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Level : 1^{st} year LMD
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Algebra 1

Series 2 : Sets and Maps

Exercise 1 : Assuming the set $A = \{w, x, y, z\}$, $B = \{x, y\}$, $C = \{x, y, z\}$ and $D = \{x, z\}$ three parts of A. Identify the elements in each set : $B^c, C^c, B \backslash C, B \backslash D, B \cap C, B \cap D, B \cap (C \cup D), (B \cap C) \cup D, B \backslash D, D \backslash B, B \times C, C \times B, B \times D, P(B)$ and $P(C)$.

Exercise 2 : Let A, B and C be three parts of a set E. Prove that :

1) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
2) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Exercise 3 : Let E be a set and A and B two parts of E. Assume that :

$$A \cap B \neq \emptyset,\ A \cup B \neq E,\ A \not\subseteq B \text{ and } B \not\subseteq A.$$

Suppose that : $A_1 = A \cap B$, $A_2 = A \cap B^c$, $A_3 = B \cap A^c$, $A_4 = (A \cup B)^c$.

1) Prove that A_1, A_2, A_3 and A_4 are not emply.
2) Prove that A_1, A_2, A_3 and A_4 are two by two disjoint.
3) Prove that $A_1 \cup A_2 \cup A_3 \cup A_4 = E$.

Exercise 4 : Let A, B and C be three parts of a set E.

1) What do you think about the implication : $(A \cup B \not\subseteq C) \Longrightarrow (A \not\subseteq C$ or $B \not\subseteq C)$?
2) Suppose that we have $A \cup B \subset A \cup C$ and $A \cap B \subset A \cap C$. Prove that $B \subset C$.

Exercise 5 : Let E a set and A and B two parts of E. Demonstrate that :

1) $F \subset G \Longleftrightarrow F \cup G = G$.
2) $F \subset G \Longleftrightarrow F \cap G^c = \emptyset$.

Exercise 6 : Let $f : I \longrightarrow J$ the function defined by $f(x) = x^2$.

1) Give sets I and J such that f will be injective but not surjective.
2) Give sets I and J such that f will be surjective but not injective.
3) Give sets I and J such that f will be neither injective nor surjective.
4) Give sets I and J such that f will be injective and surjective.

Exercise 7 : We consider the map $f : \mathbb{N} \longrightarrow \mathbb{N}$ defined by : for all $n \in \mathbb{N}$, $f(n) = n^2$.

1) Is it exist a map $g : \mathbb{N} \longrightarrow \mathbb{N}$ such that $f \circ g = Id_{\mathbb{N}}$?
2) Is it exist a map $h : \mathbb{N} \longrightarrow \mathbb{N}$ such that $h \circ f = Id_{\mathbb{N}}$?

Exercise 8 : Let E and F two sets and a map $f : E \longrightarrow F$. Let A and B two parts of E. Demonstrate that :

1) $f(A \cup B) = f(A) \cup f(B)$.
2) $f(A \cap B) \subset f(A) \cap f(B)$.

Give an example for the second property. Then prove that f is injective iff for any parts A and B of E, we have $f(A \cap B) = f(A) \cap f(B)$.

Exercise 9 : 1) Let f the map of $\{1, 2, 3, 4\}$ in it self defined by : $f(1) = 4$, $f(2) = 1$, $f(3) = 2$ and $f(4) = 2$.
Determine $f^{-1}(A)$ when $A = \{2\}$, $A = \{1, 2\}$ and $A = \{3\}$.

2) Let f the map of $\mathbb{R}$ in $\mathbb{R}$ defined by : $f(x) = x^2$. Determine $f^{-1}(A)$ when $A = \{1\}$ and $A = [1, 2]$.

Exercise 10 : 1) Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by : $f(x, y) = x$. Determine $f([0, 1] \times [0, 1])$ and $f^{-1}([-1, 1])$.

2) Let $g : \mathbb{R} \longrightarrow [-1, 1]$ defined by : $g(x) = \cos(\pi x)$. Determine $g(\mathbb{N})$, $g(2\mathbb{N})$ and $g^{-1}(\{-1, 1\})$.

Exercise 11 : Let E and F two sets and a map $f : E \longrightarrow F$. Let C and D two parts not emply of F. Demonstrate that :

1) $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$.
2) $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$.

Exercise 12 : Let E and F two sets and a map $f : E \longrightarrow F$.
1) Prove that for any part A of E we have : $A \subset f^{-1}(f(A))$.
2) Prove that for any parts B of E we have : $f(f^{-1}(B)) \subset B$.
3) Prove that f is injective iff for any part A of E we have : $A = f^{-1}(f(A))$.
4) Prove that f is surjective iff for any part B of E we have : $f(f^{-1}(B)) = B$.

Exercise 13 : 1) Let $q_1 \in \mathbb{N}_{-\{0,1\}}$ and $q_2 \in \mathbb{N}_{-\{0,1\}}$. Prove that :

$$-\frac{1}{2} < \frac{1}{q_1} - \frac{1}{q_2} < \frac{1}{2}$$

2) Let $f : \mathbb{Z} \times \mathbb{N}_{-\{0,1\}} \longrightarrow \mathbb{Q}$ the map defined by : $f(p,q) = p + \frac{1}{q}$.
a) Prove that f is injective.
b) Is f surjective ?