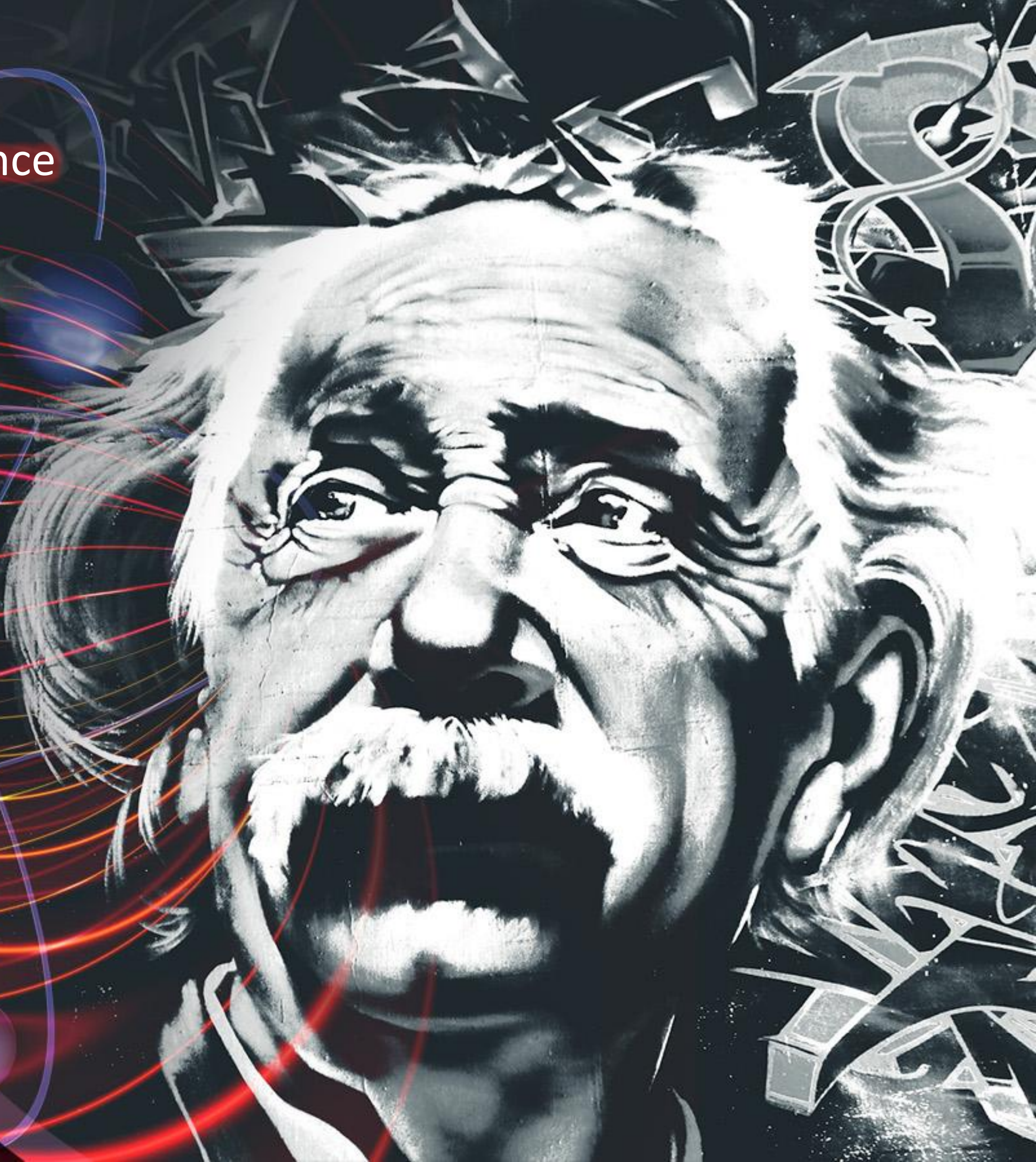
Khemis Miliana University
Faculty of Material Sciences and Computer Science
Dept. of Physics

Special Relativity

Chapter 03:
Relativistic dynamics

L3 Fundamental Physics



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Chapter 03: Relativistic dynamics

- Reminder: Dynamical quantities in classical mechanics
- The mass variation
- The Momentum four-vector and Mass-Energy equivalence
- Dynamics of a massive particle
- Transformations of Momentum 4-vector
- Applications

Dynamical quantities in classical mechanics

Let's recall the different dynamical quantities in the classical mechanics (Newton Mechanics):

- **The momentum (impulsion):** $\vec{p}\left[kg.m.s^{-1}\right] = m\vec{v}$
- **The force:** $\vec{f}[N] = \frac{d}{dt}\vec{p} = \frac{d}{dt}(m\vec{v}) = \left(\frac{dm}{dt}\right)\vec{v} + m\left(\frac{d\vec{v}}{dt}\right) = m\left(\frac{d\vec{v}}{dt}\right) = m\vec{a}$ **(**$m = Cte$**)**
- **Kinetic energy:** $T[J] = \frac{1}{2}mv^2 = \frac{p^2}{2m}$
- **Kinetic moment:** $\vec{L} = \vec{r} \wedge \vec{p} = m(\vec{r} \wedge \vec{v})$

Dynamical quantities in classical mechanics

Some quantities are conserved (under given conditions):

- Conservation of momentum:

$$\sum_i \overrightarrow{p_i} = \sum_f \overrightarrow{p_f}$$

- Conservation of kinetic energy (elastic collisions):

$$\sum_i T_i = \sum_f T_f$$

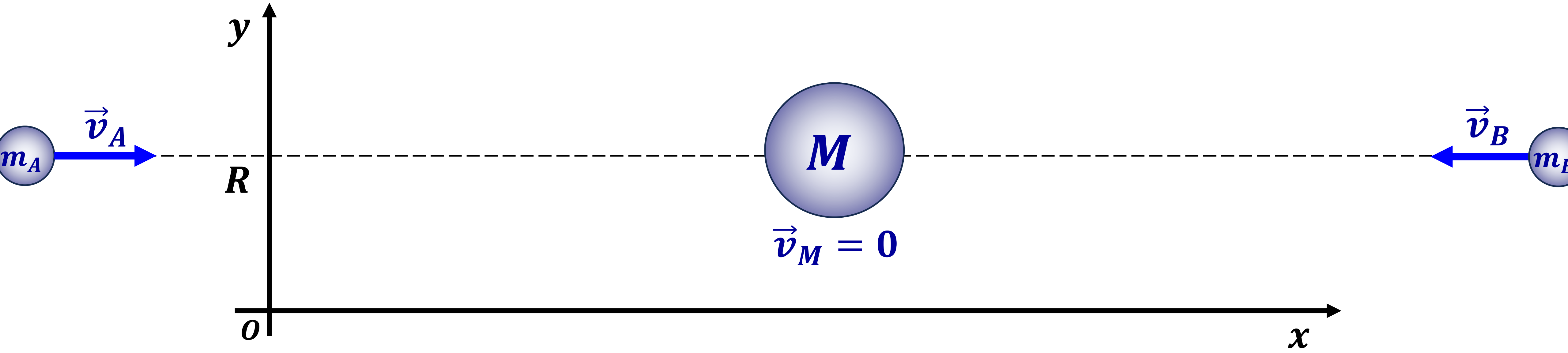
- Conservation of total energy:

$$\sum_i E_i^{tot} = \sum_f E_f^{tot}$$

The mass variation

One of the important consequence of the theory of special relativity is the dependance of the mass particle on its own velocity.

To illustrate this dependance, let's consider in a stationary frame (R), a binary collision between two particles of identical masses at rest ($m_A = m_B$), both animated with same speed in opposite directions ($\overrightarrow{v_A} = -\overrightarrow{v_B} = \vec{u}$). This collision will produce an output particle of a mass M.



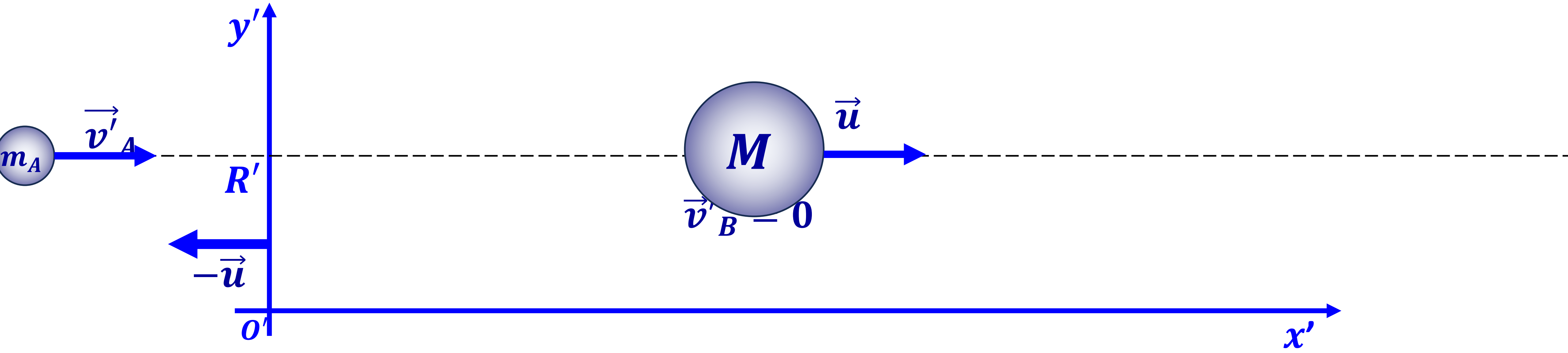
The mass variation

In other hand, let's see how a moving observer (O')related to the particle B (frame R') will consider this collision ($m'_B = m_0$):

Before impact: $m'_A\vec{v}'_A + m'_B\overrightarrow{v'}_B = m'_A\vec{v}'_A$

After impact: $M'\overrightarrow{v'}_M = (m'_A + m'_B)\vec{u} = (m'_A + m_0)\vec{u}$

The conservation of the momentum implies: $m'_A\vec{v}'_A = (m'_A + m_0)\vec{u}$



The mass variation

Besides that, we know that v'_A in the frame R' is given with velocity transformation law (T.L):

$$v'_x = \frac{v_x - u}{\left(1 - {v_x u}/{c^2}\right)}$$

with : $v'_x = v'_A;\ v_x = v_A = u$

We obtain :

$$v'_A = \frac{u - (-u)}{\left(1 - {u(-u)}/{c^2}\right)} = \frac{2u}{\left(1 + {u^2}/{c^2}\right)}$$

The mass variation

Thus, the following equation :

$$m'_A \vec{v}'_A = (m'_A + m_0)\vec{u}$$

Could be rewritten as (after doing projection):

$$m'_A \frac{2u}{\left(1 + {u^2}/{c^2}\right)} = (m'_A + m_0)u \rightarrow m'_A(2u) = (m'_A + m_0)u\left(1 + {u^2}/{c^2}\right)$$

Then, we could obtain:

$$m'_A U\left(1 - {u^2}/{c^2}\right) = m_0 U\left(1 + {u^2}/{c^2}\right)$$

Which leads to the following relevant result:

$$m'_A = m_0\left(\frac{1 + {u^2}/{c^2}}{1 - {u^2}/{c^2}}\right)$$

The mass variation

By using the definition of $v'_A = v' = \frac{2u}{\left(1+{u^2}/{c^2}\right)}$, we could easily demonstrate that:

$$\left(\frac{1+{u^2}/{c^2}}{1-{u^2}/{c^2}}\right) = \frac{1}{\sqrt{1-{v'^2}/{c^2}}}$$

This implies that is possible to link the mass of the particle with its mass in rest (R'), and its own velocity measured with respect to another frame (R):

$$m'_A = m_0\left(\frac{1+{U^2}/{c^2}}{1-{U^2}/{c^2}}\right) = \frac{m_0}{\sqrt{1-{v^2}/{c^2}}}$$

The mass variation

As main result, in general way, a particle with a rest mass m_0, moving with a velocity v will see its own mass increase according to the relation:

$$m(v) = m_0 \frac{1}{\sqrt{1 - {v^2}/{c^2}}} = m_0 \frac{1}{\sqrt{1-\beta^2}} = \gamma . m_0$$

As another implicit consequence, we are brought to redefine the momentum (impulsion) as:

$$\vec{p} = m.\vec{v} = \gamma m_0 \vec{v} = \frac{m_0 \vec{v}}{\sqrt{1-\beta^2}}; \ (\vec{p} \neq m_0 \vec{v})$$

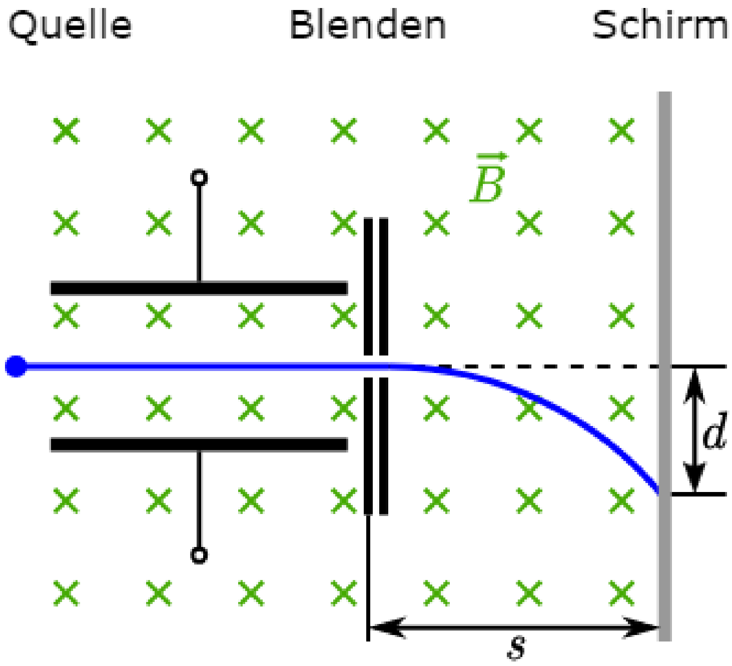
The mass variation

Bucherer experiment (1908):

In this experiment Alfred Bucherer measured the quotient e/m (similarly to the experiment of J.J. Thomson) as a function of the velocity of the β^-particle emitted by a radioactive source. A velocity selector allows to slow or to accelerate the emitted electrons in the aim to make them reach a given speed at the end of their travel. With this final speed, they will enter a space in the presence of an intense magnetic field, perpendicular to the electron's trajectory.

TABLE 3-1 BUCHERER'S RESULTS

u/c	$e/m(= u/rB)$ in coul/kg	$\frac{e}{m_0}\left(= \frac{e}{m\sqrt{1-u^2/c^2}}\right)$ in coul/kg
(Measured)	(Measured)	(Computed)
0.3173	1.661×10^{11}	1.752×10^{11}
0.3787	1.630×10^{11}	1.761×10^{11}
0.4281	1.590×10^{11}	1.760×10^{11}
0.5154	1.511×10^{11}	1.763×10^{11}
0.6870	1.283×10^{11}	1.767×10^{11}



The Momentum four-vector and Mass-Energy equivalence

Four-vector momentum-energy:

Reconsider the mass expression as a function of velocity:

$$m = m_0 \frac{1}{\sqrt{1-\beta^2}}$$

Let's rewrite this expression by taking the square of the same expression:

$$m^2\left(1 - {v^2}/{c^2}\right) = m_0^2 \rightarrow m^2(c^2 - v^2) = m_0^2 c^2 \leftrightarrow m^2c^2 - m^2v^2 = m_0^2c^2$$

Knowing that by definition: $p = mv$ and multiplying by c^2 :

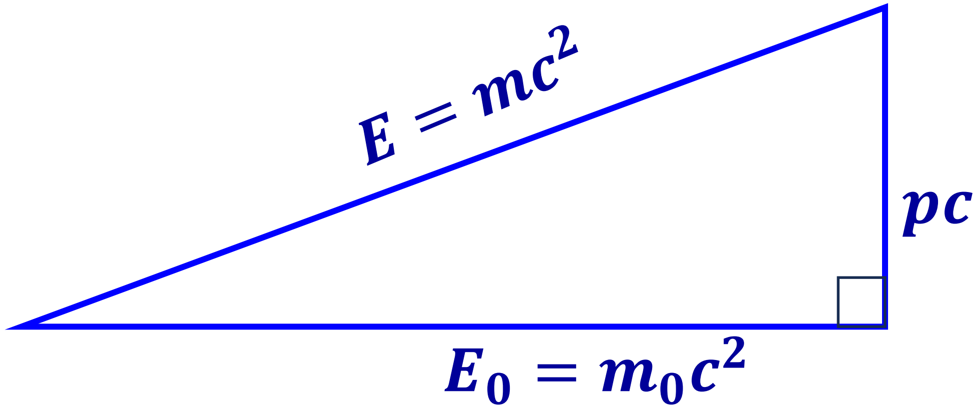
$$m^2c^4 = p^2c^2 + m_0^2c^4 \leftrightarrow E^2 = p^2c^2 + E_0^2$$

By identification, we get :

$pc = mvc$: **energy equivalence of momentum**

$E_0 = m_0c^2$: **rest energy**

$E = mc^2$: **total energy of the particle**



The Momentum four-vector and Mass-Energy equivalence

Four-vector momentum-energy:

In fact, the previous result is equivalent to define a 4-vector of momentum (momentum-energy), from the 4-vector of velocity:

$$\hat{p} = m_0\hat{V} = m_0.\gamma\begin{pmatrix} v_x \\ v_y \\ v_z \\ c \end{pmatrix} = m\begin{pmatrix} v_x \\ v_y \\ v_z \\ c \end{pmatrix} = \begin{pmatrix} mv_x \\ mv_y \\ mv_z \\ mc \end{pmatrix} \equiv (\vec{p}, mc) \equiv \left(\vec{p}, \frac{E}{c}\right)$$

The measure (the norm) of this four-vector is given by :

$$\hat{p}^2 = m_0^2\hat{V}^2 = m^2v^2 - m^2c^2 = -m_0^2c^2 \leftrightarrow m^2c^2 - m^2v^2 = m_0^2c^2$$

In a simple way, and for a particle with a rest mass $m_0 = Cte$, the previous results, inform us by the invariance of such quantity (momentum-energy):

$$E^2 - p^2c^2 = m_0^2c^4 = Cte$$

The Momentum four-vector and Mass-Energy equivalence

Equivalence Mass-Energy:

We note that the dimension of the quantities: $mc^2 \equiv mv^2 \equiv pc$ are in well concordance with energy dimension.

The usual unit in this case, is $1eV = 1.6 \times 10^{-19}J$ besides the multiples: $KeV(10^3), MeV(10^6), GeV(10^9)$

The definition of the rest energy $E_0 = m_0c^2$, and the total energy $E = mc^2$, show clearly that in special relativity an equivalence exists between the energy and the mass of particles, through the factor c^2.

Indeed, it is possible to choose a unit system, such that $c = 1$, which implies $E = m$

Dynamics of a massive particle: Force, Work and Kinetic Energy

(A) Concept of force:

We consider the general definition of the force as the time derivation of the momentum:

$$\vec{F} = \frac{d}{dt}(\vec{p})$$

Thus, the expression of the force in special relativity could be written:

$$\vec{F} = \frac{d}{dt}(m\vec{v}) = m\frac{d\vec{v}}{dt} + \vec{v}\frac{dm}{dt}$$

This expression will be examined again, after the definition of the force work, and the kinetic energy expressions.

Dynamics of a massive particle: Force, Work and Kinetic Energy

(B) Concept of work and kinetic energy:

The work of a given force, which exerting on a solid body to displace it from A to B, through a path $d\vec{l}$ is defined as:

$$W = \int_A^B \vec{F}.d\vec{l}$$

We will suppose that the displacement is on x-axis only, and this force will take the body from its rest state $(u = 0)$ to another state with a non-null velocity $(u = v)$:

$$W = \int_{u=0}^{u=v} F.dx = \int_{u=0}^{u=v} \frac{d}{dt}(mu).dx = \int_{u=0}^{u=v} d(mu).\frac{dx}{dt}$$

$$W = \int_{u=0}^{u=v} d(mu).u = \int_{u=0}^{u=v} (mdu + udm).u = \int_{u=0}^{u=v} (mudu + u^2dm)$$

Dynamics of a massive particle: Force, Work and Kinetic Energy

(B) Concept of work and kinetic energy :

Let's recall also that the body mass verify the following expression of the 4-vector momentum:

$$m^2c^2 - m^2u^2 = m_0^2c^2$$

By taking the differential of this expression: $d(m^2c^2 - m^2u^2) = d(m_0^2c^2)$ **we get:**

$$2m.dm.c^2 - 2m.dm.u^2 - 2m^2.u.du = 0$$

Which is equivalent to write (after simplifying by $2m$**):**

$$dm.c^2 - dm.u^2 - m.u.du = 0 \leftrightarrow u^2dm + m.u.du = c^2dm$$

We could already identify the integrand appearing in the expression of the work W**:**

$$W = \int_{u=0}^{u=v}(mudu + u^2dm) = c^2\int_{m_0}^{m}dm = c^2(m - m_0) = mc^2 - m_0c^2$$

Dynamics of a massive particle: Force, Work and Kinetic Energy

(B) Concept of work and kinetic energy :

This force work, will allow to the particle to acquire a kinetic energy T at the end of its path, implying that:

$$W = T = mc^2 - m_0c^2 \rightarrow T = E - E_0$$

Or, in other way:

$$E = T + E_0$$

By the same, the expression of kinetic energy as a function of mass and velocity:

$$T = m_0c^2(\gamma - 1) = m_0c^2\left[\frac{1}{\sqrt{1 - {v^2}/{c^2}}} - 1\right]$$

We note that in the case of a particle in rest:

$$v = 0 \rightarrow m(v = 0) = m_0 \rightarrow T = m_0c^2 - m_0c^2 = 0$$

Dynamics of a massive particle: Force, Work and Kinetic Energy

(B) Concept of work and kinetic energy :

Classical limit:

The relativistic expression of kinetic energy, should allow us to get the classical expression of kinetic energy: $T_c = \frac{1}{2}mv^2$

Indeed, from the relativistic expression and for $v \ll c$, with a judicious Limited Development:

$$T = m_0c^2\left[\frac{1}{\sqrt{1 - {v^2}/{c^2}}} - 1\right] = m_0c^2\left[\left(1 - {v^2}/{c^2}\right)^{-1/2} - 1\right] \cong m_0c^2\left[1 + \frac{v^2}{2c^2} - 1\right]$$

$$T \equiv T_c = \frac{1}{2}m_0v^2$$

Dynamics of a massive particle: Force, Work and Kinetic Energy

(B) Concept of work and kinetic energy :

Back to 4-vector momentum-energy

Let's take both formulas where the total energy of a particle is cited:

$$E^2 = p^2c^2 + m_0^2c^4; E = T + m_0c^2$$

This two relationships, allow us to link the momentum to the kinetic energy:

$$E^2 = \left(T + m_0c^2\right)^2 = T^2 + 2Tm_0c^2 + m_0^2c^4 = p^2c^2 + m_0^2c^4$$

$$T^2 + 2Tm_0c^2 = p^2c^2 \rightarrow p = \sqrt{\left(\frac{T^2}{m_0c^2} + 2T\right)m_0}$$

In the case of low velocities (classical) : $v < c \rightarrow T < m_0c^2 \rightarrow T^2 \ll m_0c^2 : p \cong \sqrt{2Tm_0}$

Dynamics of a massive particle: Force, Work and Kinetic Energy

(B) Concept of work and kinetic energy :

The triangle momentum-energy

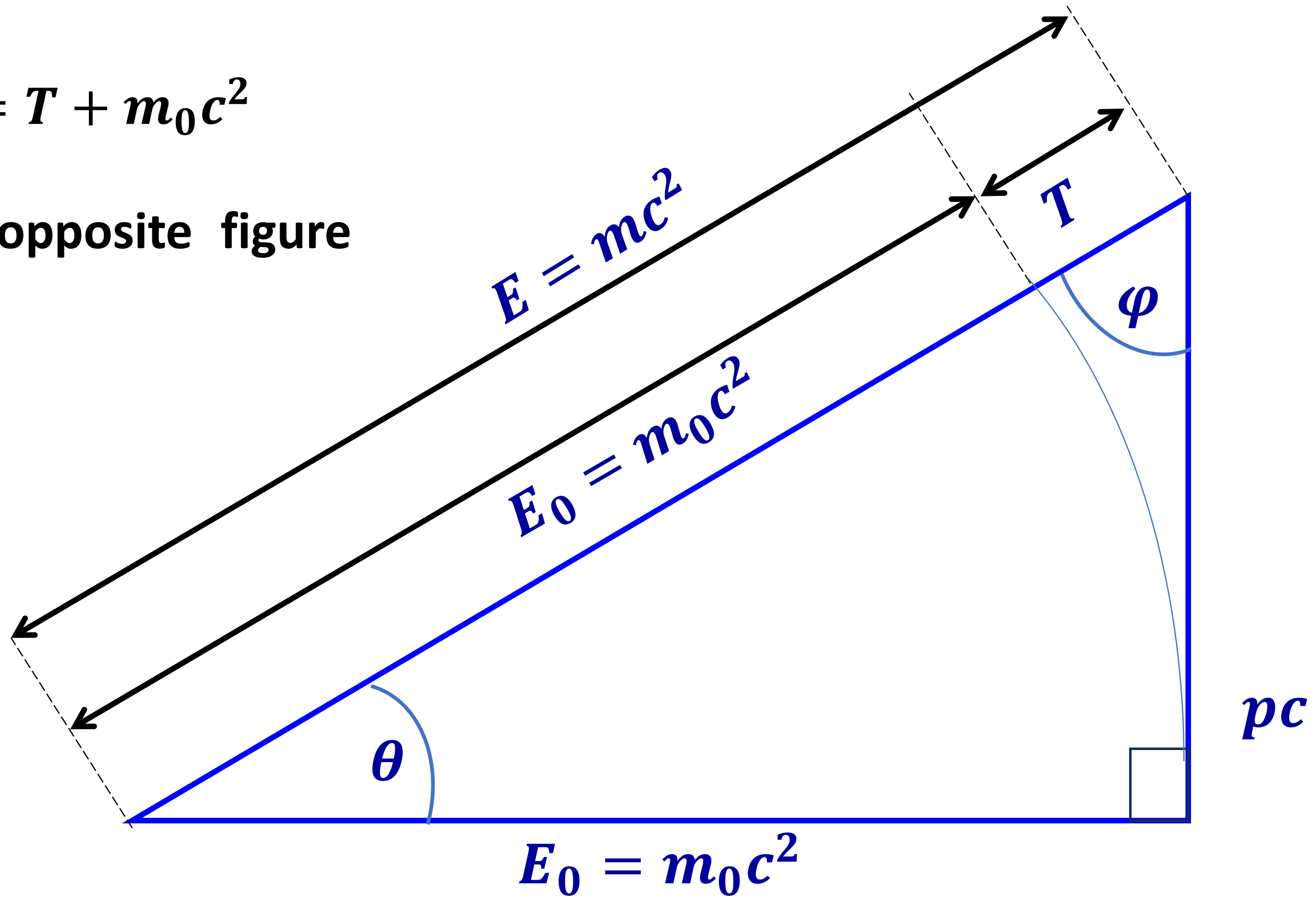
Both relations: $E^2 = p^2c^2 + m_0^2c^4; E = T + m_0c^2$

are depicted on the triangle in the opposite figure (Pythagoras theorem)

It is possible to demonstrate that:

$$\sin\theta = \beta = \frac{v}{c}$$

$$\sin\varphi = \sqrt{1 - \beta^2}$$



Dynamics of a massive particle: Force, Work and Kinetic Energy

(C) Expression of the force:

Let's reconsider again the expression of the force:

$$\vec{F} = \frac{d}{dt}(m\vec{v}) = m\frac{d\vec{v}}{dt} + \vec{v}\frac{dm}{dt}$$

To calculate the term $\frac{dm}{dt}$, we use the definition of total energy:

$$E = mc^2 \leftrightarrow m = \frac{E}{c^2} \rightarrow \frac{dm}{dt} = \frac{1}{c^2}\frac{dE}{dt} = \frac{1}{c^2}\frac{d\left(T + m_0c^2\right)}{dt} = \frac{1}{c^2}\frac{dT}{dt}$$

By recalling that:

$$W = T = \int \vec{F}.d\vec{l} \rightarrow dT = \vec{F}.d\vec{l} \rightarrow \frac{dT}{dt} = \vec{F}\frac{d\vec{l}}{dt} = \vec{F}.\vec{v}$$

We could write:

$$\frac{dm}{dt} = \frac{1}{c^2}\frac{dT}{dt} = \frac{1}{c^2}\vec{F}.\vec{v}$$

Dynamics of a massive particle: Force, Work and Kinetic Energy

(C) Expression of the force :

By replacing $\frac{dm}{dt} = \frac{1}{c^2}\vec{F}.\vec{v}$ in the expression of the force:

$$\vec{F} = \frac{d}{dt}(m\vec{v}) = m\frac{d\vec{v}}{dt} + \vec{v}\frac{dm}{dt} = m\frac{d\vec{v}}{dt} + \vec{v}\frac{(\vec{F}.\vec{v})}{c^2}$$

Since the definition of acceleration is $\vec{a} = {d\vec{v}}/{dt}$ the relativistic expression of the second Newton's principle will become:

$$\vec{F} = m\vec{a} + \vec{v}\frac{(\vec{F}.\vec{v})}{c^2}$$

Thus, the following expression inform us that $\vec{a}$ in general, is no more parallel to $\vec{F}$:

$$\vec{a} = \frac{\vec{F}}{m} - \vec{v}\frac{(\vec{F}.\vec{v})}{mc^2}$$

Transformations of 4-vector Momentum

The four-vector $\widehat{p}$:

Consider the 4-vector momentum-energy defined by its four components in steady frame (R):

$$p_x = mv_x = \frac{m_0 v_x}{\sqrt{1 - {v^2}/{c^2}}}\,; p_y = \frac{m_0 v_y}{\sqrt{1 - {v^2}/{c^2}}}\,; p_z = \frac{m_0 v_z}{\sqrt{1 - {v^2}/{c^2}}}\,; E = \frac{m_0 c^2}{\sqrt{1 - {v^2}/{c^2}}}$$

In another frame (R') moving along-x-axis with a velocity u with respect of (R) :

$$p'_x = m'v'_x = \frac{m_0 v'_x}{\sqrt{1 - {v'^2}/{c^2}}}\,; p'_y = \frac{m_0 v'_y}{\sqrt{1 - {v'^2}/{c^2}}}\,; p'_z = \frac{m_0 v'_z}{\sqrt{1 - {v'^2}/{c^2}}}\,; E' = \frac{m_0 c'^2}{\sqrt{1 - {v'^2}/{c^2}}}$$

Knowing that the L.T for velocities (from R to R') are given in this case by:

$$v_x = \frac{v'_x + u}{\left(1 + {v_x u}/{c^2}\right)}\,; v_y = \frac{v'_y\sqrt{1 - {u^2}/{c^2}}}{\left(1 + {v'_x u}/{c^2}\right)}\,; v_z = \frac{v'_z\sqrt{1 - {u^2}/{c^2}}}{\left(1 + {v'_x u}/{c^2}\right)}$$

Transformations of 4-vector Momentum

The four-vector $\widehat{p}$:

Consider the 4-vector momentum-energy defined b ...):

$$p_x = mv_x = \frac{m_0 v_x}{\sqrt{1 - v^2/\ldots}} ; p_{\ldots}$$

In another f ... (R') :

$$\ldots \frac{\ldots v'_z}{\sqrt{1 - v'^2/c^2}} ; E' = \frac{m_0 c'^2}{\sqrt{1 - v'^2/c^2}}$$

How to describe the components of $\widehat{P'}$ in terms of $\widehat{P}$?

Or in other words, what are the L.T of 4-vector momentum ?

Ki ... m R to R') are given in this case by:

$$v'_x = \frac{v_x - u}{\left(1 - v_x u/c^2\right)} ; v'_y = \frac{v_y\sqrt{1 - u^2/c^2}}{\left(1 - v_x u/c^2\right)} ; v'_z = \frac{v_z\sqrt{1 - u^2/c^2}}{\left(1 - v_x u/c^2\right)}$$

Transformations of 4-vector Momentum

The four-vector $\hat{p}$:

Initially, to make it simple, let's consider a velocity with only both components x and y, which implies that in each frame one can write: $v^2 = v_x^2 + v_y^2$ and $v'^2 = v'^2_x + v'^2_y$

By applying both L.T: $v_x = \frac{v'_x + u}{\left(1 + {v'_x u}/{c^2}\right)}$; $v_y = \frac{v'_y \sqrt{1 - {u^2}/{c^2}}}{\left(1 + {v'_x u}/{c^2}\right)}$

$$\frac{\gamma}{\left(1 + {v'_x u}/{c^2}\right)} = \gamma' \Gamma; \ \Gamma = \frac{1}{\sqrt{1 - {u^2}/{c^2}}}$$

It is possible to show that: $c^2 - v^2 = \frac{c^2(c^2 - v'^2)(c^2 - u^2)}{\left(c^2 + v'_x u\right)^2}$

If we divide throughout by c^2 , invert, and take the square root, we can find:

$$\frac{1}{\sqrt{1 - {v^2}/{c^2}}} = \frac{1 + {v'_x u}/{c^2}}{\sqrt{1 - {v'^2}/{c^2}}\sqrt{1 - {u^2}/{c^2}}} \leftrightarrow \frac{1}{\sqrt{1 - {v^2}/{c^2}}\left(1 + {v'_x u}/{c^2}\right)} = \frac{1}{\sqrt{1 - {v'^2}/{c^2}}\sqrt{1 - {u^2}/{c^2}}}$$

Transformations of 4-vector Momentum

The four-vector $\widehat{p}$:

Consequently, starting from the expression p_x :

$$p_x = \frac{m_0}{\sqrt{1 - v^2/c^2}} \frac{v'_x + u}{\left(1 + {v'_x u}/{c^2}\right)} = \frac{m_0 (v'_x + u)}{\sqrt{1 - v'^2/c^2}\sqrt{1 - u^2/c^2}} = \Gamma(m' v'_x + m' u) = \Gamma\left(p'_x + u\frac{E'}{c^2}\right)$$

$$p_y = \frac{m_0 v_y}{\sqrt{1 - v^2/c^2}} = \frac{m_0}{\sqrt{1 - v^2/c^2}} \frac{v'_y \sqrt{1 - u^2/c^2}}{\left(1 + {v'_x u}/{c^2}\right)} = \frac{m_0 v'_y \sqrt{1 - u^2/c^2}}{\sqrt{1 - v'^2/c^2}\sqrt{1 - u^2/c^2}} = \frac{m_0 v'_y}{\sqrt{1 - v'^2/c^2}} = p'_y$$

In the same way, one can find: $p_z = p'_z$. Also, we have:

$$E = \frac{m_0 c^2}{\sqrt{1 - v^2/c^2}} = \Gamma m' c^2 \left(1 + {v_x u}/{c^2}\right) = \Gamma(E' + u . p'_x)$$

Transformations of 4-vector Momentum

$$\Gamma = \frac{1}{\sqrt{1 - {u^2}/{c^2}}}$$

Four-vector $\widehat{p}$ Lorentz Transformations

$T.L: S \rightarrow S'$	$T.L: S' \rightarrow S$
$p'_x = \Gamma\left(p_x - u\frac{E}{c^2}\right)$	$p_x = \Gamma\left(p'_x + u\frac{E'}{c^2}\right)$
$p'_y = p_y$	$p_y = p'_y$
$p'_z = p_z$	$p_z = p'_z$
$E' = \Gamma(E - u.p_x)$	$E = \Gamma(E' + u.p'_x)$

Applications

(a) Massless particles: Photons

The particular case of photon where $m_0 = m_{ph} = 0$, knowing that the photon energy is given as a function of its frequency (wave length) by the Einstein's relation: $E = h\nu = \frac{hc}{\lambda}$

This will bring us to redefine an equivalent of the photon momentum:

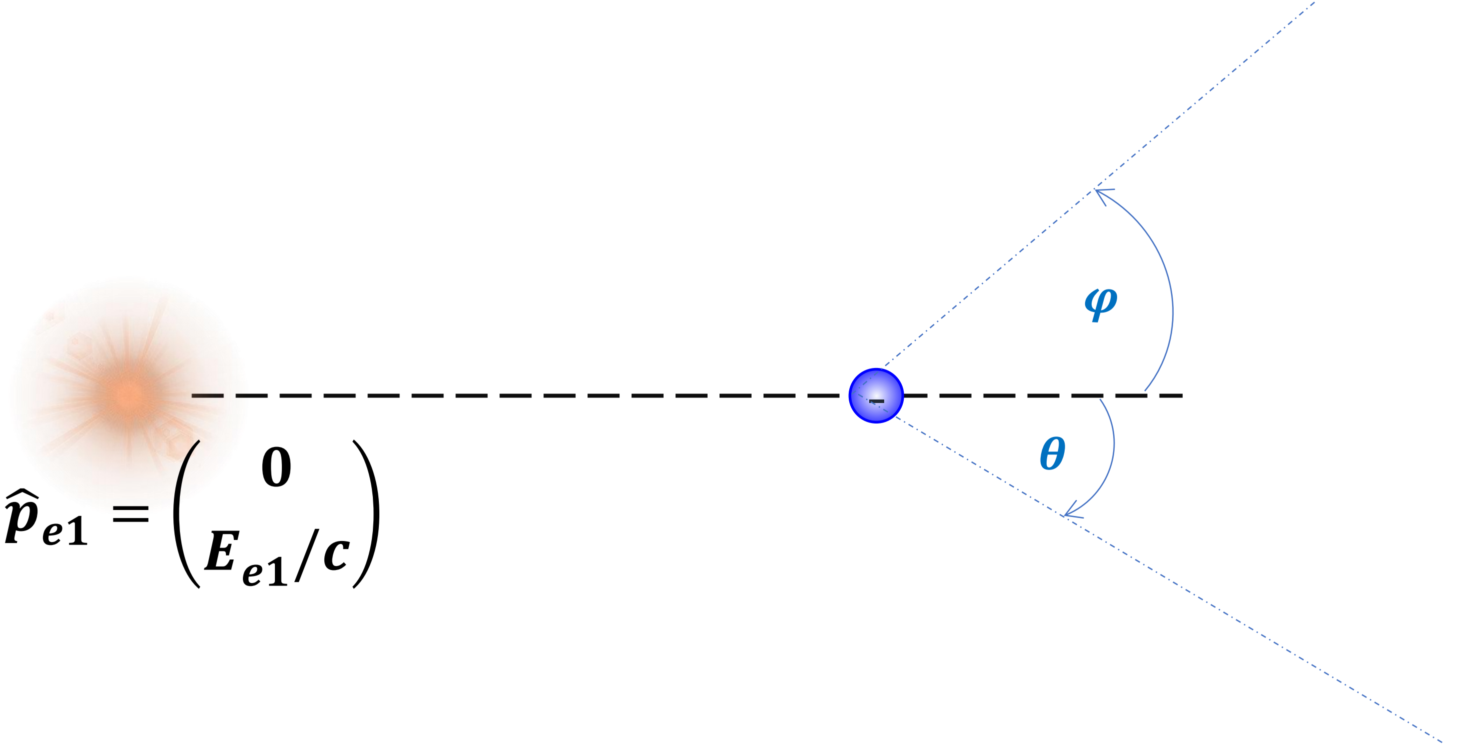
$$p = \frac{E}{c} = \frac{h\nu}{c} = \frac{h}{\lambda} = \frac{h}{2\pi}\frac{2\pi}{\lambda} = \hbar k \rightarrow \vec{p} = \hbar\vec{k}$$

The photonic four-vector could be then defined as:

$$\hat{p}_{ph} = \begin{pmatrix} \hbar k \\ 0 \\ 0 \\ h\nu/c \end{pmatrix} = \begin{pmatrix} \vec{p} \\ \frac{h\nu}{c} \end{pmatrix} \equiv \left(\vec{p}, \frac{h\nu}{c}\right)$$

Applications

(b) Compton Effect :



Before interaction (electron at rest):

$$\hat{p}_{ph1} = \begin{pmatrix} \vec{p}_{ph1} \\ \frac{h\nu_1}{c} \end{pmatrix} ; \hat{p}_{e1} = \begin{pmatrix} 0 \\ E_{e1}/c \end{pmatrix}$$

After interaction:

$$\hat{p}_{ph2} = \begin{pmatrix} \vec{p}_{ph2} \\ \frac{h\nu_2}{c} \end{pmatrix} ; \hat{p}_{e2} = \begin{pmatrix} \vec{p}_{e2} \\ E_{e2}/c \end{pmatrix}$$

The invariance of the total four-vector (electron + photon):

$$\hat{p}_{ph1} + \hat{p}_{e1} = \hat{p}_{ph2} + \hat{p}_{e1} \leftrightarrow \hat{p}_{ph1} + \hat{p}_{e1} - \hat{p}_{ph2} = \hat{p}_{e2}$$

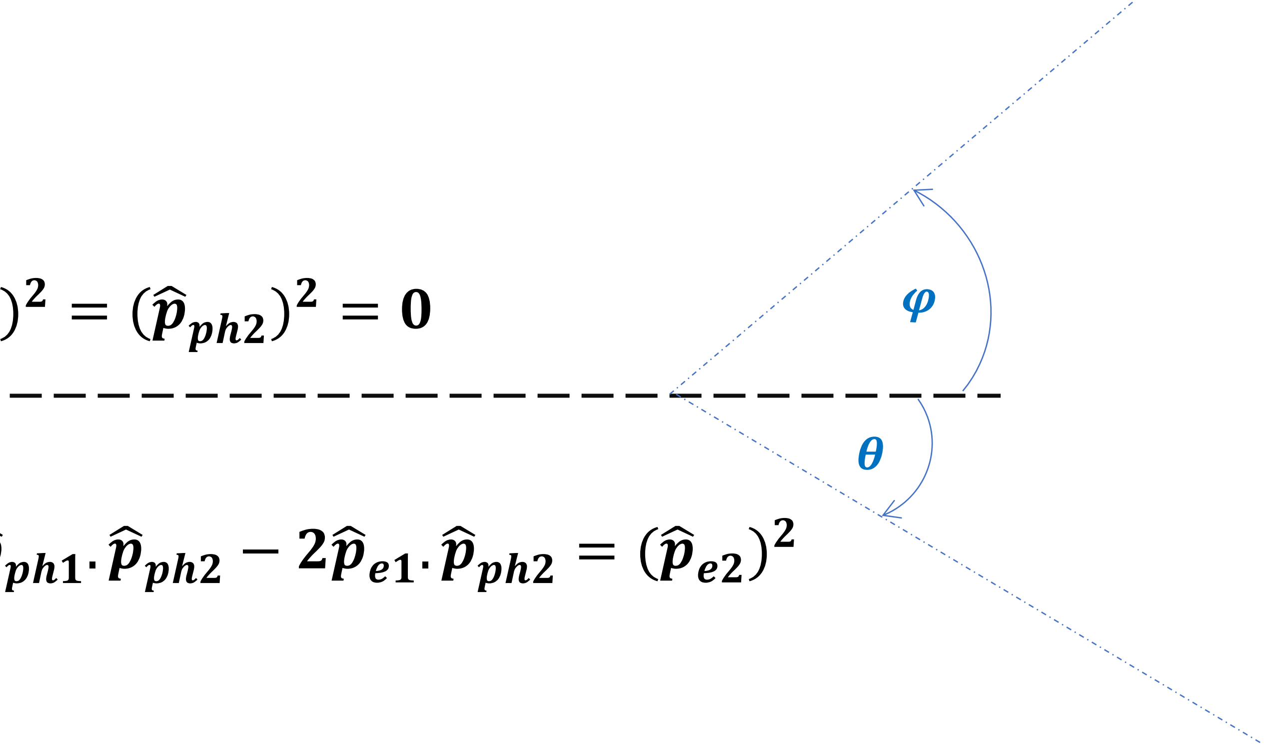
And by squaring both sides:

$$\left[\hat{p}_{ph1} + \hat{p}_{e1} - \hat{p}_{ph2}\right]^2 = (\hat{p}_{e2})^2$$

Applications

(b) Compton Effect :

We know that: $(\hat{p}_{e1})^2 = (\hat{p}_{e2})^2 = m_e^2 c^4; (\hat{p}_{ph1})^2 = (\hat{p}_{ph2})^2 = 0$



Thus, we obtain:

$$(\hat{p}_{e1})^2 + (\hat{p}_{ph1})^2 + (\hat{p}_{ph2})^2 + 2\hat{p}_{ph1}\cdot\hat{p}_{e1} - 2\hat{p}_{ph1}\cdot\hat{p}_{ph2} - 2\hat{p}_{e1}\cdot\hat{p}_{ph2} = (\hat{p}_{e2})^2$$

$$2\hat{p}_{ph1}\cdot\hat{p}_{e1} - 2\hat{p}_{ph1}\cdot\hat{p}_{ph2} - 2\hat{p}_{e1}\cdot\hat{p}_{ph2} = 0$$

Working on the dot products:

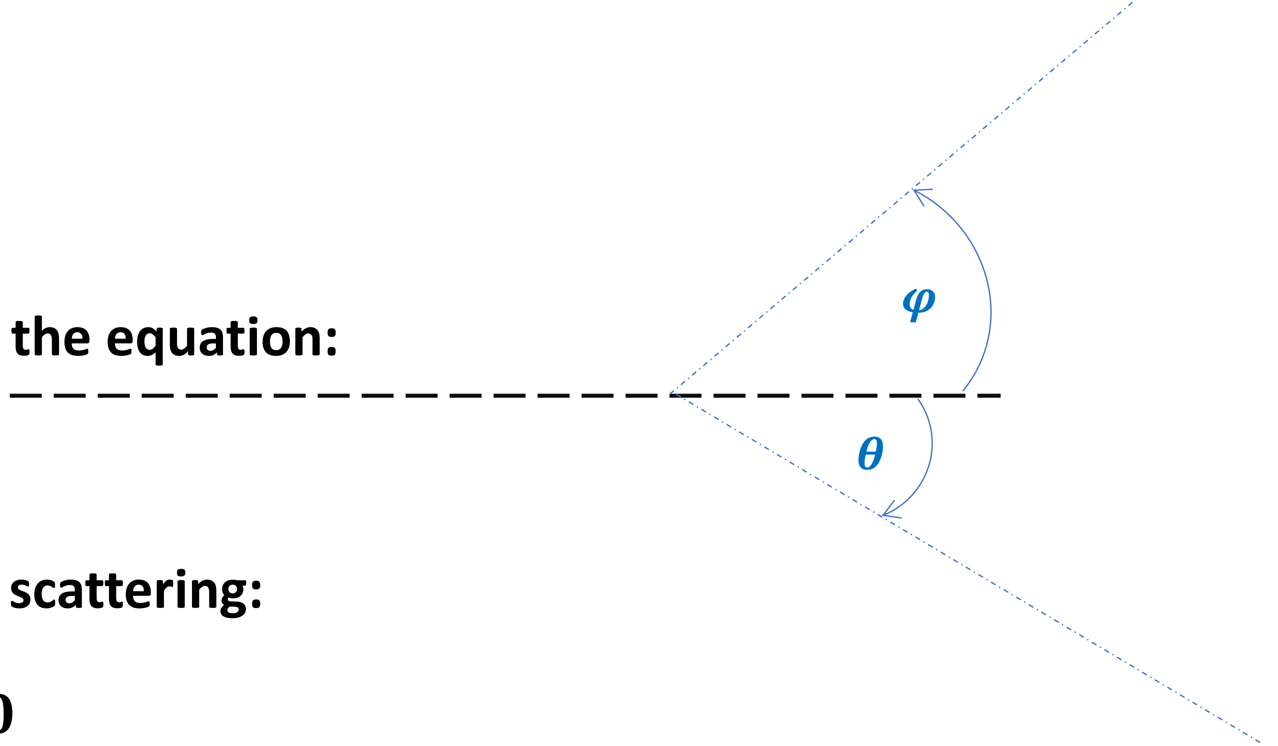
$$\hat{p}_{ph1}\cdot\hat{p}_{e1} = \begin{pmatrix} \vec{p}_{ph1} \\ \frac{h\nu_1}{c} \end{pmatrix}\cdot\begin{pmatrix} 0 \\ m_e c \end{pmatrix} = -m_e h\nu_1; \hat{p}_{e1}\cdot\hat{p}_{ph2} = \begin{pmatrix} 0 \\ m_e c \end{pmatrix}\cdot\begin{pmatrix} \vec{p}_{ph2} \\ \frac{h\nu_2}{c} \end{pmatrix} = -m_e h\nu_2$$

$$\hat{p}_{ph1}\cdot\hat{p}_{ph2} = \begin{pmatrix} \vec{p}_{ph1} \\ \frac{h\nu_1}{c} \end{pmatrix}\cdot\begin{pmatrix} \vec{p}_{ph2} \\ \frac{h\nu_2}{c} \end{pmatrix} = \vec{p}_{ph1}\cdot\vec{p}_{ph2} - \frac{h\nu_1}{c}\frac{h\nu_2}{c} = \frac{h\nu_1}{c}\frac{h\nu_2}{c}(\cos\varphi - 1)$$

Applications

(b) Compton Effect :

We insert the former results of dot products in the equation:



$$\hat{p}_{ph1} \cdot \hat{p}_{e1} - \hat{p}_{ph1} \cdot \hat{p}_{ph2} - \hat{p}_{e1} \cdot \hat{p}_{ph2} = 0$$

We obtain the wellknown relation of Compton scattering:

$$-m_e h\nu_1 + m_e h\nu_2 - \frac{h\nu_1}{c}\frac{h\nu_2}{c}(\cos\varphi - 1) = 0$$

$$\frac{h\nu_1}{c}\frac{h\nu_2}{c}(1 - \cos\varphi) = m_e(h\nu_1 - h\nu_2)$$

$$\left(\frac{1}{h\nu_2} - \frac{1}{h\nu_1}\right) = \frac{1}{m_e c^2}(1 - \cos\varphi)$$

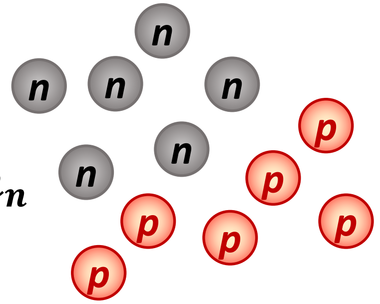
In terms of wave length the same relation could be written:

$$(\lambda_2 - \lambda_1) = \frac{hc}{m_e c^2}(1 - \cos\varphi) \leftrightarrow \Delta\lambda = 2\lambda_c \sin^2(\varphi/2)\,;\, \lambda_c = \frac{hc}{m_e c^2}$$

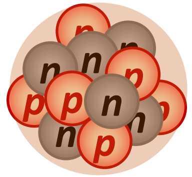
Applications

(c) Nuclear binding energy:

$Zm_p + Nm_n$



$M_N = A$



- The rest mass of a given nucleus made from Z protons and N neutrons, is given by the mass number$A$, which could be grossly considered as the sum of rest masses of this constituents:

$$M_{th} = Zm_p + Nm_n$$

- But experimentally speaking, the rest mass of the same nucleus is given by a measured value: $M_{exp} = M_N$ which is slightly different from the theoretical one $M_{th} = Zm_p + Nm_n$

- The existing difference between this two values $\Delta M = M_{th} - M_{exp} = Zm_p + Nm_n - M_N$ is called as "Mass excess", and its equivalent in energy :

$$E_B(Z, N) = \Delta M(Z, A)c^2[MeV]$$

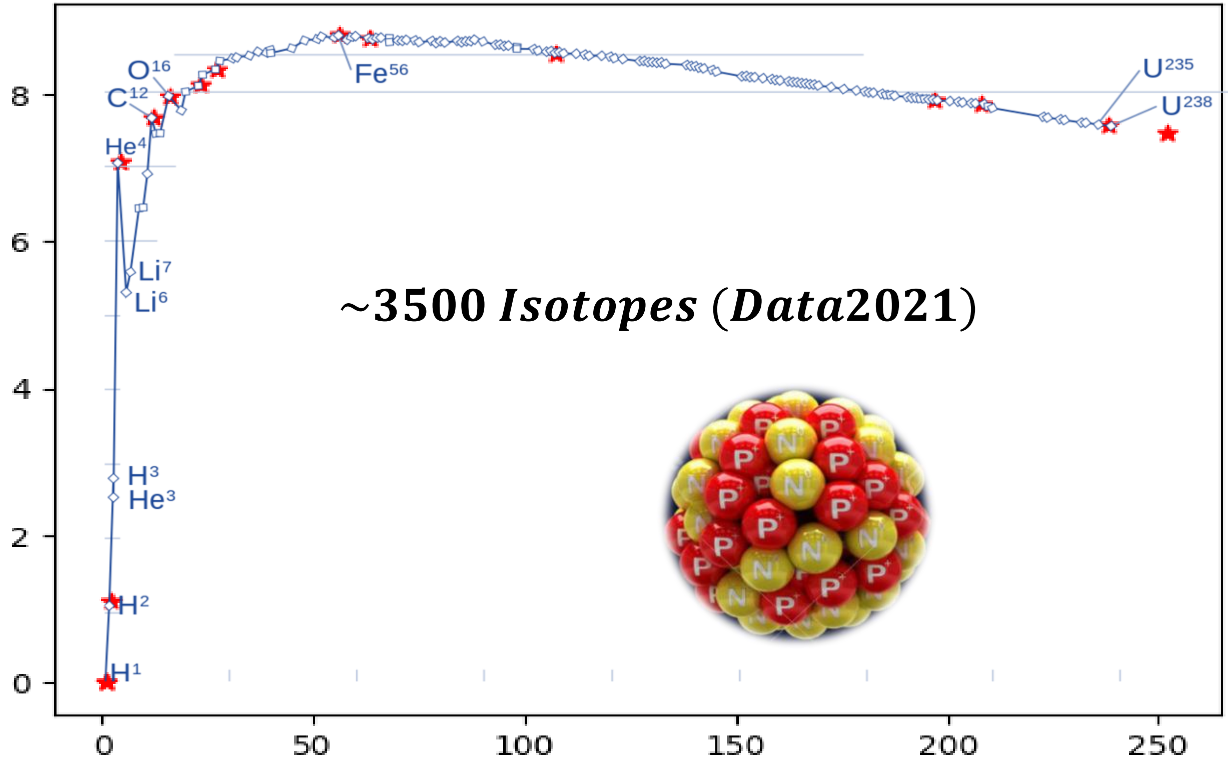
- *This is the nuclear binding energy of the nucleus. It represents the contribution of each nucleon with an amount of an equivalent energy to preserve the nucleus cohesion and stability.*

Applications

(c) Nuclear binding energy :

We can perform the calculation of the average binding energy per nucleon for any given isotope, since we have the experimental measure of its rest mass:

$$E_B/_A = \left(\frac{Zm_p + Nm_n}{A} - 1\right)c^2$$



Isotope(Z,N)	Mass [u.m.a]
${}^{1}_{0}n$	1.008665
${}^{1}_{1}H$	1.007825
${}^{2}_{1}D$	2.014102
${}^{4}_{2}He$	4.002603
${}^{12}_{6}C$	12.000000
${}^{16}_{8}O$	15.994915
${}^{23}_{11}Na$	22.989770
${}^{27}_{13}Al$	26.981538
${}^{56}_{26}Fe$	55.934942
${}^{63}_{29}Cu$	62.929601
${}^{107}_{47}Ag$	106.905093
${}^{197}_{79}Au$	196.966552
${}^{208}_{82}Pb$	207.976636
${}^{238}_{92}U$	238.050783
${}^{252}_{99}Es$	252.082972

Applications

(d) Nuclear fission energy:

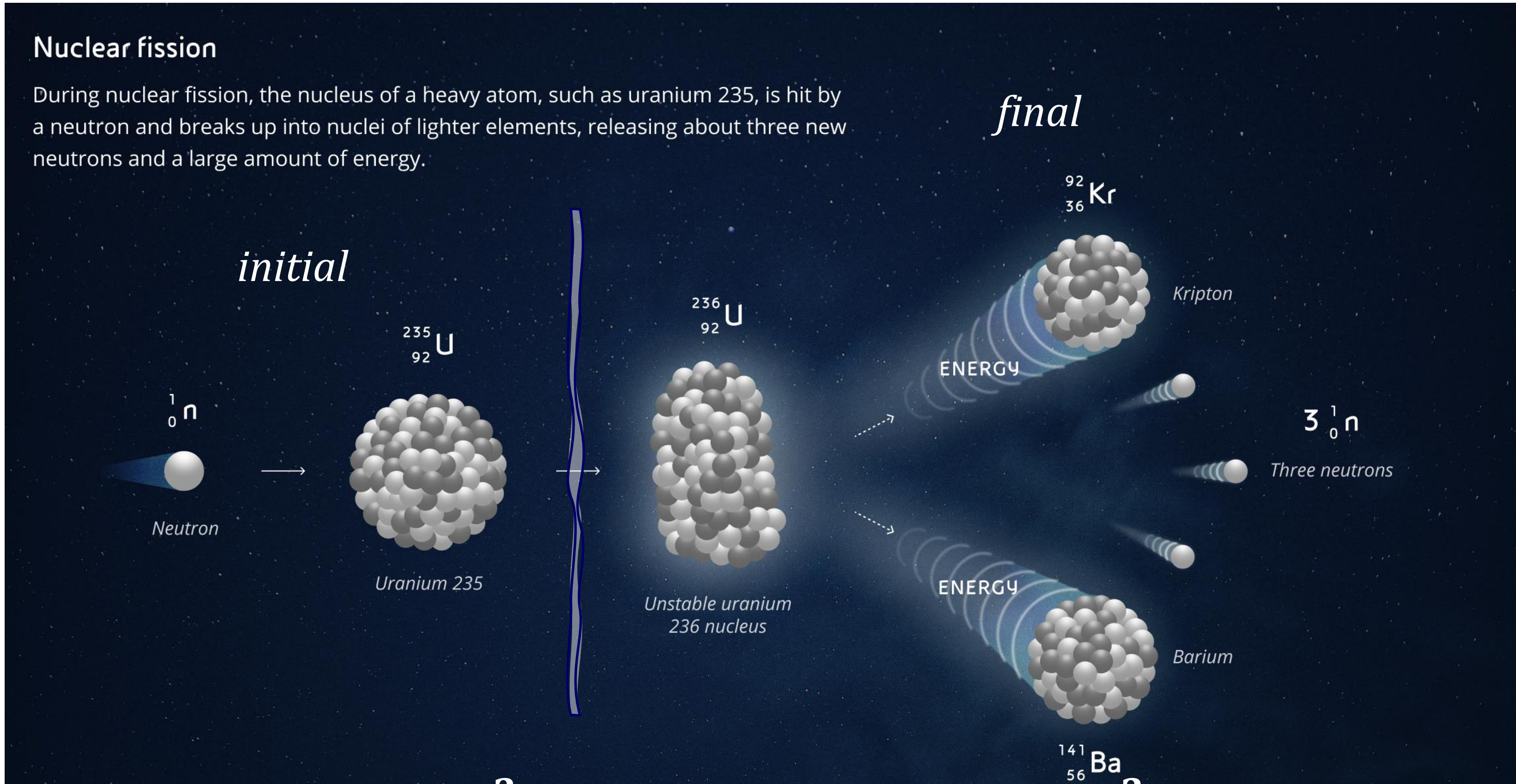
The nuclear fission reaction consists of the nucleus scission (heavy nuclei) into two fragments called Fission Products (FPs) with few emitted neutrons, and gammas:

$$ {}^{1}_{0}n + {}^{235}_{92}U \rightarrow {}^{236}_{92}U^{*} \rightarrow {}^{92}_{36}Kr + {}^{141}_{56}Ba + 3n + Q $$

The Q-value is the energy balance of the fission reaction, and it is defined by :

$$ \boldsymbol{Q = \sum_{i} m_i\, c^2 - \sum_{f} m_f c^2} $$

Isotope(Z, N)	$M_i[u.m.a]$
${}^{235}_{92}U$	235.043928
${}^{92}_{36}Kr$	91.926173
${}^{141}_{56}Ba$	140.914404



$$ \boldsymbol{Q = (m_n + M_{235})c^2 - (3m_n + M_{92} + M_{141})c^2} $$

Applications

(d) Nuclear fission energy :

Find the Q-value for each reaction?

$$^{235}_{92}U + ^{1}_{0}n \longrightarrow ^{236}_{92}U \longrightarrow ^{90}_{38}Sr + ^{144}_{54}Xe + 2\ ^{1}_{0}n$$

$$^{235}_{92}U + ^{1}_{0}n \longrightarrow ^{236}_{92}U \longrightarrow ^{87}_{35}Br + ^{146}_{57}La + 3\ ^{1}_{0}n$$

$$^{235}_{92}U + ^{1}_{0}n \longrightarrow ^{236}_{92}U \longrightarrow ^{96}_{37}Rb + ^{137}_{55}Cs + 3\ ^{1}_{0}n$$

$$^{235}_{92}U + ^{1}_{0}n \longrightarrow ^{236}_{92}U \longrightarrow ^{137}_{52}Te + ^{97}_{40}Zr + 2\ ^{1}_{0}n$$

$$^{235}_{92}U + ^{1}_{0}n \longrightarrow ^{236}_{92}U \longrightarrow ^{141}_{56}Ba + ^{92}_{36}Kr + 3\ ^{1}_{0}n$$

m_n	*1.00866*	***Q-value [MeV]***
M_{U235}	*235.0439*	
M_{Sr90}	*89.90773*	
M_{Xe144}	*143.93894*	
M_{Br87}	*86.92067*	
M_{La146}	*145.92569*	
M_{Rb96}	*95.93413*	
M_{Cs137}	*136.90709*	
M_{Zr97}	*96.91096*	
M_{Te137}	*136.9256*	
M_{Kr92}	*91.92617*	
M_{Ba141}	*140.9144*	

Applications

(d) Nuclear fission energy :

$${}^{235}_{92}U + {}^{1}_{0}n \longrightarrow {}^{236}_{92}U \longrightarrow {}^{90}_{38}Sr + {}^{144}_{54}Xe + 2\ {}^{1}_{0}n$$

$${}^{235}_{92}U + {}^{1}_{0}n \longrightarrow {}^{236}_{92}U \longrightarrow {}^{87}_{35}Br + {}^{146}_{57}La + 3\ {}^{1}_{0}n$$

$${}^{235}_{92}U + {}^{1}_{0}n \longrightarrow {}^{236}_{92}U \longrightarrow {}^{96}_{37}Rb + {}^{137}_{55}Cs + 3\ {}^{1}_{0}n$$

$${}^{235}_{92}U + {}^{1}_{0}n \longrightarrow {}^{236}_{92}U \longrightarrow {}^{137}_{52}Te + {}^{97}_{40}Zr + 2\ {}^{1}_{0}n$$

$${}^{235}_{92}U + {}^{1}_{0}n \longrightarrow {}^{236}_{92}U \longrightarrow {}^{141}_{56}Ba + {}^{92}_{36}Kr + 3\ {}^{1}_{0}n$$

m_n	*1.00866*	***Q-value [MeV]***
M_{U235}	*235.0439*	
M_{Sr90}	*89.90773*	***175.67***
M_{Xe144}	*143.93894*	
M_{Br87}	*86.92067*	***167.89***
M_{La146}	*145.92569*	
M_{Rb96}	*95.93413*	***172.67***
M_{Cs137}	*136.90709*	
M_{Zr97}	*96.91096*	***185.09***
M_{Te137}	*136.9256*	
M_{Kr92}	*91.92617*	***173.28***
M_{Ba141}	*140.9144*	

Applications

(d) Nuclear fission energy :

Products	Emitted energy [MeV]
Prompt energy (instantaneous)	**175 (~87%)**
Fission fragments	**163**
Fission neutrons	**5**
γ emission	**7**
Delayed energy (radioactivity)	**27 (~13%)**
β decay (electrons)	**8**
ν emission (neutrinos)	**12**
γ emission	**7**
Total	**202**

Applications

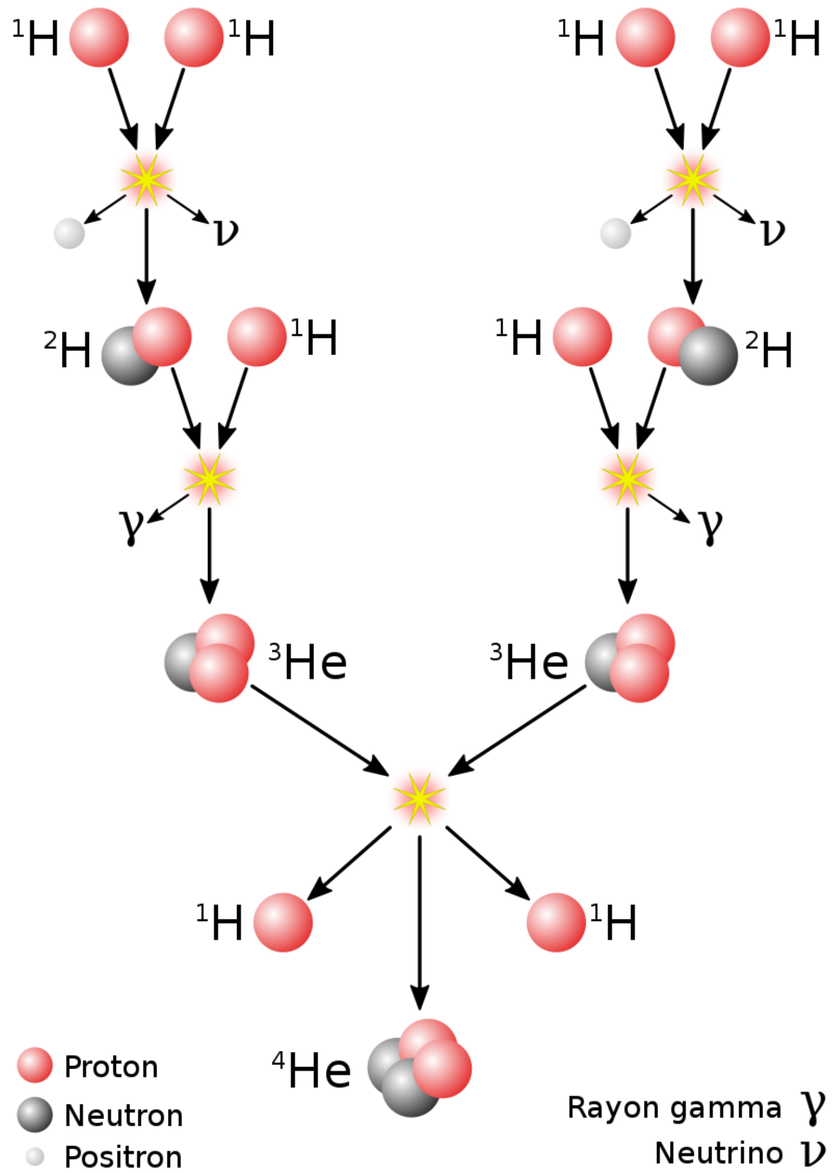
(e) Nuclear fusion energy:

The nuclear fusion reaction, in the contrary of the fission reaction, consists to merge two nucleus (light nuclei) to form a new one, and this is accompanied by a release of energy with few particles like neutrons and protons:

$$^{2}_{1}D + ^{2}_{1}D \rightarrow ^{3}_{2}He + ^{1}_{0}n + Q$$

$$Q = (2m_D)c^2 - (m_{He3} + m_n)c^2$$

Isotope(Z, N)	$M_i[u.m.a]$
$^{2}_{1}D$	235.043928
$^{3}_{2}He$	91.926173
$Q[MeV]$	3.27



Applications

(e) Nuclear fusion energy :

Find the Q-value for each reaction?

The nuclear fusion reaction, in the contrary of the fission reaction, consists to merge two nucleus (light nuclei) to form a new one, and this is accompanied by a release of energy with few particles like neutrons and protons:

$$^{2}_{1}D + ^{2}_{1}D \rightarrow ^{3}_{2}He + ^{1}_{0}n + Q$$

$$^{2}_{1}D + ^{2}_{1}D \rightarrow ^{3}_{1}T + ^{1}_{1}p + Q$$

$$^{2}_{1}D + ^{3}_{1}T \rightarrow ^{4}_{2}He + ^{1}_{0}n + Q$$

$$^{2}_{1}D + ^{3}_{2}He \rightarrow ^{4}_{2}He + ^{1}_{1}p + Q$$

m_n	*1.008664*
m_p	*1.007825*
m_D	*2.014101*
m_T	*3.016049*
m_α	*4.002603*
m_{He3}	*3.016029*

Applications

(e) Nuclear fusion energy :

The nuclear fusion reaction, in the contrary of the fission reaction, consists to merge two nucleus (light nuclei) to form a new one, and this is accompanied by a release of energy with few particles like neutrons and protons:

$$ {}_1^2D + {}_1^2D \rightarrow {}_2^3He(0.82MeV) + {}_0^1n(2.45MeV): Q = 3.27MeV $$

$$ {}_1^2D + {}_1^2D \rightarrow {}_1^3T(1.01MeV) + {}_1^1p(3.03MeV): Q = 4.04MeV $$

$$ {}_1^2D + {}_1^3T \rightarrow {}_2^4He(3.52MeV) + {}_0^1n(14.1MeV): Q = 17.61MeV $$

$$ {}_1^2D + {}_2^3He \rightarrow {}_2^4He(3.67MeV) + {}_1^1p(14.7MeV): Q = 18.37MeV $$