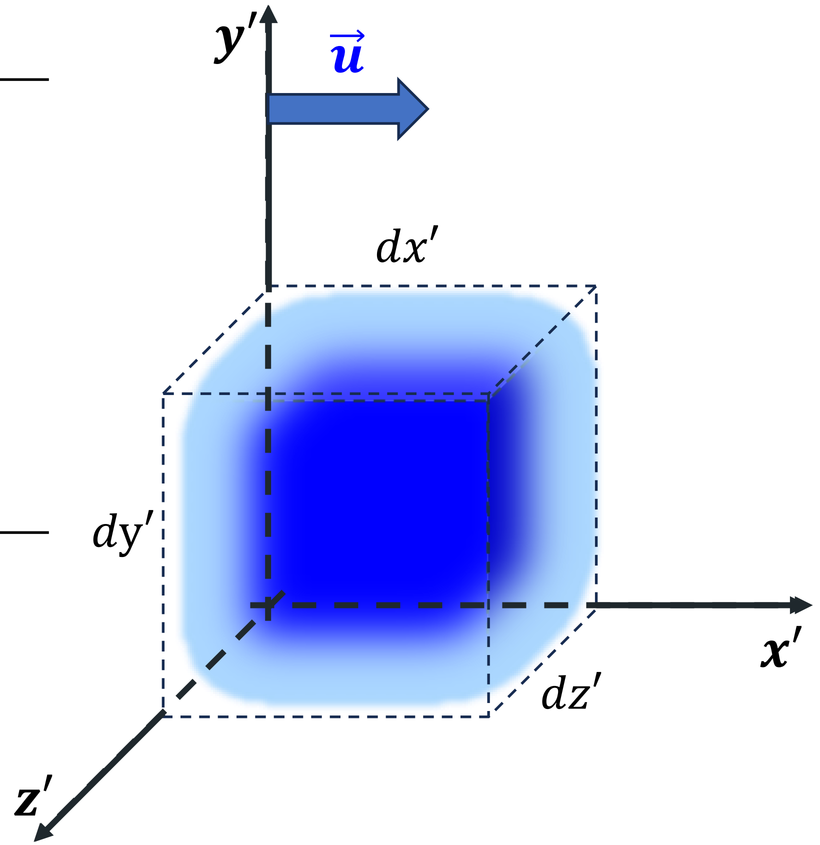


$$T.L: (S' \leftrightarrow S)$$

$$dx = \frac{dx'}{\gamma} = dx'\sqrt{1 - {u^2}/{c^2}}$$

$$dy = dy'$$

$$dz = dz'$$



$$dV = dx.dy.dz = dx'.dy'.dz'\sqrt{1 - {u^2}/{c^2}} = dV'\sqrt{1 - {u^2}/{c^2}}$$

Four-vector charge-current

Let's recall that the proper length is defined with respect to the moving frame (R'), such as:

$$\rho_0 = \frac{dQ}{dV'}$$

Thus, the charge density in the frame (R) could be deduced as:

$$\rho_0 = \frac{dQ}{dV'} = \rho' = \frac{dQ}{dV/\sqrt{1 - u^2/c^2}} = \frac{dQ}{dV}\sqrt{1 - u^2/c^2} = \rho\sqrt{1 - u^2/c^2}$$

$$\rho = \frac{\rho_0}{\sqrt{1 - u^2/c^2}} = \gamma\rho_0$$

Four-vector charge-current

With respect to (S) this charge density is moving with a velocity $\vec{u}$, implying a current density:

$$\vec{j} = \rho\vec{u} = \frac{\rho_0\vec{u}}{\sqrt{1 - {u^2}/{c^2}}}$$

In a general case, where we have: $\vec{u} = u_x\vec{e}_x + u_y\vec{e}_y + u_z\vec{e}_z$

$$\vec{j} = \rho\vec{u} = \frac{\rho_0\vec{u}}{\sqrt{1 - {u^2}/{c^2}}} = \begin{pmatrix} \gamma\rho_0 u_x \\ \gamma\rho_0 u_y \\ \gamma\rho_0 u_z \end{pmatrix}$$

With : $\rho = \gamma\rho_0$

We remark that:

$$\gamma = \frac{\rho}{\rho_0} = \frac{m}{m_0} \rightarrow \begin{cases} \rho = \dfrac{\rho_0}{m_0} m \\ \vec{j} = \dfrac{\rho_0}{m_0} m\vec{u} = \dfrac{\rho_0}{m_0}\vec{p} \end{cases} \leftrightarrow \begin{matrix} \rho c^2 = \dfrac{\rho_0}{m_0} mc^2 \\ \vec{j}c = \dfrac{\rho_0}{m_0}\vec{p}c \end{matrix} \rightarrow \hat{j} = \begin{pmatrix} \vec{j} \\ \rho c \end{pmatrix}$$

Four-vector charge-current

Therefore, by analogy we can define a four-vector charge-current:

$$\hat{j} = \begin{pmatrix} \vec{j} \\ \rho c \end{pmatrix}$$

With an invariant measure:

$$\hat{j}^2 = \vec{j}^2 - \rho^2 c^2 = \left(\frac{\rho_0}{m_0}\right)^2 \left[p^2 - m^2 c^2\right] \times \frac{c^2}{c^2} = \left(\frac{\rho_0}{m_0}\right)^2 \frac{\left[p^2 c^2 - m^2 c^4\right]}{c^2} = \left(\frac{\rho_0}{m_0}\right)^2 \frac{-m_0^2 c^4}{c^2} = -\rho_0^2 c^2 = Cte$$

Thus, both quantities ρ and $\vec{j}$ will transform similarly from a frame to another, by obeying similar L.T of those of four-vector momentum.

Quadrivecteur $\hat{j}$	
$T.L: S \rightarrow S'$	$T.L: S' \rightarrow S$
$j'_x = \gamma(j_x - \rho u)$	$j_x = \gamma(j'_x + \rho' u)$
$j'_y = j_y$	$j_y = j'_y$
$j'_z = j_z$	$j_z = j'_z$
$\rho' = \gamma\left(\rho - u.j_x/c^2\right)$	$\rho = \gamma\left(\rho' + u.j'_x/c^2\right)$