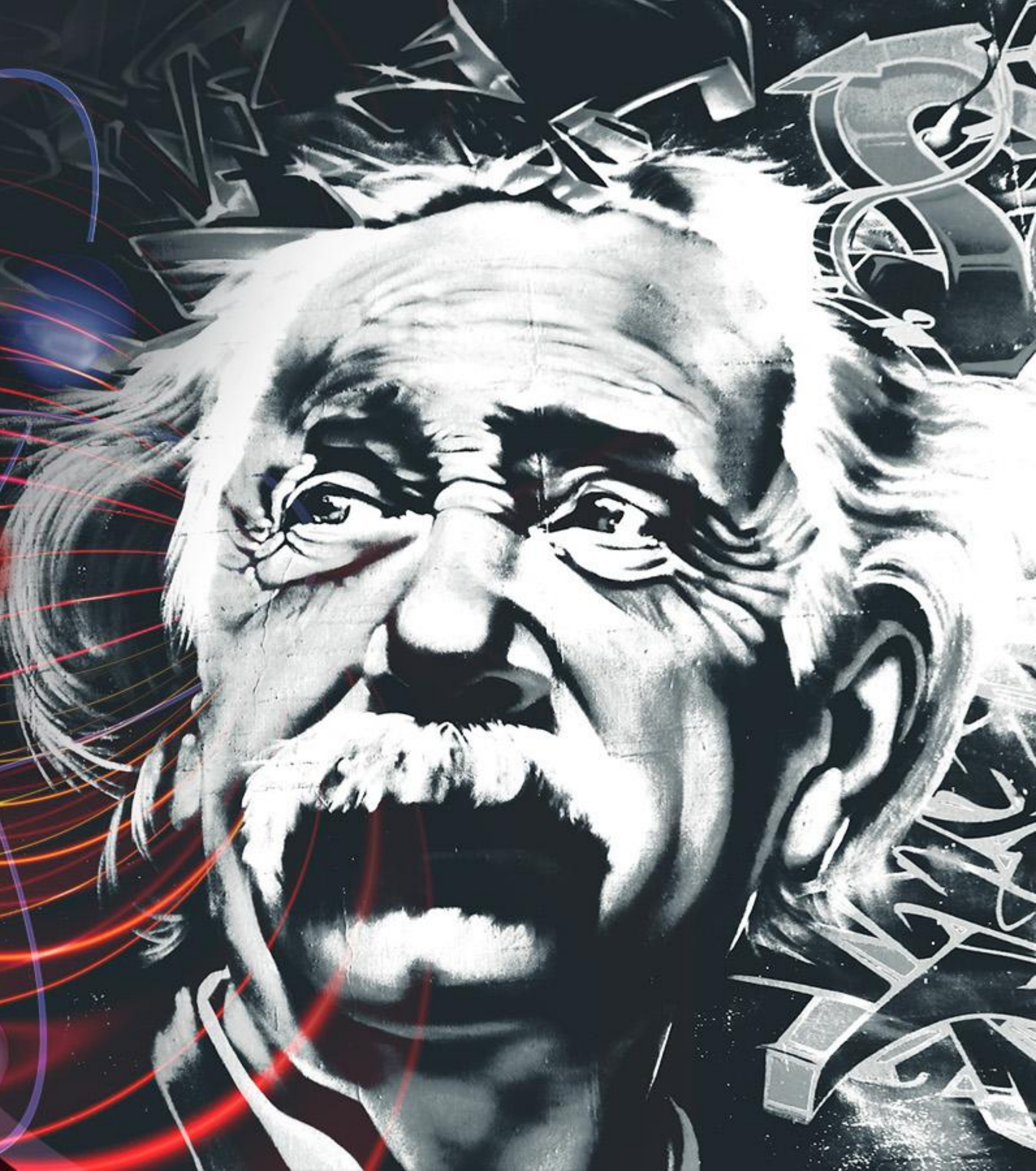
Khemis Miliana University
Faculty of Science and Technology
Dept. Science of the Matter

Relativité Restreinte

Chapitre 02:
Cinématique Relativiste

L3 Fundamental Physics

Dr. S.E. Bentridi



Chapter 02: Relativistic kinematics

- Lorentz Transformations
- Distance contraction and time dilatation
- Proper length and proper time
- Velocity Transformations
- Applications: Optical Aberration and relativistic Doppler effect
- Minkowski Space-Time and Quadri-vectors

Reminder: Einstein postulates

The Space-time invariant for Einstein:

$$\boldsymbol{s = \sum_i \Delta x_i^2 - c^2 t^2 = s' = \sum_i \Delta x'^2_i - c^2 t'^2 = 0}$$

By introducing the « time-light » dimension: $\boldsymbol{l = ct}$ and defining an imaginary coordinate $\boldsymbol{x_4 = il}$, the invariant $\boldsymbol{s}$ could be written in general form:

$$\boldsymbol{s = \sum_{i=1}^{4} \Delta X_i^2 = \Delta X_1^2 + \Delta X_2^2 + \Delta X_3^2 + \Delta X_4^2 = 0}$$

$$\boldsymbol{s' = \sum_{i=1}^{4} \Delta X'^2_i = \Delta X'^2_1 + \Delta X'^2_2 + \Delta X'^2_3 + \Delta X'^2_4 = 0}$$

$$\boldsymbol{S(x, y, z, t) = S'(x', y', z', t') = 0}$$

By identifying: $X_1 \equiv x$; $X_2 \equiv y$; $X_3 \equiv z$; $X_4 \equiv ict$ (similarly for x'_i)

Lorentz Transformations

According to the definition of Einstein invariant, it is possible to establish a set of transformations by respecting the equivalence principle of physical laws, to allow the passage from a stationary frame to a moving one. Thus, in general, one can write that:

$$x' = x'(x,y,z,t),\ y' = y'(x,y,z,t),\ z' = z'(x,y,z,t),\ t' = t'(x,y,z,t)$$

Thus, it is necessary that these transformations are linear (Space homogeneity):

$$x' = a_{11}x + a_{12}y + a_{13}z + a_{14}t$$

$$y' = a_{21}x + a_{22}y + a_{23}z + a_{24}t$$

$$z' = a_{31}x + a_{32}y + a_{33}z + a_{34}t$$

$$t' = a_{41}x + a_{42}y + a_{43}z + a_{44}t$$

Lorentz Transformations

The main objective for now, is to determine the different coefficients a_{ij} by considering Einstein postulates, in addition of the space homogeneity and reciprocity of measures in both frames.

- **To do it in a simple way, let consider that the frame (R') is moves always with respect to the stationary frame (R), along the x-axis with a constant velocity $\vec{u}$;**
- **At the initial instant $t = t' = 0$, both frames are coincided with each other at the same referential point $O = O'$;**
- **Since the relative displacement is only along OX', the coordinates (y, z) and (y', z') are completely independent of (x, t) and (x', t'), respectively;**

Lorentz Transformations

- Reciprocally, we can also consider the same for (x, t) and (x', t') to be independent from (y, z) and (y', z');
- Since the planes XY transforms always into $X'Y'$ by keeping the same orientation of $y - axes$ and $z - axes$, one can write directly that : $y' = y$ and $z' = z$;

All the previous considerations will lead to the following form of transformations:

$$\begin{cases} x' = a_{11}x + a_{12}y + a_{13}z + a_{14}t \\ y' = a_{21}x + a_{22}y + a_{23}z + a_{24}t \\ z' = a_{31}x + a_{32}y + a_{33}z + a_{34}t \\ t' = a_{41}x + a_{42}y + a_{43}z + a_{44}t \end{cases}$$

Lorentz Transformations

Therefore, let's consider the following set of transformations:

$$\boldsymbol{x' = a_{11}x + a_{14}t}$$

$$\boldsymbol{y' = y}$$

$$\boldsymbol{z' = z}$$

$$\boldsymbol{t' = a_{41}x + a_{44}t}$$

Since we have that $\boldsymbol{x'_0 = 0\ (R')}$, corresponds always to the position $\boldsymbol{x_0 = ut\ (R)}$, it follows that: $\boldsymbol{x' - x'_0 = x' = a_{11}(x - ut) \equiv a_{11}x + a_{14}t \rightarrow a_{14} = -a_{11}u}$:

$$\boldsymbol{x' = a_{11}x - a_{11}t}$$

$$\boldsymbol{t' = a_{41}x + a_{44}t}$$

Lorentz Transformations

We need now to determine the three coefficients: a_{11}, a_{41}, a_{44}!!!

To do that, we use the Einstein invariant formula: S': $x'^2 + y'^2 + z'^2 = c^2t'^2$

Indeed, by replacing x', y', z', t' in terms of x, y, z, t we get:

$$a_{11}^2(x - ut)^2 + y^2 + z^2 = c^2(a_{41}x + a_{44}t)^2$$

After necessary rearrangement, we obtain:

$$\left(a_{11}^2 - c^2a_{41}^2\right)x^2 + y^2 + z^2 - 2\left(ua_{11}^2 + c^2a_{41}a_{44}\right)xt = \left(c^2a_{44}^2 - u^2a_{11}^2\right)t^2$$

To preserve the same invariant in (R) S: $x^2 + y^2 + z^2 = c^2t^2$, it is necessary that:

$$a_{11}^2 - c^2a_{41}^2 = 1$$

$$ua_{11}^2 + c^2a_{41}a_{44} = 0$$

$$c^2a_{44}^2 - u^2a_{11}^2 = c^2$$

Lorentz Transformations

Homework (to hand over on 28 oct. 2024):

Solve the following system of equations, to find the expressions of a_{11}, a_{41}, a_{44} in terms of u and c :

$$\boldsymbol{a_{11}^2 - c^2 a_{41}^2 = 1} \qquad (1)$$

$$\boldsymbol{u a_{11}^2 + c^2 a_{41} a_{44} = 0} \qquad (2)$$

$$\boldsymbol{c^2 a_{44}^2 - u^2 a_{11}^2 = c^2} \qquad (3)$$

***Indication:** find the expressions of $(a_{44} + u a_{41})^2$ and $(a_{44} - u a_{41})(a_{44} + u a_{41})$ in terms of u^2 and c^2.*

Lorentz Transformations

$$a_{11} = a_{44} = \frac{1}{\sqrt{1 - u^2/_{c^2}}} \qquad a_{41} = -\frac{u}{c^2\sqrt{1 - u^2/_{c^2}}}$$

Finally, the transformations verifying the invariance between R and R':

$$\begin{matrix} x' = a_{11}x + a_{14}t \\ y' = y \\ z' = z \\ t' = a_{41}x + a_{44}t \end{matrix} \rightarrow \begin{matrix} x' = \frac{x - ut}{\sqrt{1 - u^2/_{c^2}}} \\ y' = y \\ z' = z \\ t' = \frac{t - \frac{u}{c^2}x}{\sqrt{1 - u^2/_{c^2}}} \end{matrix} \leftrightarrow \begin{matrix} x' = \gamma(x - ut) \\ y' = y \\ z' = z \\ t' = \gamma\left(t - \frac{u}{c^2}x\right) \end{matrix}$$

Hendrik Lorentz (1853-1928, NL)

Known as "Lorentz Transformations". We denote:

Relative (reduced) velocity: $\boldsymbol{\beta} = \frac{u}{c}$; *Contraction factor* $\boldsymbol{\alpha} = \sqrt{1 - \beta^2}$;

Lorentz factor: $\boldsymbol{\gamma} = \frac{1}{\sqrt{1 - \beta^2}}$

Lorentz Transformations

It is obvious that the same transformations are obtained when passing from R' to R, by inversing the sign of the velocity u into $-u$:

$$x = \frac{x' + ut'}{\sqrt{1 - {u^2}/{c^2}}} \qquad\qquad x = \gamma(x' + ut')$$

$$y = y' \qquad \leftrightarrow \qquad y = y'$$

$$z = z' \qquad\qquad z = z'$$

$$t = \frac{t' + \frac{u}{c^2}x'}{\sqrt{1 - {u^2}/{c^2}}} \qquad\qquad t = \gamma\left(t' + \frac{u}{c^2}x'\right)$$

Hendrik Lorentz
(1853-1928, NL)

Lorentz Transformations

Lorentz transformation (LT) could be summarized in the table below, with Galilean transformations (GT) as the result of the classical approximation:

$L.T: R \to R'$	$L.T: R' \to R$	$u \ll c$	$G.T: S \to S'$
$x' = \gamma(x - ut)$	$x = \gamma(x' + ut')$	$\beta = \frac{u}{c} \ll 1$	$x' \cong x - ut$
$y' = y$	$y = y'$	$\gamma = \frac{1}{\sqrt{1-\beta^2}} \cong 1$	$y' = y$
$z' = z$	$z = z'$		$z' = z$
$t' = \gamma\left(t - \frac{u}{c^2}x\right)$	$t = \gamma\left(t' + \frac{u}{c^2}x'\right)$	$\frac{u}{c^2} \lll 1$	$t' \cong t$

Hendrik Lorentz (1853-1928, NL)

Exercise 01 (Series n°02)

Test (30 min)

Study both factors $\alpha = \sqrt{1 - {v^2}/{c^2}}$ and $\gamma = \frac{1}{\sqrt{1 - {v^2}/{c^2}}}$, by completing the following table, then plot the variation of these factors as a function of the relative velocity $\beta = v/c$.

We give: $\boldsymbol{c = 299792458}\ [\boldsymbol{m.s^{-1}}]$

v	0.05c	0.10c	0.30c	0.50c	0.75c	0.80c	0.85c	0.90c	0.95c	0.99c	0.999c
α		0.995				0.600	0.523	0.436		0.141	
γ		1.005		1.15		1.66			3.2	7.1	

Exercise 01 (Series n°02)

Study both factors $\alpha = \sqrt{1 - {v^2}/{c^2}}$ and $\gamma = \frac{1}{\sqrt{1 - {v^2}/{c^2}}}$, by completing the following table, then plot the variation of these factors as a function of the relative velocity $\beta = v/c$.

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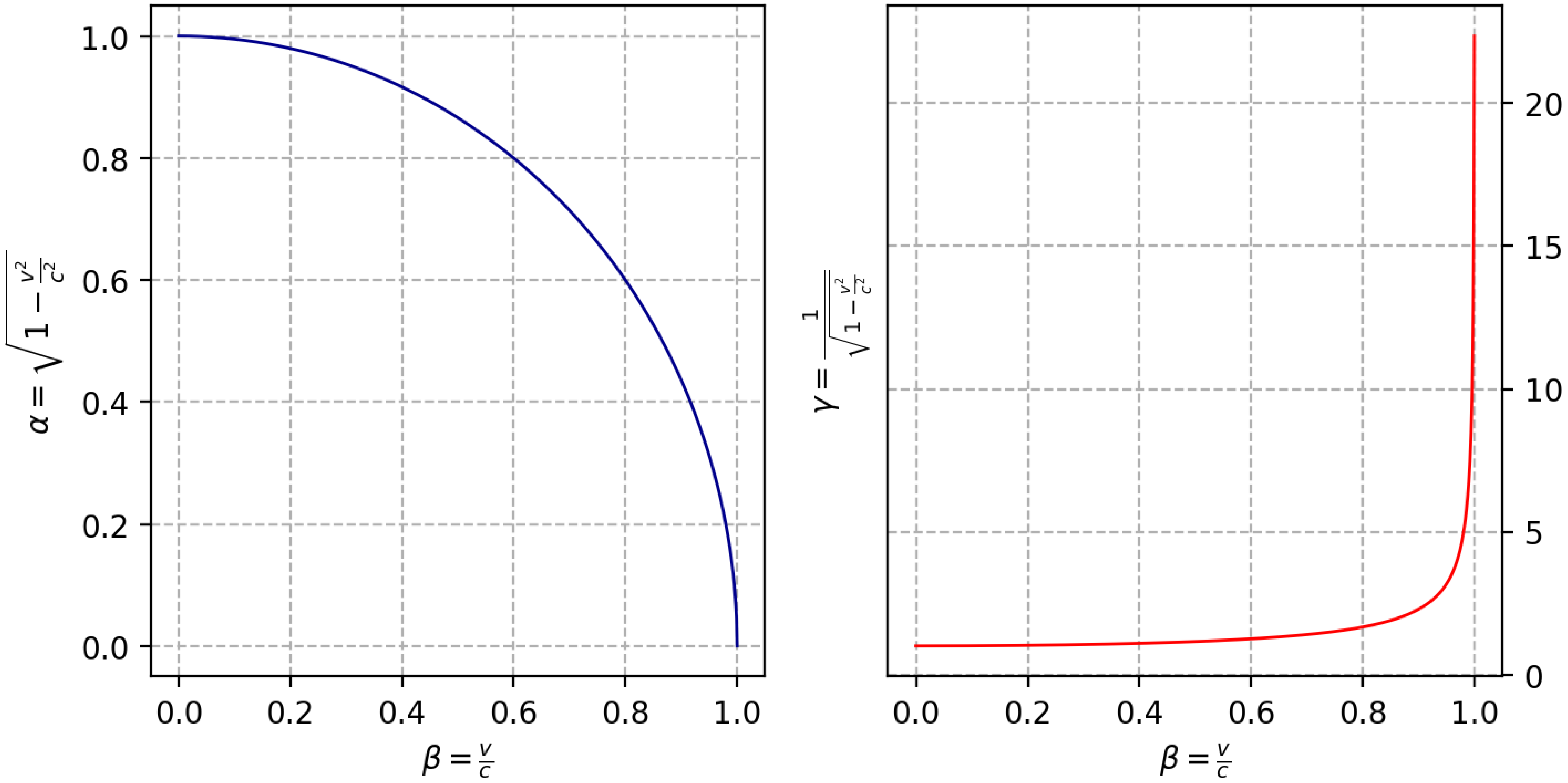
v	0.05c	0.10c	0.30c	0.50c	0.75c	0.80c	0.85c	0.90c	0.95c	0.99c	0.999c
α	0.999	0.995	0.954	0.866	0.661	0.600	0.523	0.436	0.312	0.141	0.045
γ	1.001	1.005	1.05	1.15	1.51	1.66	1.91	2.92	3.2	7.1	22.36

Exercise 01 (Series n°02)



Lorentz Transformations

One of the most important results of Lorentz Transformations (T.L) is the simultaneity principle:

Two events $A(x_A, t_A); B(x_B, t_B)$ are said simultaneous for a given observer, if this latter could measure these two events at the same time.

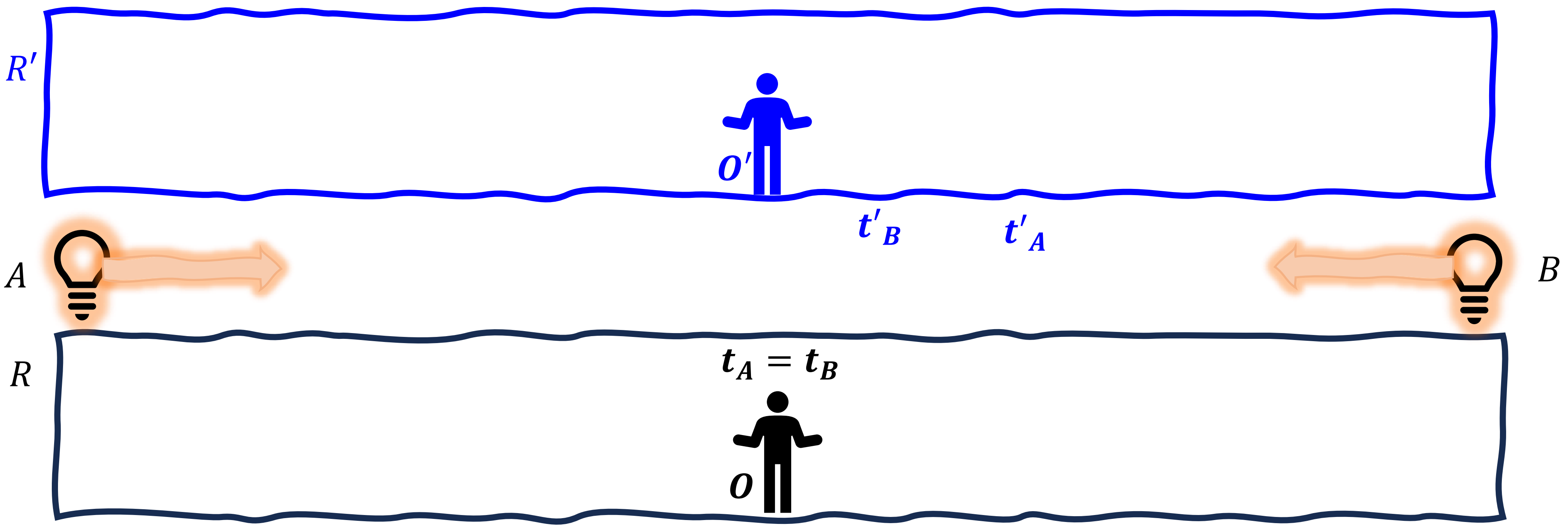
In classical physics, if an observer notes that two events are simultaneous, this implies that $t = t'$ according to Galilean Transformations (G.T), which also implies that any other observer will note that both events are simultaneous.

In opposite, in relativistic physics, if two events $A(x'_A, t'_A); B(x'_B, t'_B)$ are said simultaneous for a given moving observer $(O': u)$, they are not necessarily for another one (O) unless they occur in the same location $x'_A = x'_B$:

$$\boldsymbol{(O'): t'_A(x'_A) \equiv t'_B(x'_B) \rightarrow (O): t_B - t_A = \frac{(v/c^2)(x'_B - x'_A)}{\sqrt{1 - (v^2/c^2)}}}$$

Lorentz Transformations

Let's examine the following example where an observer O in a stationary frame (R) could observe two simultaneous and opposite light signals approaching him as shown in the figure below. Both signals will arrive at the same time to the observer O (distances are supposed equals).In other hand, another observer O' in a moving frame $(R$') with respect to (R) *will not see these two events at the same time.*



Exercise 02 (Series n°02)

- If a man (S’) on a moving train (with a constant velocity $\boldsymbol{u = 20[m.s^{-1}]}$) lights two cigarettes, one ten minutes after the other, then these events occur at the same place on his reference frame (the train). A ground observer (S), however, would assert that these same events occur at different places in his reference system (the ground). By using LT find what is the distance separation observed by (S). compare the findings with those given by GT.
- Suppose that (S’), seated at the center of a moving railroad car in the same previous train, observes that two men, on at each end of the car, light cigarettes simultaneously. The ground observer S, watching the railroad car go by, would assert (if he could make precise enough measurements) that the man in the back of the car lit his cigarette a little before the man in the front of the car lit his. Assuming that the distance separation in (S’) is $d = 25[m]$, what is the time separation observed by (S).

Exercise 02 (Series n°02)

1. If a man (S') on a moving train (with a constant velocity $\boldsymbol{u = 20[m.s^{-1}]}$) lights two cigarettes, one ten minutes after the other, then these events occur at the same place on his reference frame (the train). A ground observer (S), however, would assert that these same events occur at different places in his reference system (the ground). By using LT find what is the distance separation observed by (S). compare the findings with those given by GT.

Using T.L to pass from an observer $O'(x'_2 = x'_1)$ to another O :

$$x_2 - x_1 = \gamma\left(x'_2 - x'_1 + u(t'_2 - t'_1)\right) \cong u(t'_2 - t'_1) = 12000m = 12km$$

2. Suppose that (S'), seated at the center of a moving railroad car in the same previous train, observes that two men, on at each end of the car, light cigarettes simultaneously. The ground observer S, watching the railroad car go by, would assert (if he could make precise enough measurements) that the man in the back of the car lit his cigarette a little before the man in the front of the car lit his. Assuming that the distance separation in (S') is $d = 25[m]$, what is the time separation observed by (S).

Similarly, by using T.L, while in this case we know that: $t'_2 = t'_1$:

$$t_2 - t_1 = \gamma\left(t'_2 - t'_1 + \frac{u}{c^2}(x'_2 - x'_1)\right) \cong \frac{u}{c^2}(x'_2 - x'_1) = 5.56 \times 10^{-15}s \equiv 0$$

Length contraction and time dilatation

From L.T it follows two major considerations. The first one concerns the length of a solid body when measured by more than one observer.

Indeed, after the first equation from L.T, the measure of both ends for a given length in the frame (R') as a function of the measure in the frame (R) could be written as follows:

$$x'_1 = \gamma(x_1 + ut_1); x'_2 = \gamma(x_2 + ut_2)$$

Thus, the distance between the both measured points is given by:

$$l' = x'_2 - x'_1 = \gamma\big(x_2 - x_1 + u(t_2 - t_1)\big)$$

Since the length measure is done simultaneously $(t_2 = t_1)$:

$$l' = x'_2 - x'_1 = \gamma(x_2 - x_1) = \gamma l \rightarrow l = \alpha l' = l'\sqrt{1 - (u^2/c^2)}$$

This is the length contraction consequence of L.T: a measured length within proper frame where it is considered at rest, seems shorter than its measure taken by another observer in stationary frame for whom the measured length is considered in motion.

Length contraction and time dilatation

The second consequence of T.L is related to the time measure between two observers, one is moving with respect to the other.

Let's consider now, two events observed by (O') attached to the proper frame (R') in the same spatial point but a different instants: $A(x', t'_1)$ et $B(x', t'_2)$. These two event will be measured by an external observer (O) attached to the laboratory (stationary) frame (R) in the following way:

$$t_1 = \gamma\left(t'_1 - \frac{u}{c^2}x'\right); t_2 = \gamma\left(t'_2 - \frac{u}{c^2}x'\right)$$

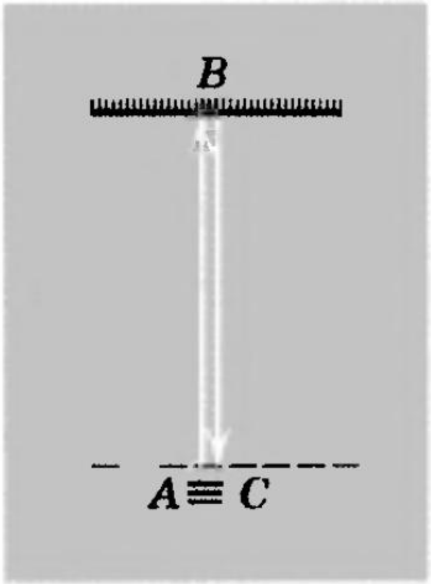
Thus, the time shift between both events could be given as:

$$\Delta t = t_2 - t_1 = \gamma(t'_2 - t'_1) = \gamma\Delta t' = \frac{\Delta t'}{\sqrt{1 - (u^2/c^2)}}$$

This is called the time dilatation: two non-proper synchronized clocks will measure a longer time interval than a clock related to both events happening at the same location but at different instants.

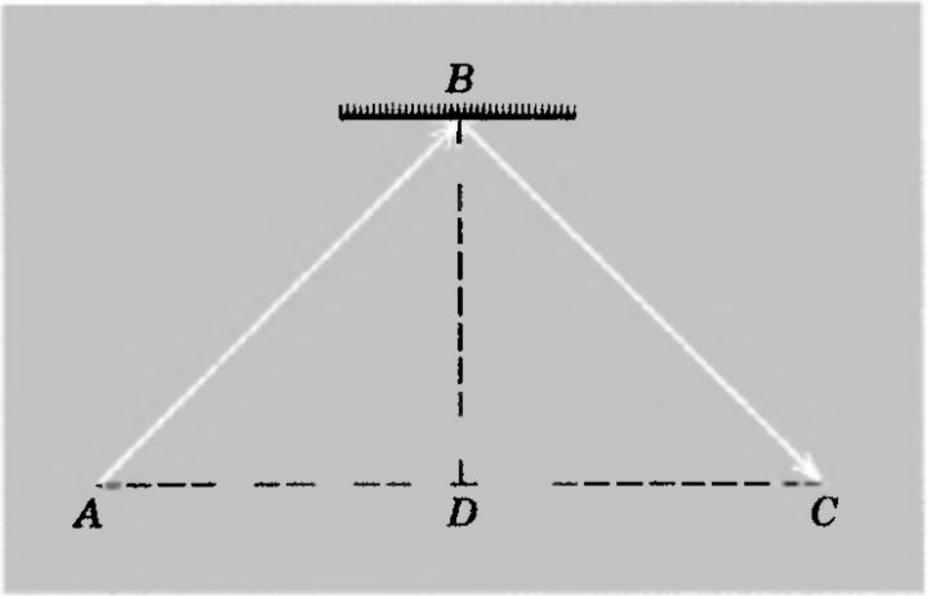
Length contraction and time dilatation

R'



(a)

(b)



(c)

R

(d)

Proper length and proper time

We will retain that the length measure of a rigid body will be done at rest in the proper frame of the body. This length denoted $l' = L_0$ is called the ***"proper length"***

This length measured by another observer which is not in the proper frame, is denoted L and it is related to the proper length by the relation of length contraction, known ***"Lorentz-Fitzgerald contraction"***:

$$\boldsymbol{L} = \boldsymbol{L_0}\sqrt{1 - (u^2/c^2)}$$

Proper length and proper time

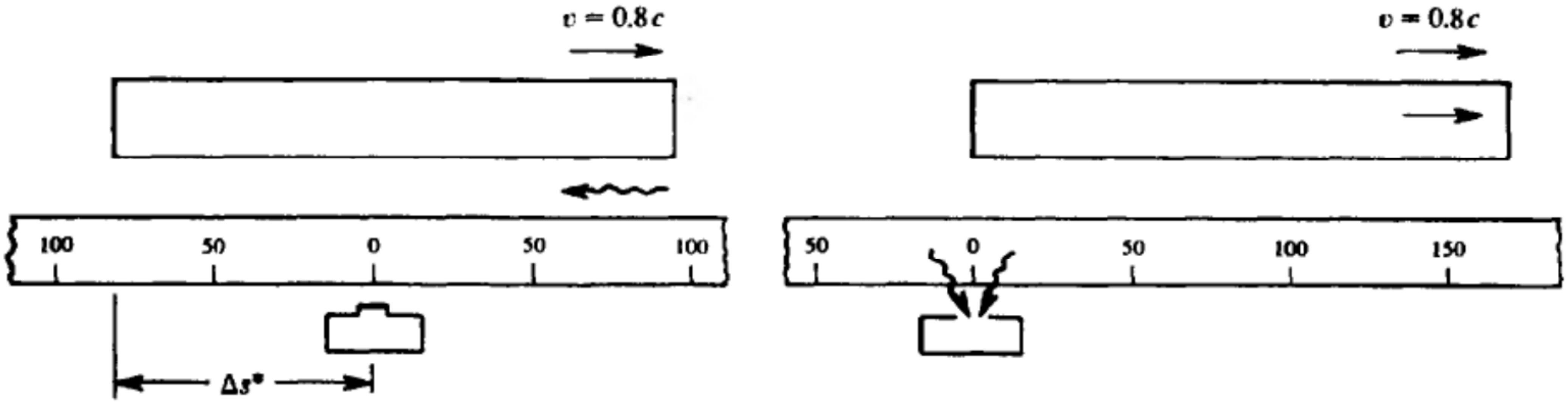
In the same way, a clock measuring a time interval between two events happening at the same point of the space, within the proper frame where both events are considered at rest, will give the "***proper time***" of these two events, when measured by the same clock, denoted : $\boldsymbol{\Delta\tau}$ **or** $\boldsymbol{\Delta t_0}$.

This time duration measured by another observer moving with respect to the stationary frame, will require two synchronized clocks, since the observer will see that the two events will occur at different places and at different instants:

$$\boldsymbol{\Delta t = \frac{\Delta\tau}{\sqrt{1-(u^2/c^2)}}}$$

Exercise 03 (Series n°02)

A rod moving from left to right. When the left end of the rod passes a camera, a picture is taken of the end together with a stationary calibrated meterstick. In the developed picture the left end of the rod coincides with the zero mark and the right end coincides with the 0.90m mark on the meterstick. If the rod is moving at 0.8c with respect to the camera, determine the actual length of the rod.



Exercise 03 (Series n°02)

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In order that the light signal from the right end of the rod will be recorded by the camera, it must have started from the 0.90m mark at an earlier time given by:

$$\Delta t = \frac{\Delta s}{c} = \frac{0.9}{3 \times 10^8} = 3 \times 10^{-9} s$$

During this time interval the left end of the rod will advance through a distance ΔS^* given by:

$$\Delta S^* = v.\Delta t = 0.8 \times 3 \times 10^8 \times 3 \times 10^{-9} = 0.72m$$

Therefore, the actual length of the rod is $L = 0.90 + 0.72 = 1.62m$

This result illustrates that photographing a moving rod will not give its correct length.

Exercise 04 (Series n°02)

Among the particles of high-energy physics are charged pions, particles of mass between that of the electron and the proton and of positive or negative electronic charge $\pi^{\pm}(q_{\pi} = \pm q_{e}, m_{\pi} = 139.57 MeV/c^{2})$. They can be produced by bombarding a suitable target in an accelerator with high-energy protons, the pions leaving the target with speeds close to that of light. It is found that the pions are radioactive and, when they are brought to rest, their half-life is measured to be $\tau_{\pi} = 2.6 \times 10^{-8} s$. A collimated pion beam, leaving the accelerator target at a speed $u = 0.99c$, is found to drop to half its original intensity around 55m from the target.

1. Are these results consistent with kinetic calculations?
2. Show how the time dilatation accounts for the measurements.
3. Show how the length contraction accounts for the measurements.

Exercise 04 (Series n°02)

$\tau_\pi = 2.6 \times 10^{-8} s$

1. Are these results consistent with kinetic calculations?

By using a simple calculation, during a time interval $\Delta t' = \tau_\pi$ the pions will travel a distance : $d' = u.\Delta t' = u.\tau_\pi = 0.99c \times 2.6 \times 10^{-8} = 7.71m$

The obtained results of classical kinematic calculations does not match with measurement of 55m.

2. Show how the time dilatation accounts for the measurements.

If the laboratory observation register a distance of $d = 55 \cong 7d'$, this indicates that the necessary time to the decay of the half of the initial beam, it is much longer than the proper period (measured within the proper frame of pions), thus, we observe a time dilatation due to relativistic effects $(\beta = \frac{u}{c} = 0.99)$

Exercise 04 (Series n°02)

$\tau_\pi = 2.6 \times 10^{-8} s$

3. Show how the length contraction accounts for the measurements.

By using the length dilatation law, one can find:

$$T_\pi = \frac{\tau_\pi}{\sqrt{1-\beta^2}} = 7.09 \times 2.6 \times 10^{-8} = 18.34 \times 10^{-8} s$$

$$d = u.\Delta t = uT_\pi = 54.6m \cong 55m$$

Exercise 05 (Series n°02)

A train $800m$ long (as measured by an observer on the train) is traveling at a speed of $u = 200km/h$. Two lightning bolts strike the ends of the train simultaneously ($t_A = t_B$) as determined by an observer O on the ground. What is the time separation as measured by an observer O' on the train?

We already have L.T allowing us to calculate the time laps (measured in R) $\Delta t = t_2 - t_1$, as a function of the measure on the train :

$$t_B = \gamma\left(t'_B + \frac{u}{c^2}x'_B\right); t_A = \gamma\left(t'_A + \frac{u}{c^2}x'_A\right) \rightarrow \Delta t = t_B - t_A = \gamma\left(t'_B - t'_A + \frac{u}{c^2}(x'_B - x'_A)\right)$$

Since : $t_B = t_A$, we get:

$$0 = \gamma\left(t'_B - t'_A + \frac{u}{c^2}(x'_B - x'_A)\right) \rightarrow \Delta t' = t'_B - t'_A = \frac{-u}{c^2}(x'_B - x'_A) = -\frac{200 \times 10^3 \times 800}{9 \times 10^{16} \times 3600}$$
$$\cong -5 \times 10^{-13} s$$

The observer O' on the train will see the A event happening before the B event.

Exercise 06 (Series n°02)

The space-time coordinates of two events as measured by O are:

$$A(x_1 = 6 \times 10^4 m, y_1 = z_1 = 0, t_1 = 2 \times 10^{-4} s);$$
$$B(x_2 = 12 \times 10^4 m, y_1 = z_1 = 0, t_2 = 10^{-4} s)$$

1. What must be the velocity of O' with respect to O if O' measures the two events to occur simultaneously?
2. What is the spatial separation of the two events as measured by O' ?

1. Let's use L.T to deduce the velocity of O' with respect to O:

$$t'_2 - t'_1 = 0 = \gamma\left(t_2 - t_1 - \frac{u}{c^2}(x_2 - x_1)\right) \rightarrow u = c^2\left(\frac{t_2 - t_1}{x_2 - x_1}\right) = -1.5 \times 10^8 \left[\frac{m}{s}\right] \leftrightarrow \frac{v}{c} = -0.5$$

The observer O' should move with a speed $v = 0.5c$ in the direction of the event A $(-OX)$ to see both events occur simultaneously

2. The observer O', will see both events separated by a distance $x'_2 - x'_1$:

$$x'_2 - x'_1 = \gamma(x_2 - x_1 - u(t_2 - t_1))$$
$$x'_2 - x'_1 = 1.15\left(6 \times 10^4 + 1.5 \times 10^8 \times \left(-1 \times 10^{-4}\right)\right) = 6.92 \times 10^4 m$$

Exercise 08 (Series n°02)

A $\mu - meson$ (μ) with an average lifetime $\tau_\mu = 2.2\mu s$ is created in the upper atmosphere at an elevation $h = 6000m$. When it is created it has a velocity $u = 0.998c$ in a direction toward the earth.

1. What is the average distance that it will travel before decaying, as determined by an observer O on the earth?
2. Consider an observer O' at rest with respect to the $\mu - meson$. How far will he measure the earth to approach him before the $\mu - meson$ disintegrates? Compare this distance with the distance he measures from the point of creation of the $\mu - meson$ to the earth.

Exercise 09 (Series n°02)

Suppose an observer O determines that two events are separated by a distance $x_B - x_A = 3.6 \times 10^8 m$ and occur with a delay $\Delta t = t_B - t_A = 2s$ apart.

1. What should be the velocity of an observer O' to see both events happening simultaneously?
2. Therefore, what is the proper time interval $(\Delta t')$ between the occurrence of these two events as measured by the observer O'?

1. If a given observer O' could observe two events A and B in his proper frame, then the observations of O and O' are related with L.T relationships:

$$x'_B - x'_A = 0 = \gamma\big(x_B - x_A - u(t_B - t_A)\big) \rightarrow x_B - x_A = u(t_B - t_A)$$

$$u = \frac{x_B - x_A}{t_B - t_A} = \frac{3.6 \times 10^8}{2} = 1.8 \times 10^8 [m/s] = 0.6c$$

2. In this case, the proper time interval $\Delta\tau$ could be deduced by using the time dilatation equation:

$$\Delta t = \frac{\Delta\tau}{\sqrt{1 - u^2/c^2}} \rightarrow \Delta\tau\sqrt{1 - u^2/c^2} = 2 \times 0.8 = 1.6s$$

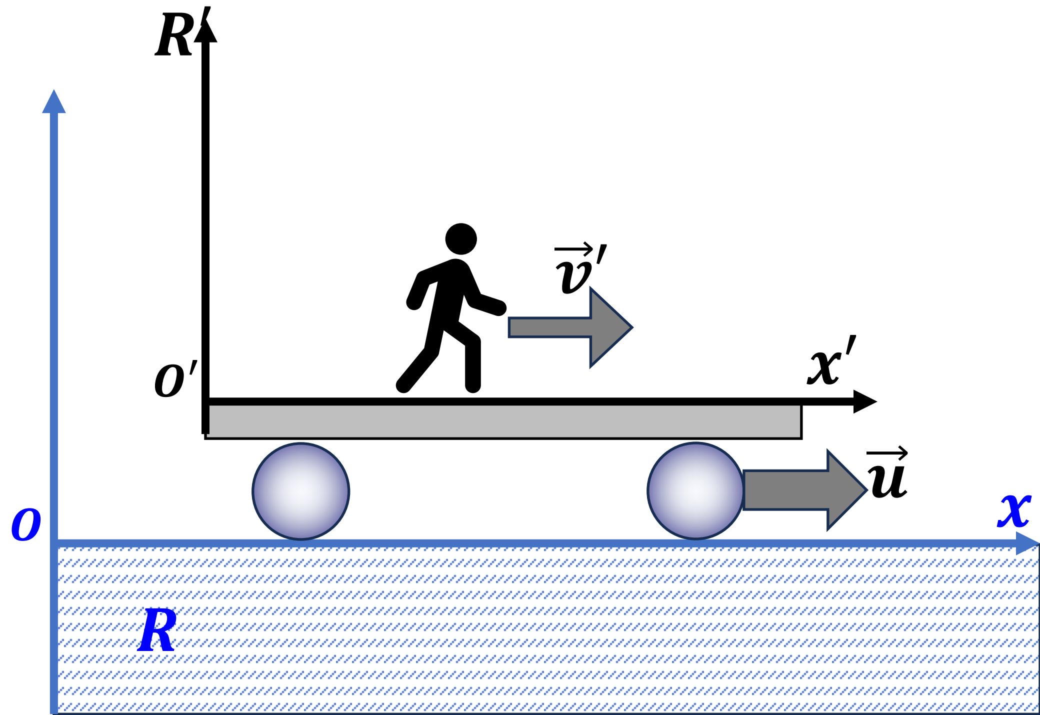
Velocity transformations

Let's remember that for the classical mechanics and according to G.T, the velocity addition theorem between two velocities from a moving frame (R') with respect to a stationary (R), is given by:

$$\overrightarrow{\boldsymbol{v}'} = \vec{\boldsymbol{v}} - \vec{\boldsymbol{u}} \leftrightarrow \vec{\boldsymbol{v}} = \overrightarrow{\boldsymbol{v}'} + \vec{\boldsymbol{u}}$$

What about relativistic mechanics?

Consider a passenger walking with a velocity $\overrightarrow{v'}$ inside the railroad car of train (as measured by an observer O' attached to the train R'). This train is moving with a velocity $\vec{u}$ with respect to a ground observer O. The whole motion is along $OX//O'X'$.



Velocity transformations

From the definition the velocity, which is the same in both frames:

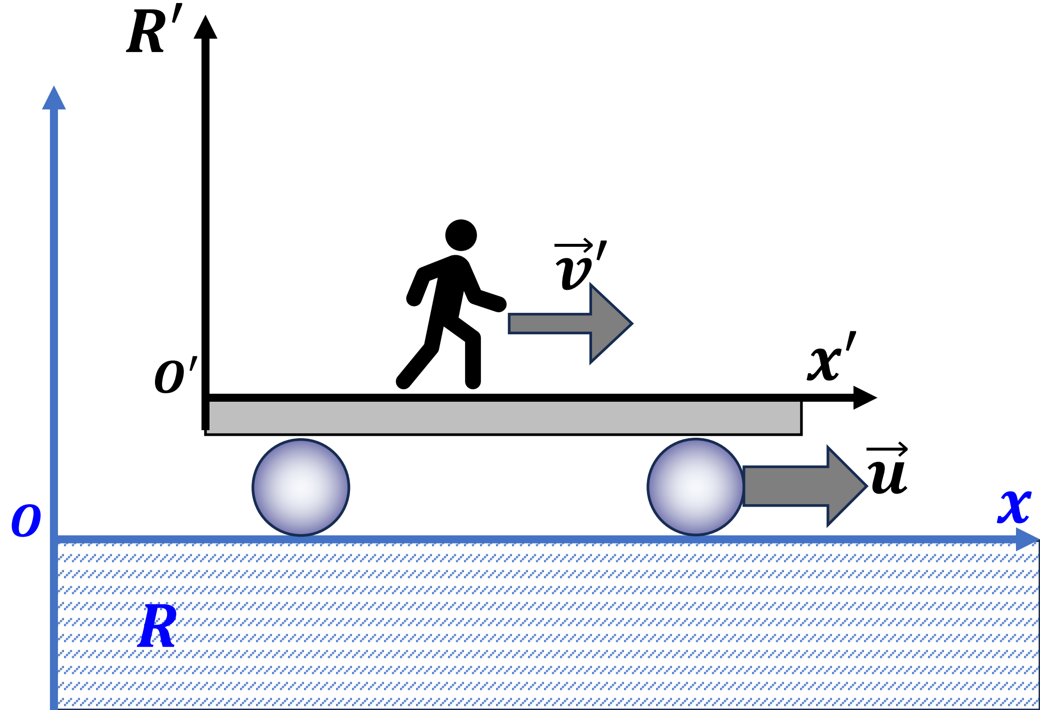
$$v_x = \frac{dx}{dt};\ v'_x = \frac{dx'}{dt'}$$

with: $dx = \gamma(dx' + udt')$ et $dt = \gamma(dt' + {}^{u}/_{c^2}\, dx')$

And by using L.T, we obtain for the x-component:

$$\boldsymbol{v_x = \frac{dx}{dt} = \frac{\gamma(dx' + udt')}{\gamma\left(dt' + {}^{u}/_{c^2}\, dx'\right)} = \frac{v'_x + u}{\left(1 + {}^{v'_x u}/_{c^2}\right)}}$$

We get the addition law of velocities according the Special Relativity of Einstein.



Velocity transformations

Similarly, this is could be used to calculate in general way the other components v_y et v_z.

Since: $y = y' \rightarrow dy = dy'$ and $z = z' \rightarrow dz = dz'$, we could write:

$$v_y = \frac{dy}{dt} = \frac{dy'}{\gamma\left(dt' + {}^{u}/_{c^2}\, dx'\right)} = \frac{dy'/dt'}{\gamma\left(1 + {}^{u}/_{c^2}\, dx'/dt'\right)} = \frac{v'_y\sqrt{1 - {}^{u^2}/_{c^2}}}{\left(1 + {}^{v'_x u}/_{c^2}\right)}$$

$$v_z = \frac{dz}{dt} = \frac{dz'}{\gamma\left(dt' + {}^{u}/_{c^2}\, dx'\right)} = \frac{dz'/dt'}{\gamma\left(1 + {}^{u}/_{c^2}\, dx'/dt'\right)} = \frac{v'_z\sqrt{1 - {}^{u^2}/_{c^2}}}{\left(1 + {}^{v'_x u}/_{c^2}\right)}$$

Velocity transformations

We summarize these velocity transformations in the following table:

$T.L: (R') \rightarrow (R)$	$T.L: (R) \rightarrow (R')$	$T.G: u \ll c$
$v_x = \dfrac{v'_x + u}{\left(1 + {v'_x u}/{c^2}\right)}$	$v'_x = \dfrac{v_x - u}{\left(1 - {v_x u}/{c^2}\right)}$	$v'_x \cong v_x - u$
$v_y = \dfrac{v'_y \sqrt{1 - {u^2}/{c^2}}}{\left(1 + {v'_x u}/{c^2}\right)}$	$v'_y = \dfrac{v_y \sqrt{1 - {u^2}/{c^2}}}{\left(1 - {v_x u}/{c^2}\right)}$	$v'_y \cong v_y$
$v_z = \dfrac{v'_z \sqrt{1 - {u^2}/{c^2}}}{\left(1 + {v'_x u}/{c^2}\right)}$	$v'_z = \dfrac{v_z \sqrt{1 - {u^2}/{c^2}}}{\left(1 - {v_x u}/{c^2}\right)}$	$v'_z \cong v_z$

Velocity transformations: $c + c = c$!!!

Let's apply the L.T of velocities for a photon as observed in a frame R' $(v' = c)$. This latter is moving with a velocity u in the same direction of the photon. So, could be observed in another stationary frame S with another velocity, different from light celerity c ?

$$v = \frac{v' + u}{\left(1 + {v'u}/{c^2}\right)} = \frac{c + u}{1 + {cu}/{c^2}} = \frac{c + u}{1 + u/c} = c\,\frac{c + u}{c + u} = c$$

If now, the frame R' is just another photon (with celerity $u = c$), how the first photon, moving with a velocity $(v' = c)$ with respect to the photon frame, could be observed by a stationary observe at R ?

$$v = \frac{v' + u}{\left(1 + {v'u}/{c^2}\right)} = \frac{c + c}{1 + {c.c}/{c^2}} = \frac{2c}{1 + 1} = c$$

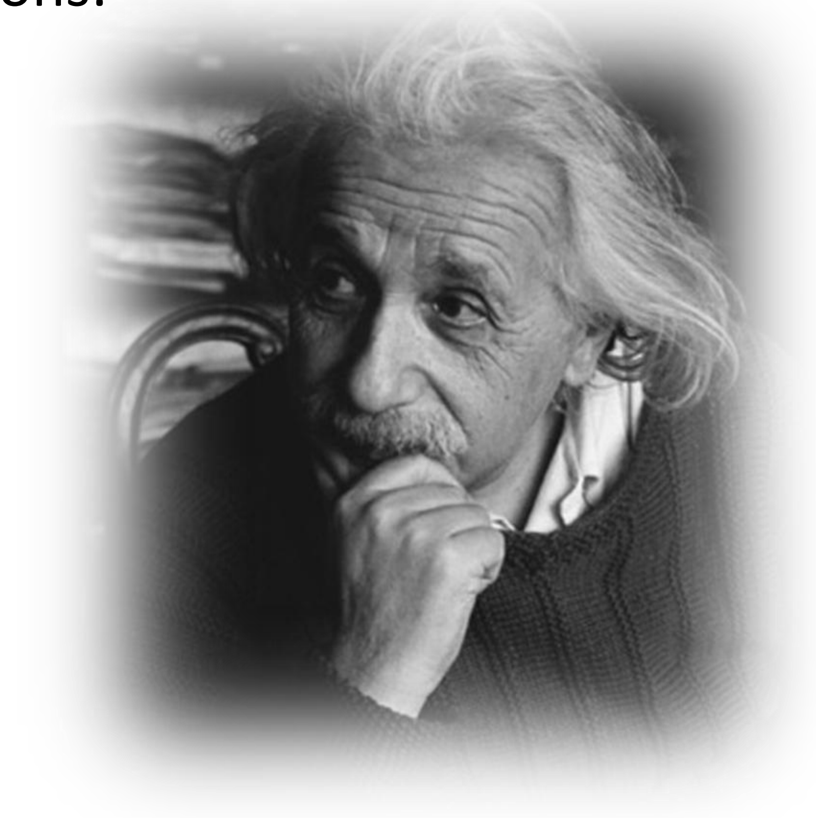
acceleration transformations

It could be an interesting exam exercise !!!

In the same way than previously, one could deduce the relativistic transformations for accelerations, but with a little bit tedious calculations:

$$a_x = \frac{dv_x}{dt} = \frac{d}{dt}\left(\frac{\boldsymbol{v'}_x + \boldsymbol{u}}{\left(\mathbf{1} + {}^{\boldsymbol{v'}_x \boldsymbol{u}}\!/_{\boldsymbol{c}^2}\right)}\right)$$

$$a_{y,z} = \frac{dv_{y,z}}{dt} = \frac{d}{dt}\left(\frac{\boldsymbol{v'}_{y,z}\sqrt{\mathbf{1} - {}^{\boldsymbol{u}^2}\!/_{\boldsymbol{c}^2}}}{\left(\mathbf{1} + {}^{\boldsymbol{v'}_x \boldsymbol{u}}\!/_{\boldsymbol{c}^2}\right)}\right)$$



with :

$$\frac{dv'_x}{dt'} = a'_x; \frac{dv'_y}{dt'} = a'_y; \frac{dv'_z}{dt'} = a'_z$$

Exercise 10 (Series n°02)

Two electrons are ejected in opposite directions from radioactive atoms in a sample of radioactive material at rest in the laboratory. Each electron has a speed of 0.67c as measured by a laboratory observer O.

1. **What is the speed of one electron as measured from the other considered as a moving observer O', according to the classical velocities' addition law? Comment!**

2. **Reexamine this problem by using the relativistic law of velocities' addition.**

Solution:

1. Using GT transformations (on the same axis):

$$v = v' + u \rightarrow v' = v - u = 0.67c - (-0.67c) = 1.34c$$

This is no-sense result, since there are no particles with speed over light speed in free space!!!

2. Let's use now LT transformations:

$$v' = \frac{v - u}{1 - vu/c^2} = \frac{0.67c + 0.67c}{1 + (0.67)^2} = \frac{1.34}{1.45} = 0.92c$$

Exercise 11 (Series n°02)

Rocket A travels to the right and rocket B travels to the left, with velocities $\vec{v}_A = 0.8c\vec{\imath}$ et $\vec{v}_B = -0.6c\vec{\imath}$, respectively, relative to the earth. What is the velocity of rocket A measured from rocket B?

- Repeat the previous question, if rocket A travels now with same speed in the +y-direction relative to the earth.

Solution:

1. Using the velocities L.T along OX:

$$v'_A = \frac{v_A - v_B}{\left(1 - {v_A v_B}/{c^2}\right)} = \frac{0.8c - (-0.6c)}{(1 - 0.8(-0.6))} = 0.946c$$

Exercise 11 (Series n°02)

Rocket A travels to the right and rocket B travels to the left, with velocities $\vec{v}_A = 0.8c\vec{\imath}$ et $\vec{v}_B = -0.6c\vec{\imath}$, respectively, relative to the earth. What is the velocity of rocket A measured from rocket B?

- Repeat the previous question, if rocket A travels now with same speed in the +y-direction relative to the earth.

2. In this second case, both components of A could be calculated as follows:

$$v'_{Ax} = \frac{v_{Ax} - v_B}{\left(1 - {v_{Ax}v_B}/{c^2}\right)} = \frac{0 - (-0.6c)}{(1 - 0(-0.6))} = 0.6c;\ v'_{Ay} = \frac{v_{Ay}\sqrt{1 - {v_B}^2/c^2}}{\left(1 - {v_{Ax}v_B}/{c^2}\right)} = \frac{0.8c \times 0.8}{1} = 0.64c$$

We get:

$$\overrightarrow{v'_A} = 0.6c\vec{i'} + 0.64c\vec{j'} \rightarrow v'_A = \sqrt{v'^2_{Ax} + v'^2_{Ay}} = \sqrt{0.36 + 0.4096} = 0.877c$$

$$\tan\varphi' = \frac{v'_{Ay}}{v'_{Ax}} = \frac{0.64}{0.6} = 1.0667 \rightarrow \varphi' \cong 46°85'$$

Exercise 12 (Series n°02)

Consider a radioactive nucleus that moves with a constant speed $u = 0.5c$ relative to the laboratory. The nucleus decays and emits an electron e^- with a speed $v'_e = 0.9c$ relative to the nucleus along the direction of motion. Find the velocity of the electron in the laboratory frame.

Now, suppose that the nucleus decays by emitting an electron with the same speed in a direction perpendicular to the direction (the laboratory) motion as determined by an observer at rest with respect to the nucleus. Find the velocity of the electron as measured by an observer in the laboratory frame.

Solution:

By using the L.T for velocities (from R’ to R), one can find

$$v_e = \frac{v'_e + u}{\left(1 + {v'_e u}/{c^2}\right)} = \frac{0.9c + 0.5c}{1 + 0.9 \times 0.5} = 0.965c$$

Exercise 12 (Series n°02)

Now, suppose that the nucleus decays by emitting an electron with the same speed in a direction perpendicular to the direction motion as determined by an observer at rest (the laboratory) with respect to the nucleus. Find the velocity of the electron as measured by an observer in the laboratory frame.

Again, by using the L.T for velocities (from R' to R), but with new directions now:

$$v_x = \frac{v'_x + u}{\left(1 + {v'_x u}/{c^2}\right)} = \frac{0 + 0.5c}{1 + 0} = 0.5c$$

$$v_y = \frac{v'_y\sqrt{1 - {u^2}/{c^2}}}{\left(1 + {v'_x u}/{c^2}\right)} = \frac{0.9c\sqrt{1 - (0.5)^2}}{1 + 0} = 0.779c$$

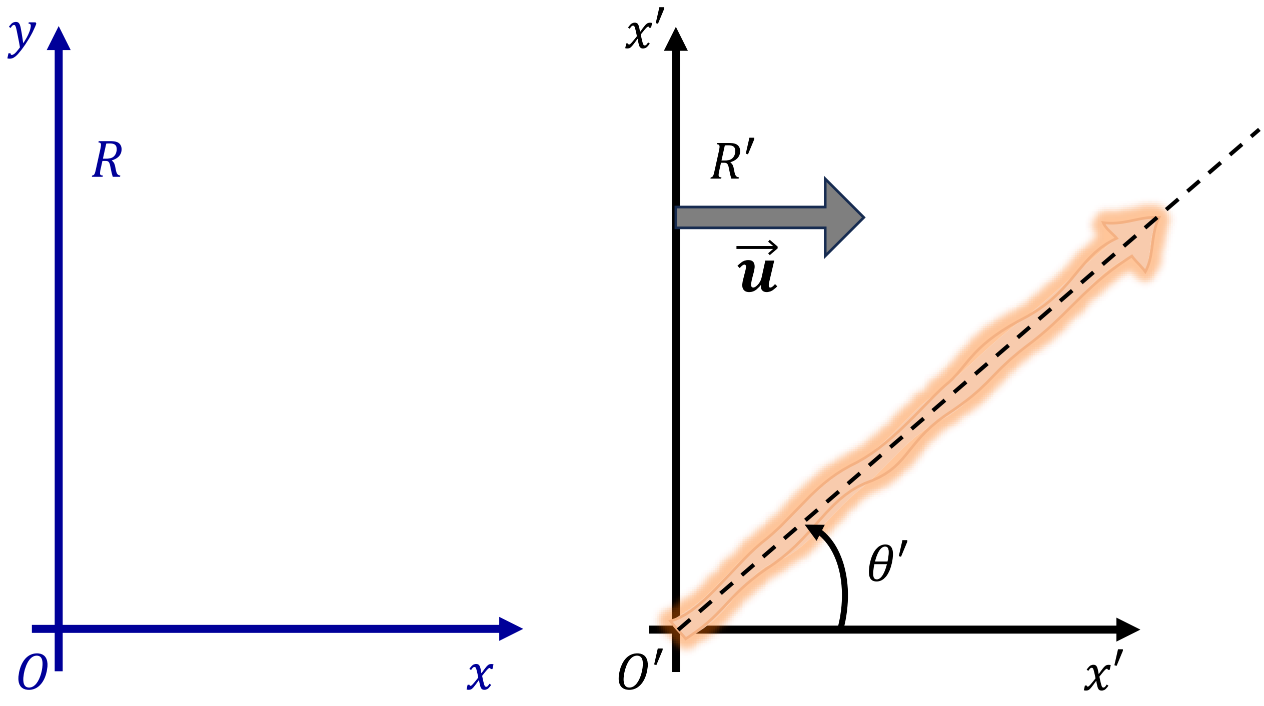
$v = \sqrt{v_x^2 + v_y^2} = 0.926c$ and $\tan\varphi = {v_y}/{v_x} = 1.56 \rightarrow \varphi \cong 57.3°$

Applications: Optical Aberration and Doppler effect

Consider the light wave propagation in a frame R' moving with a velocity u with respect to another stationary frame R.

The light ray propagates since on the plane $x' - y'$ from O', as shown in the opposite figure.

$$\boldsymbol{F(x,y,t) \propto \cos\left(\left[\vec{r}.\vec{k} - \omega t\right]\right)}$$



The wave function representing this propagation is a sinusoidal function, which could be considered under the form: ($k' = \frac{2\pi}{\lambda'}$; $\omega' = 2\pi\nu'$; $\lambda'\nu' = c$):

$$\boldsymbol{F(x',y',t') \propto \cos\left(\left[\vec{r'}.\vec{k'} - \omega' t'\right]\right) \equiv \cos\left(2\pi\left[\frac{\cos\theta'}{\lambda'}x' + \frac{\sin\theta'}{\lambda'}y' - \nu' t'\right]\right)}$$

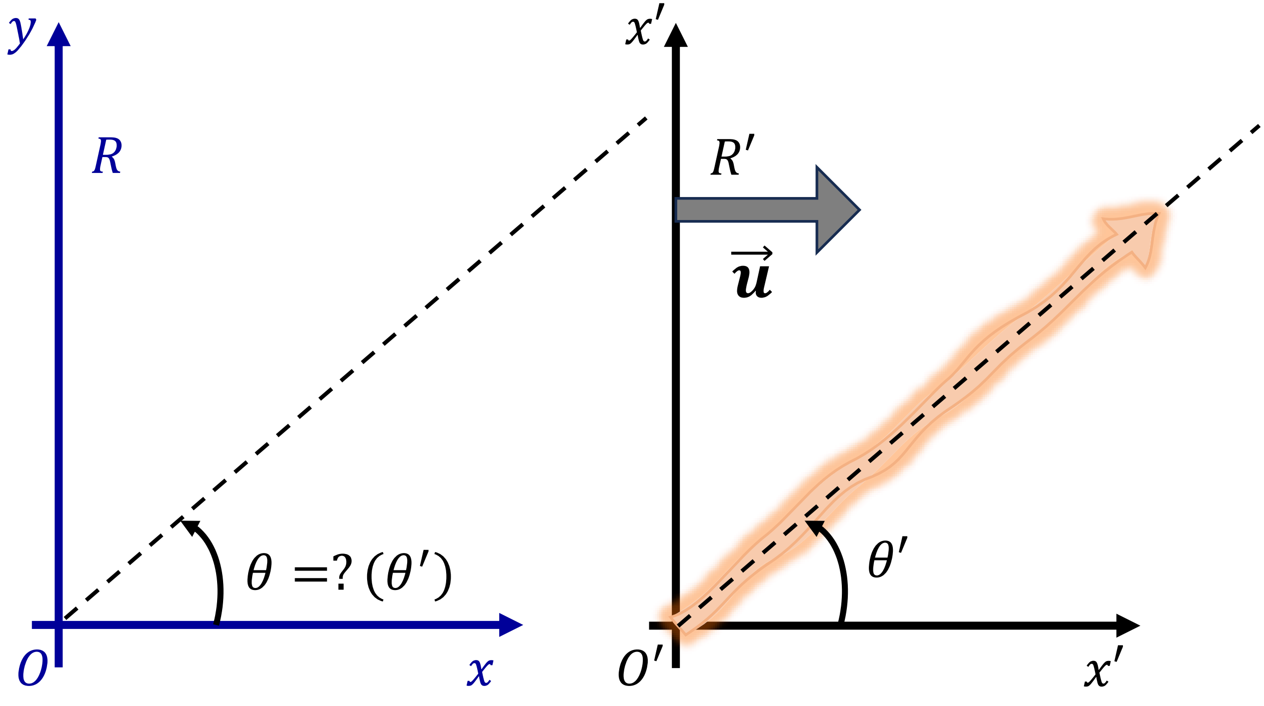
Applications: Optical Aberration and Doppler effect

How an observer in R, will observe the propagation of this wave?

Knowing that the wave function is invariant under L.T (Einstein postulates) :

$$x' = \gamma(x - ut)$$

$$t' = \gamma\left(t - \frac{u}{c^2}x\right)$$



Thus, by replacing the expressions of x' and t' as a function of x and t, then after rearrangements we could get the following expression:

$$F(x, y, t) \propto \cos\left(2\pi\left[\frac{\cos\theta' + \beta}{\lambda'\sqrt{1-\beta^2}}x + \frac{\sin\theta'}{\lambda'}y - \frac{(\beta\cos\theta' + 1)\nu'}{\sqrt{1-\beta^2}}t\right]\right)$$

Applications: Optical Aberration and Doppler effect

Now it is up to compare both expressions and identify the coefficients:

$$F(x,y,t) \propto \cos\left(2\pi\left[\frac{\cos\theta}{\lambda}x + \frac{\sin\theta}{\lambda}y - \nu t\right]\right)$$

$$F(x,y,t) \propto \cos\left(2\pi\left[\frac{\cos\theta' + \beta}{\lambda'\sqrt{1-\beta^2}}x + \frac{\sin\theta'}{\lambda'}y - \frac{(1+\beta\cos\theta')\nu'}{\sqrt{1-\beta^2}}t\right]\right)$$

This implies:

$$\frac{\cos\theta}{\lambda} = \frac{\cos\theta' + \beta}{\lambda'\sqrt{1-\beta^2}};\ \frac{\sin\theta}{\lambda} = \frac{\sin\theta'}{\lambda'};\ \nu = \frac{(1+\beta\cos\theta')\nu'}{\sqrt{1-\beta^2}}$$

With the relation: $\lambda'\nu' = \lambda\nu = c$

$$\theta =?\ \theta' \rightarrow \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sin\theta'\sqrt{1-\beta^2}}{\cos\theta' + \beta}$$

Applications: Optical Aberration and Doppler effect

- **Relativisitic optical aberration:**

This is a relativistic optical phenomenon expected by the theory of special relativity. It stipulates that a moving luminous object (star) with a constant speed u with respect to a stationary observer O, could be seen at a location slightly shifted from its real position (optical image). This shift is given by the following relationship deduced previously:

$$\boldsymbol{tan\,\theta = \frac{sin\,\theta}{cos\,\theta} = \frac{sin\,\theta'\sqrt{1-\beta^2}}{cos\,\theta' + \beta}}$$

The inverse relation is also obtained by replacing θ by θ' and u by $-u$:

$$\boldsymbol{tan\,\theta' = \frac{sin\,\theta'}{cos\,\theta'} = \frac{sin\,\theta\sqrt{1-\beta^2}}{cos\,\theta - \beta}}$$

Exercise 13 (Series n°02)

By using the relativistic aberration relationship given below, deduce the shift between angles θ' and θ, for $\theta' = \frac{\pi}{2}$ and $u = 3 \times 10^4 [m/s]$

$$tan\,\theta = \frac{sin\,\theta' \sqrt{1-\beta^2}}{cos\,\theta' + \beta}$$

Solution:

$$\theta = \frac{\pi}{2} = 1.57079632 \rightarrow sin\left(\frac{\pi}{2}\right) = 1; cos\left(\frac{\pi}{2}\right) = 0$$

$$\beta = \frac{u}{c} = \frac{3 \times 10^4}{3 \times 10^8} = 10^{-4} \lll 1 \rightarrow \sqrt{1-\beta^2} \cong 1$$

$$tan\,\theta' = \frac{\sqrt{1-\beta^2}}{\beta} = \frac{\sqrt{1-\beta^2}}{\beta} \cong \frac{1}{\beta} \equiv \frac{c}{u} = 10^4 \rightarrow \theta' = 1.57069632$$

$$\Delta\theta = \theta - \theta' = 10^{-4} \cong 0.0057°$$

Exercise 14 (Series n°02)

At $t = 0$ observer O emits a photon traveling at speed c in a direction $\theta = 60°$ with the x-axis. A second observer O', travels with a speed $u = 0.6c$ along the common x-x' axis. Using the relationship below, find what angle does the photon make with the x'-axis of O'?

$$tan\ \theta' = \frac{sin\ \theta\sqrt{1-\beta^2}}{cos\ \theta - \beta}$$

Solution:

$$\theta = 60° \rightarrow sin(60°) = 0.866; cos(60°) = 0.5$$

$$\beta = \frac{u}{c} = 0.6 \rightarrow \sqrt{1-\beta^2} = 0.8$$

$$tan\ \theta' = \frac{sin\ \theta\sqrt{1-\beta^2}}{cos\ \theta - \beta} = \frac{0.866 \times 0.8}{0.5 - 0.6} = -6.928 \rightarrow \theta' = -81°.78'$$

Applications: Optical Aberration and Doppler effect

- **Relativistic Doppler effect:**

Consider now, the last coefficient obtained by identification between both expressions of wave function (in R and R', respectively):

$$\boldsymbol{\nu} = \frac{(\mathbf{1} + \boldsymbol{\beta} \boldsymbol{cos}\, \boldsymbol{\theta}')\boldsymbol{\nu}'}{\sqrt{\mathbf{1} - \boldsymbol{\beta}^{2}}}$$

This expression translates the well-known phenomenon; Relativistic Doppler Effect.

Inversely, we could write for an observer O with respect to another one O':

$$\boldsymbol{\nu}' = \frac{(\mathbf{1} - \boldsymbol{\beta} \boldsymbol{cos}\, \boldsymbol{\theta})\boldsymbol{\nu}}{\sqrt{\mathbf{1} - \boldsymbol{\beta}^{2}}}$$

Applications: Optical Aberration and Doppler effect

- **Relativistic Doppler effect:**

We could also verify for very small velocities (with respect to light celerity), that the relativistic expression:

$$\boldsymbol{\nu} = \frac{(\mathbf{1} + \boldsymbol{\beta} \boldsymbol{cos}\ \boldsymbol{\theta}')\boldsymbol{\nu}'}{\sqrt{\mathbf{1} - \boldsymbol{\beta}^2}}$$

Will be reduced to the classical expression of Doppler effect ($\beta \ll 1 \rightarrow \sqrt{1 - \beta^2} \cong 1$)

$$\boldsymbol{\nu} \cong (\mathbf{1} + \boldsymbol{\beta} \boldsymbol{cos}\ \boldsymbol{\theta})\boldsymbol{\nu}'$$

Pour $\theta = 0$ (The source is moving from O along x-axis in the direction OX): $\boldsymbol{\nu} \cong (\mathbf{1} + \boldsymbol{\beta})\boldsymbol{\nu}'$

Pour $\theta = \pi$ (The source is moving toward O along x-axis in the direction OX): $\boldsymbol{\nu} \cong (\mathbf{1} - \boldsymbol{\beta})\boldsymbol{\nu}'$

Exercise 15 (Series n°02)

For both limit cases of $\theta = 0°$ et $\theta = 180°$, demonstrate that the relativistic Doppler effect will be reduced to the following expressions:

$$\nu' = \nu\sqrt{\frac{c-u}{c+u}}\,;\ \nu' = \nu\sqrt{\frac{c+u}{c-u}}$$

Examine also the case of $\theta = 90°$.

Solution:

1. Partant de l'expression: $\nu' = \frac{(1-\beta\cos\theta)\nu}{\sqrt{1-\beta^2}}$

Pour $\theta = 0 \rightarrow \nu' = \nu\frac{(1-\beta)}{\sqrt{1-\beta^2}} = \nu\sqrt{\frac{(1-\beta)^2}{(1-\beta)(1+\beta)}} = \nu\sqrt{\frac{1-\beta}{1+\beta}} = \nu\sqrt{\frac{c-u}{c+u}}$

De même pour $\theta = 180° \rightarrow \nu' = \nu\frac{(1+\beta)}{\sqrt{1-\beta^2}} = \nu\sqrt{\frac{(1+\beta)^2}{(1-\beta)(1+\beta)}} = \nu\sqrt{\frac{c+u}{c-u}}$

2. Pour le cas $\theta = 90° \rightarrow \nu' = \nu\frac{1}{\sqrt{1-\beta^2}} \leftrightarrow \nu = \nu'\sqrt{1-\beta^2}$

Exercise 16 (Series n°02)

1. A star is receding from the earth at a speed $\boldsymbol{u = 5 \times 10^{-3} c}$. What is the wavelength shift for the sodium $\boldsymbol{D_2}$ line ($\boldsymbol{\lambda_{D2} = 5890\text{Å}}$)?
2. Suppose that the Doppler shift in the sodium $\boldsymbol{D_2}$ line is $\boldsymbol{100\text{Å}}$ when the light is observed from a distant star. Determine the star's velocity of recession.

Solution:

1. Partant de l'expression($\theta = 0$):

$$\nu' = \nu\sqrt{\frac{c-u}{c+u}} \leftrightarrow \frac{c}{\lambda'} = \frac{c}{\lambda}\sqrt{\frac{c-u}{c+u}} \rightarrow \lambda' = \lambda\sqrt{\frac{c+u}{c-u}} = 5890\sqrt{\frac{1+0.005}{1-0.005}} = 5919.5\text{Å}$$

$$\Delta\lambda = \lambda' - \lambda = \mathbf{29.5\text{Å}}$$

2. Dans le second cas, reprenons la même expression:

$$\lambda' = \lambda\sqrt{\frac{c+u}{c-u}} \leftrightarrow 5990 = 5890\sqrt{\frac{c+u}{c-u}} \rightarrow 1.0342 = \frac{c+u}{c-u} \rightarrow u = 0.0168c$$

Minkowski Space-Time and Quadri-vectors

In 1907, H. Minkowski proposed the unification of the three spatial coordinates with the unique time coordinate, to form a space-time continuum, known also as Minkowski space

In fact, we already saw that with the definition of the "time-light" dimension$l = ct$ as well as the imaginary variable $x_4 = ict = il$, the Einstein invariant could be written in more concise and elegant way:

$$\boldsymbol{s} = \sum_{i=1}^{4} \Delta \boldsymbol{X}_i^2 = \Delta \boldsymbol{X}_1^2 + \Delta \boldsymbol{X}_2^2 + \Delta \boldsymbol{X}_3^2 + \Delta \boldsymbol{X}_4^2 = \boldsymbol{s}' = \sum_{i=1}^{4} \Delta \boldsymbol{X}'^2_i = \Delta \boldsymbol{X}'^2_1 + \Delta \boldsymbol{X}'^2_2 + \Delta \boldsymbol{X}'^2_3 + \Delta \boldsymbol{X}'^2_4 = \boldsymbol{0}$$

Hermann MINKOWSKI (1864-1909, Prussia)

By identifying: $\chi_1 \equiv x$; $\chi_2 \equiv y$; $\chi_3 \equiv z$; $\chi_4 \equiv ict$

Minkowski Space-Time and Quadri-vectors

- **Four-vector in the space-time:**

Any position vector in this space could be written:

$$\hat{\chi} = \hat{\chi}(\chi_1, \chi_2, \chi_3, \chi_4) \equiv \hat{\chi}(x, y, z, ict) \equiv \hat{\chi}(\vec{r}, ict) \equiv \begin{pmatrix} \chi_1 \\ \chi_2 \\ \chi_3 \\ ict \end{pmatrix}$$

Which is called the **“Four-vector position”**, defining an **“event”**.

The L.T could be then rewritten with the new notation:

$\boldsymbol{T.L: R \rightarrow R'}$	$\boldsymbol{T.L: R' \rightarrow R}$
$\chi'_1 = \boldsymbol{\gamma}(\chi_1 + \boldsymbol{i\beta}\chi_4)$	$\chi_1 = \boldsymbol{\gamma}(\chi'_1 - \boldsymbol{i\beta}\chi'_4)$
$\chi'_2 = \chi_2; \chi'_3 = \chi_3$	$\chi_2 = \chi'_2; \chi_3 = \chi'_3$
$\chi'_4 = \boldsymbol{\gamma}(\chi_4 - \boldsymbol{i\beta}\chi_1)$	$\chi_4 = \boldsymbol{\gamma}(\chi'_4 + \boldsymbol{i\beta}\chi'_1)$

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Minkowski Space-Time and Quadri-vectors

It is also usual to use tensor notation, where the sum operation between two 4-vectors is defined as:

$$\sum_{\mu,\nu=1}^{4} S^{\nu}\, T_{\mu} = \delta_{\mu}^{\nu} \sum_{\mu} S^{\mu} T_{\mu}\, , \delta_{\mu}^{\mu} = 1, \delta_{\mu}^{\nu \neq \mu} = 0$$

Then, to obtain "the measure" of a 4-vector:

$$\hat{A}^2 = \sum_{\mu 1}^{4} \hat{A}^{\mu}\, \hat{A}_{\mu} = A^1 A_1 + A^2 A_2 + A^3 A_3 + A^4 A_4$$

with: $A^i = A_i (i = 1,2,3); A^4 = -A_4$

In some notations, one can find:$A^0 = A_0; A^i = -A_i (i = 1,2,3)$

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Minkowski Space-Time and Quadri-vectors

Using tensor notation (without imaginary variable) :

$$\hat{X} = \hat{X}(X_1, X_2, X_3, X_4) \equiv \hat{X}(x, y, z, ct) \equiv \hat{X}(\vec{r}, ct) \equiv \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ ct \end{pmatrix}$$

Thus, the same four-vector position will verify the following L.T:

$\boldsymbol{T.L: R \to R'}$	$\boldsymbol{T.L: R' \to R}$
$\boldsymbol{X'_1 = \gamma(X_1 - \beta X_4)}$	$\boldsymbol{X_1 = \gamma(X'_1 + \beta X'_4)}$
$\boldsymbol{X'_2 = X_2; X'_3 = X_3}$	$\boldsymbol{X_2 = X'_2; X_3 = X'_3}$
$\boldsymbol{X'_4 = \gamma(X_4 - \beta X_1)}$	$\boldsymbol{X_4 = \gamma(X'_4 + \beta X'_1)}$

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In this notation, it is possible to see the symmtery of L.T (between X_1, X_4 and X'_1, X'_4)!!!

Minkowski Space-Time and Quadri-vectors

- **Four-vector velocity:**

A moving body with a velocity $\vec{u}$ with respect to another observer, will measure its proper time (with respect to its proper frame R') and the relation between the measured time by the observer and the proper time is given by the dilatation time relation: $\boldsymbol{t = \gamma\tau \rightarrow dt = \gamma d\tau}$

Therefore, we define a four-vector velocity as the derivative of the four-vector position with respect to its proper time:

$$\boldsymbol{\hat{V} = \frac{d\hat{X}}{d\tau} = \frac{dt}{d\tau}\frac{d\hat{X}}{dt} = \gamma\left(\frac{d\vec{r}}{dt}, \frac{d(ct)}{dt}\right) = \gamma(\vec{u}, c) \equiv \gamma\begin{pmatrix} u_x \\ u_y \\ u_z \\ c \end{pmatrix}}$$

The norm of this four-vector (magnitude) is given by:

$$\boldsymbol{\hat{V}^2 = \sum_{\mu=1}^{4} V_\mu^2 = \gamma^2\left(u^2 - c^2\right) = c^2\frac{u^2/c^2 - 1}{1 - u^2/c^2} = -c^2}$$

This is an invariant (the minus sign (-) comes from the adopted notation)

Minkowski Space-Time and Quadri-vectors

- **Four-vector acceleration :**

In the same way, we could retrieve the four-vector acceleration:

$$\boldsymbol{\hat{A}} = \frac{\boldsymbol{d\hat{U}}}{\boldsymbol{d\tau}} = \frac{\boldsymbol{dt}}{\boldsymbol{d\tau}}\frac{\boldsymbol{d\hat{U}}}{\boldsymbol{dt}} = \boldsymbol{\gamma}\frac{\boldsymbol{d}}{\boldsymbol{dt}}(\boldsymbol{\gamma\vec{u}}, \boldsymbol{\gamma c}) \equiv \boldsymbol{\gamma}\frac{\boldsymbol{d}}{\boldsymbol{dt}}\begin{pmatrix} \gamma u_x \\ \gamma u_y \\ \gamma u_z \\ \gamma c \end{pmatrix}$$

By applying the same derivation rules, knowing that:

$$\frac{d}{dt}\gamma = \frac{d}{dt}\left(1 - {u^2}/{c^2}\right)^{-1/2} = \gamma^2\frac{u}{c^2}; \vec{u} = \frac{d\vec{r}}{dt}; \vec{a} = \frac{d\vec{u}}{dt}$$

Finally, we obtain:

$$\boldsymbol{\hat{A}} = \frac{\boldsymbol{d\hat{U}}}{\boldsymbol{d\tau}} = \frac{\boldsymbol{dt}}{\boldsymbol{d\tau}}\frac{\boldsymbol{d\hat{U}}}{\boldsymbol{dt}} = \begin{pmatrix} \boldsymbol{\gamma^4\frac{\vec{u}.\vec{a}}{c^2}\vec{u} + \gamma^2\vec{a}} \\ \boldsymbol{\gamma^4\frac{\vec{u}.\vec{a}}{c^2}} \end{pmatrix}$$

In the tangent frame of the object ($\vec{u} = 0$), the norm of $\hat{A}$: $\hat{A} = \begin{pmatrix} \gamma^2\vec{a} \\ 0 \end{pmatrix} \rightarrow \hat{A}^2 = a^2(\gamma = 1)$

Minkowski Space-Time and Quadri-vectors

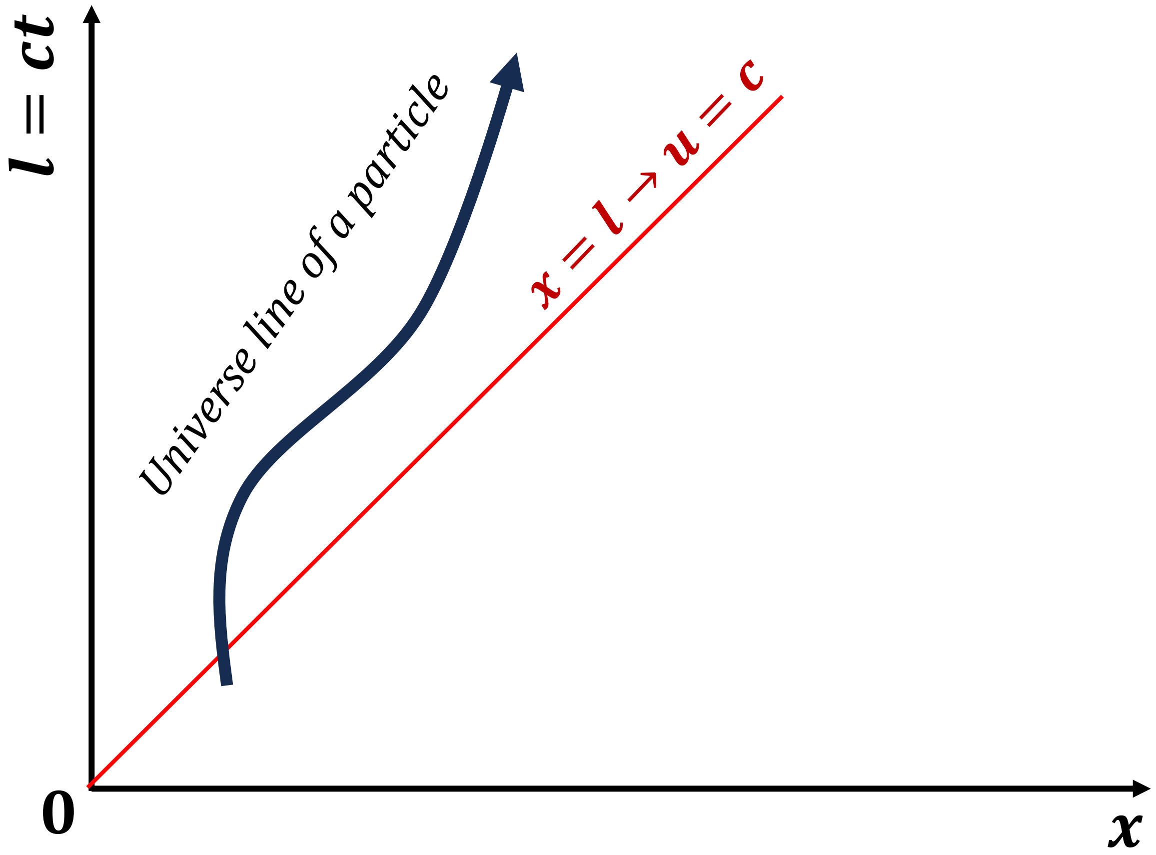
- **The $x - ct$ representation of the space-time:**

To make it simple, we consider only both dimensions $(x, l = ct)$ to represent the 2D space-time. In this case, the L.T will be reduced to:

$$\boldsymbol{x' = \gamma(x - \beta l)} \qquad \boldsymbol{x = \gamma(x' + \beta l')}$$

$$\boldsymbol{l' = \gamma(l - \beta x)} \qquad \boldsymbol{l = \gamma(l' + \beta x)}$$

Thus, the travelling of a material particle $(m \neq 0)$ is given by the curve "Universe line", which represents all the locations of this particle within this space-time: ***All events lived by this particle***



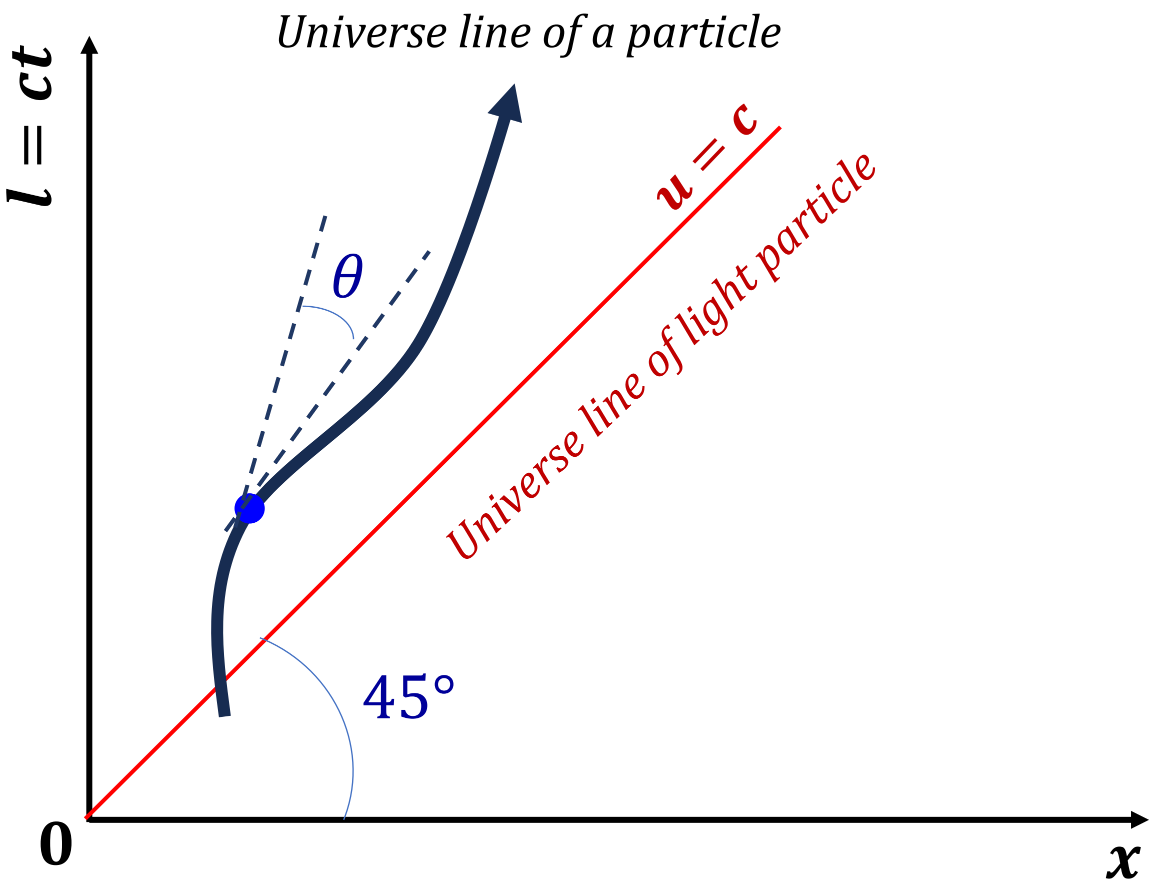
Minkowski Space-Time and Quadri-vectors

- **The $\boldsymbol{x}$–$\boldsymbol{ct}$ representation of the space-time:**

The tangent of the universe line of the given particle at each point is given by:

$\tan\theta = dx/dl = dx/cdt = u/c$

Since $u < c \rightarrow \theta < 45°$ for each material particle



In other hand, the universe line of a photon is a straight line with an angle of $45°$ with respect to OX ($u = c$)

Minkowski Space-Time and Quadri-vectors

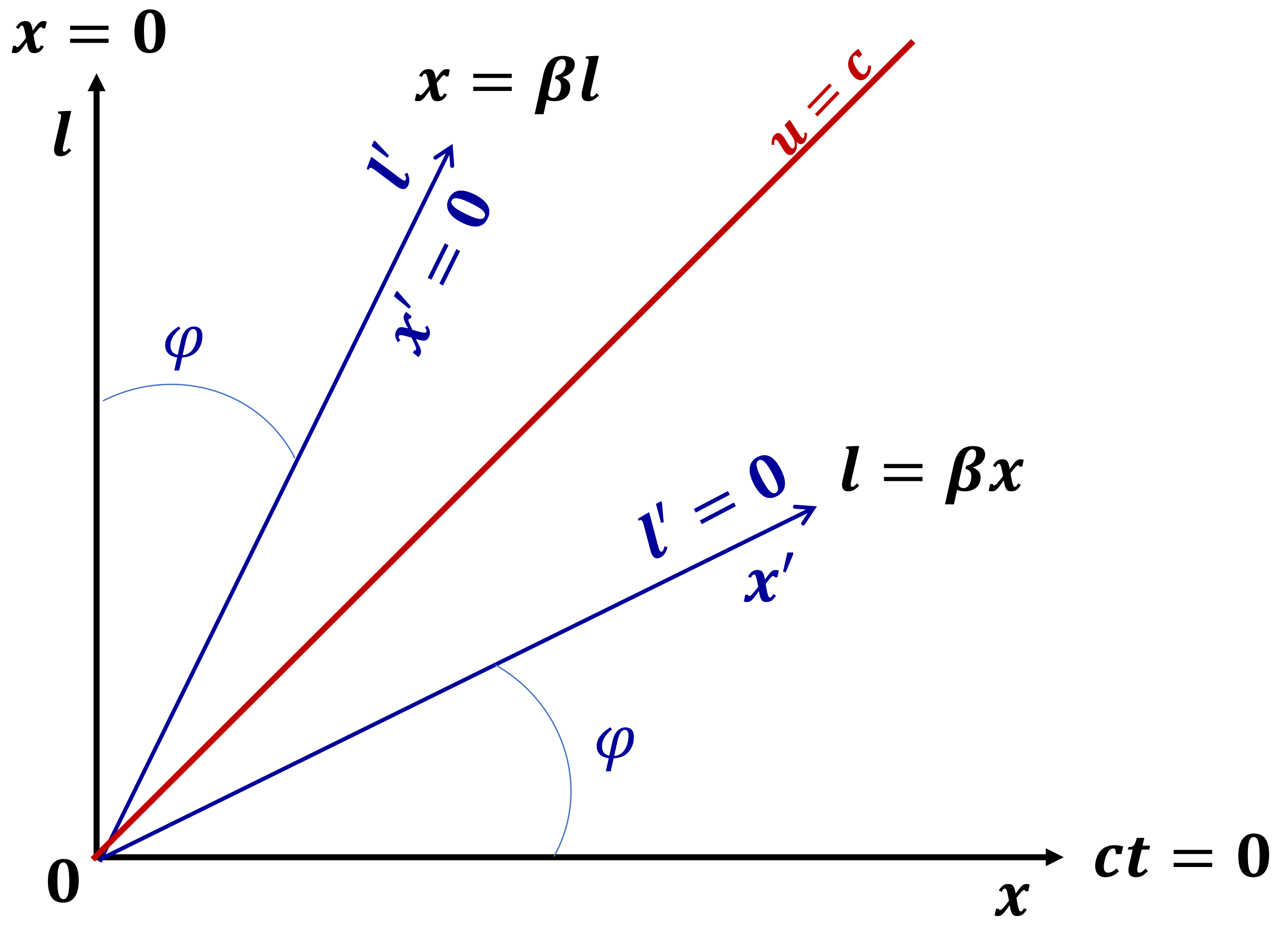
$$x' = \gamma(x - \beta l)$$
$$l' = \gamma(l - \beta x)$$

- **The x–ct representation of the space-time:**

Now, we will construct the frame $R'(x', l' = ct')$, with respect to the existing frame $R(x, l = ct)$:

- **The $l' - axis$ is obtained for $x' = 0$, and according to L.T: $x = \beta l$ $(\varphi < 45°)$**
- **The $x' - axis$ is obtained for $l' = 0$, similarly by using L.T: $l = \beta x$**

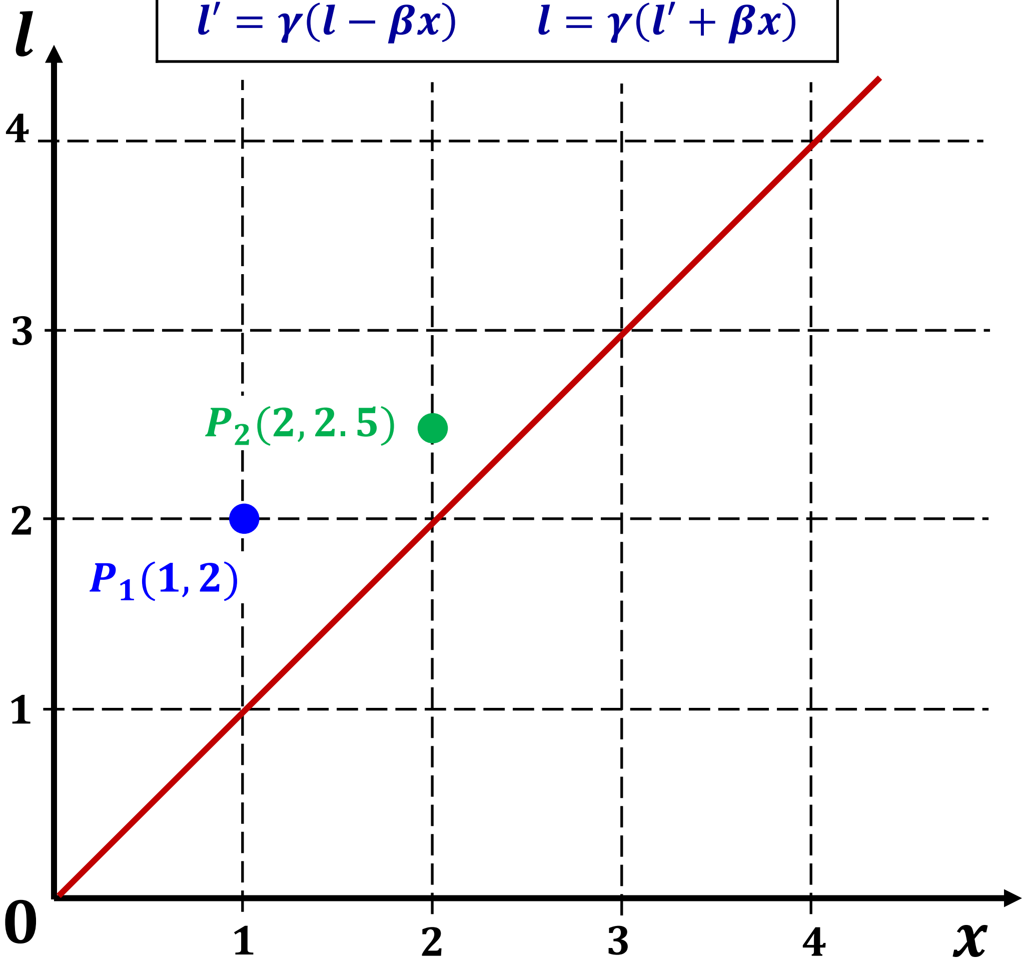
The L.T will transform an orthogonal frame into a non-orthogonal one.

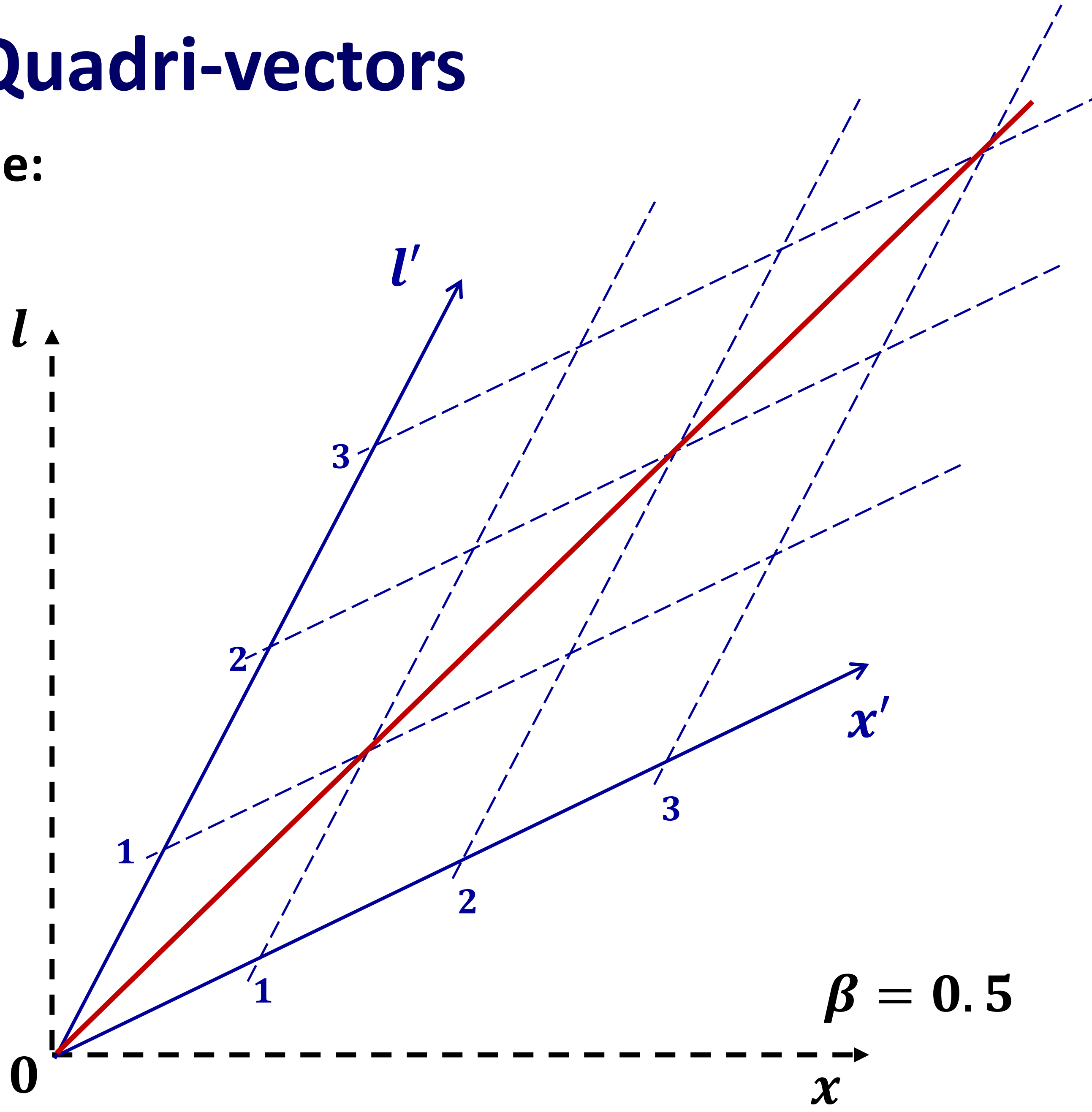


Minkowski Space-Time and Quadri-vectors

- The $x\text{–}ct$ representation of the space-time:

$$x' = \gamma(x - \beta l) \qquad x = \gamma(x' + \beta l')$$
$$l' = \gamma(l - \beta x) \qquad l = \gamma(l' + \beta x)$$



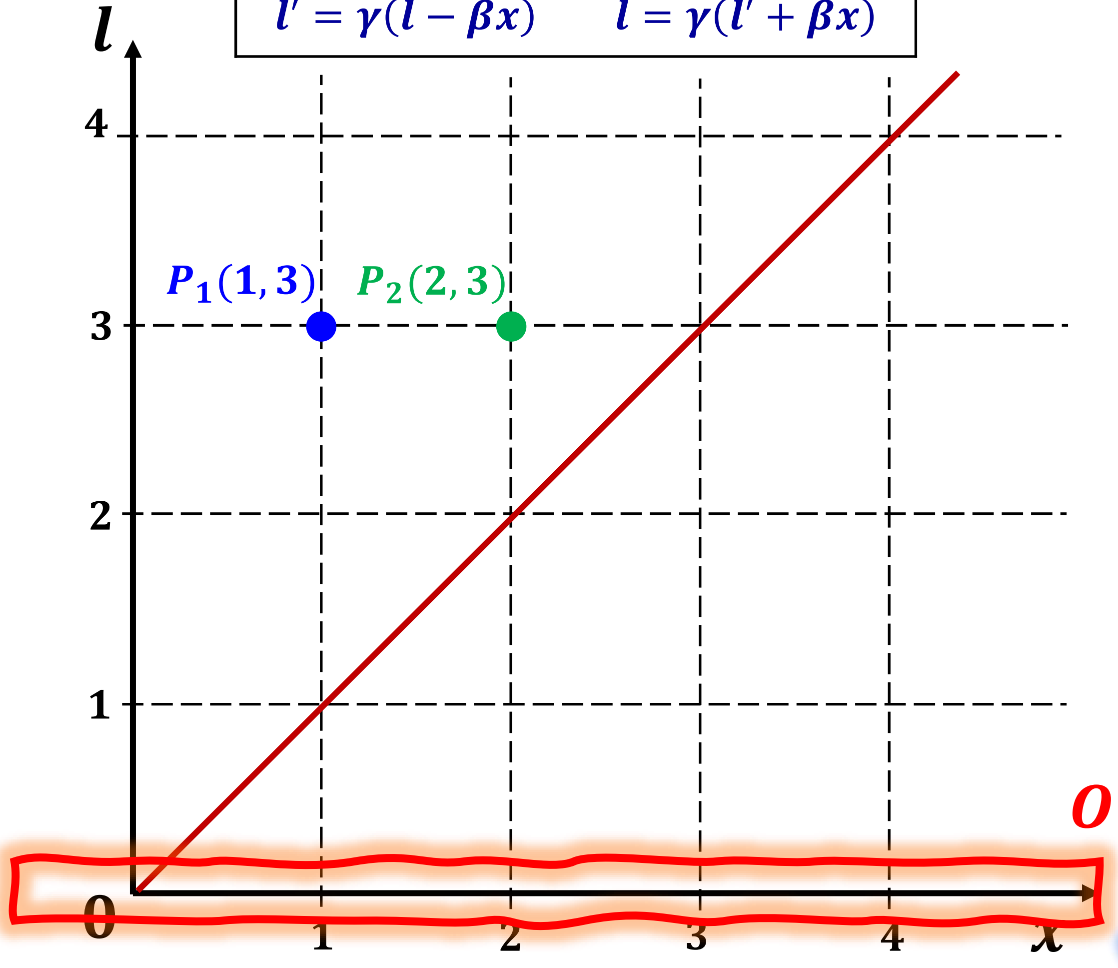


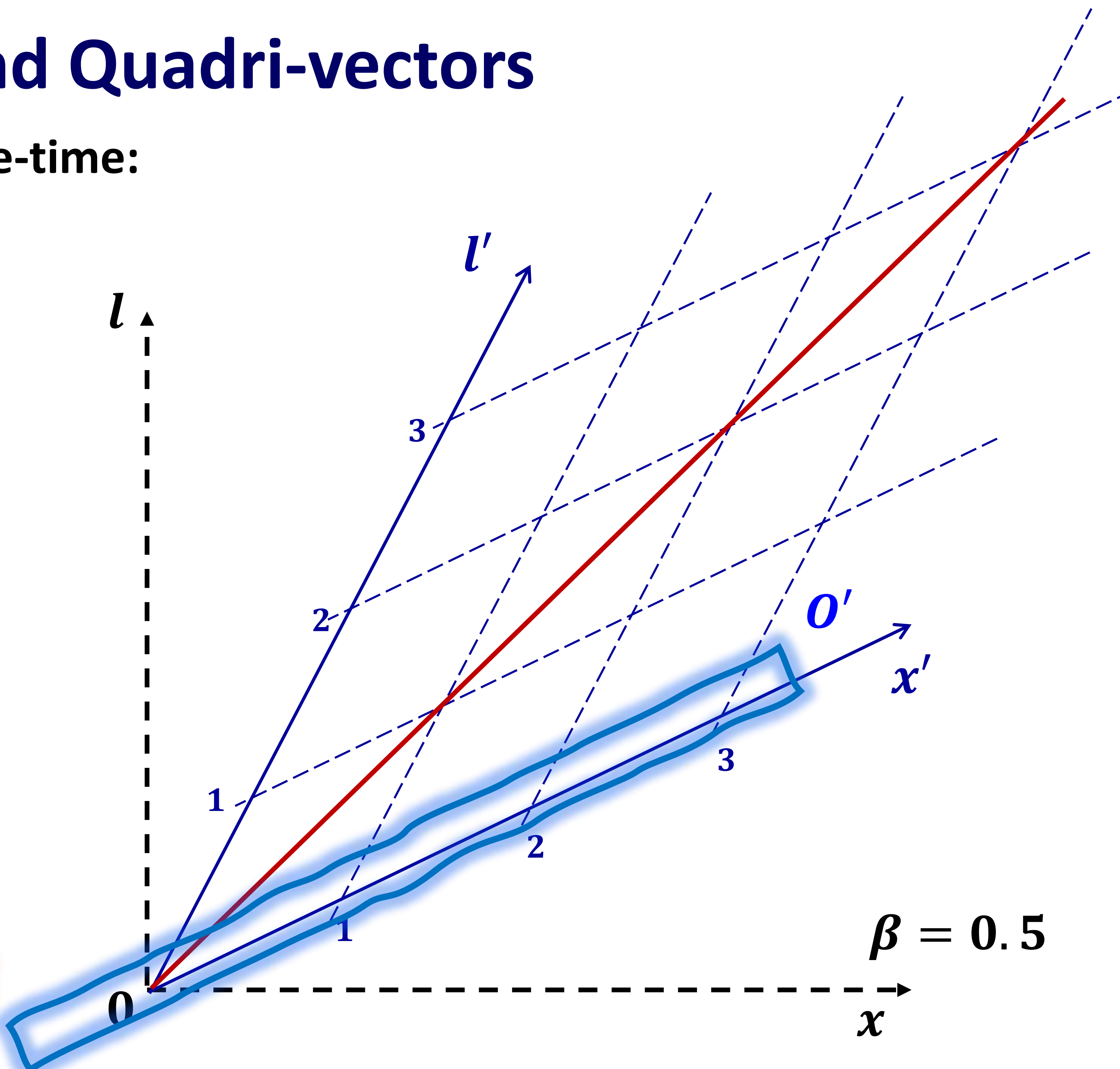
Minkowski Space-Time and Quadri-vectors

- The x–ct representation of the space-time:

$$x' = \gamma(x - \beta l) \qquad x = \gamma(x' + \beta l')$$

$$l' = \gamma(l - \beta x) \qquad l = \gamma(l' + \beta x)$$



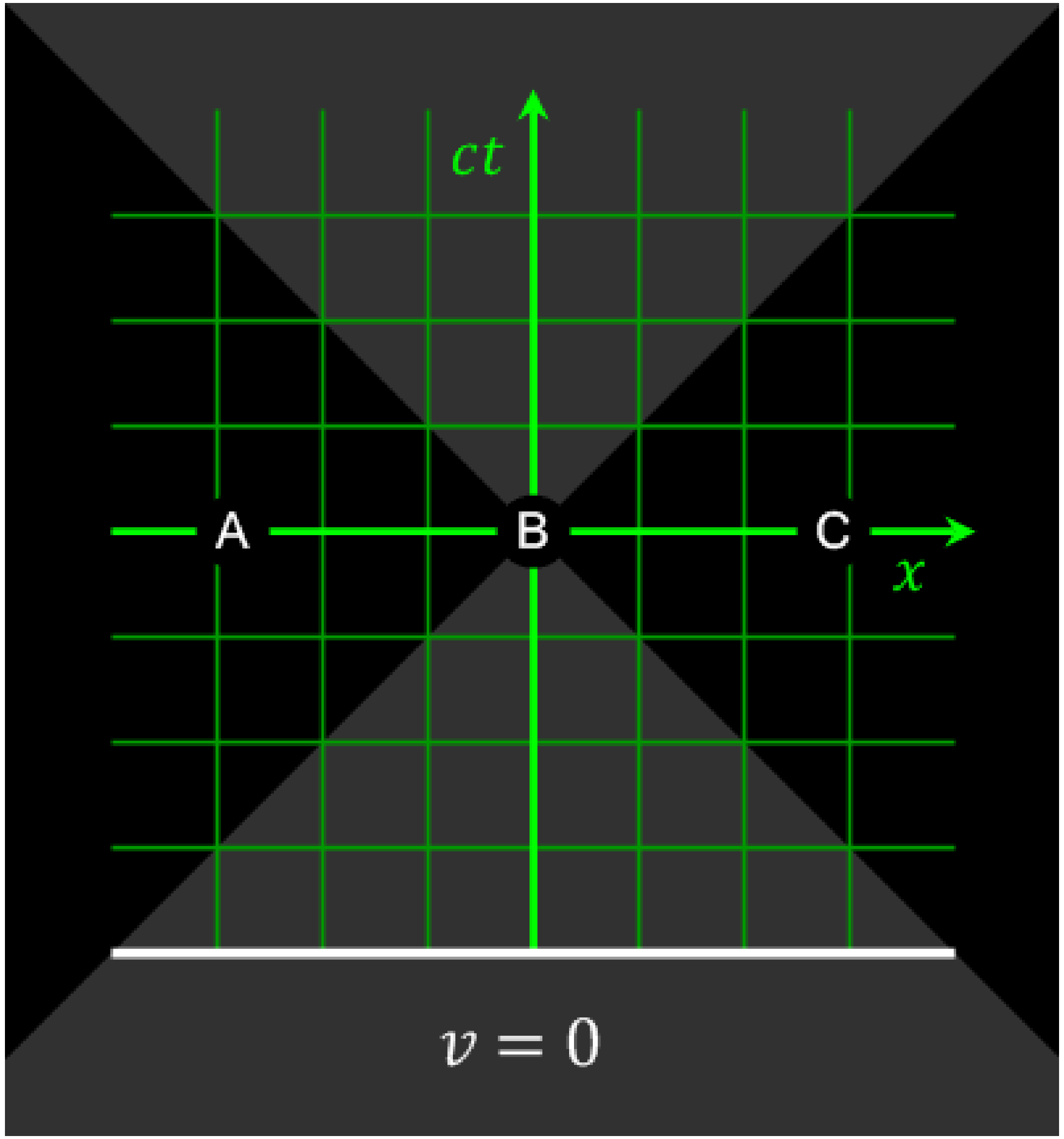


Minkowski Space-Time and Quadri-vectors

- **The x–ct representation of the space-time:**

Consequently, the Minkowski space allows a better understanding of the simultaneity question within the special relativity theory, as well as the length contraction and time dilatation consequences.

As it were Minkowski succeeded to geometrize the physics of space and time



Minkowski Space-Time and Quadri-vectors

- **The light cone or the time order of events**

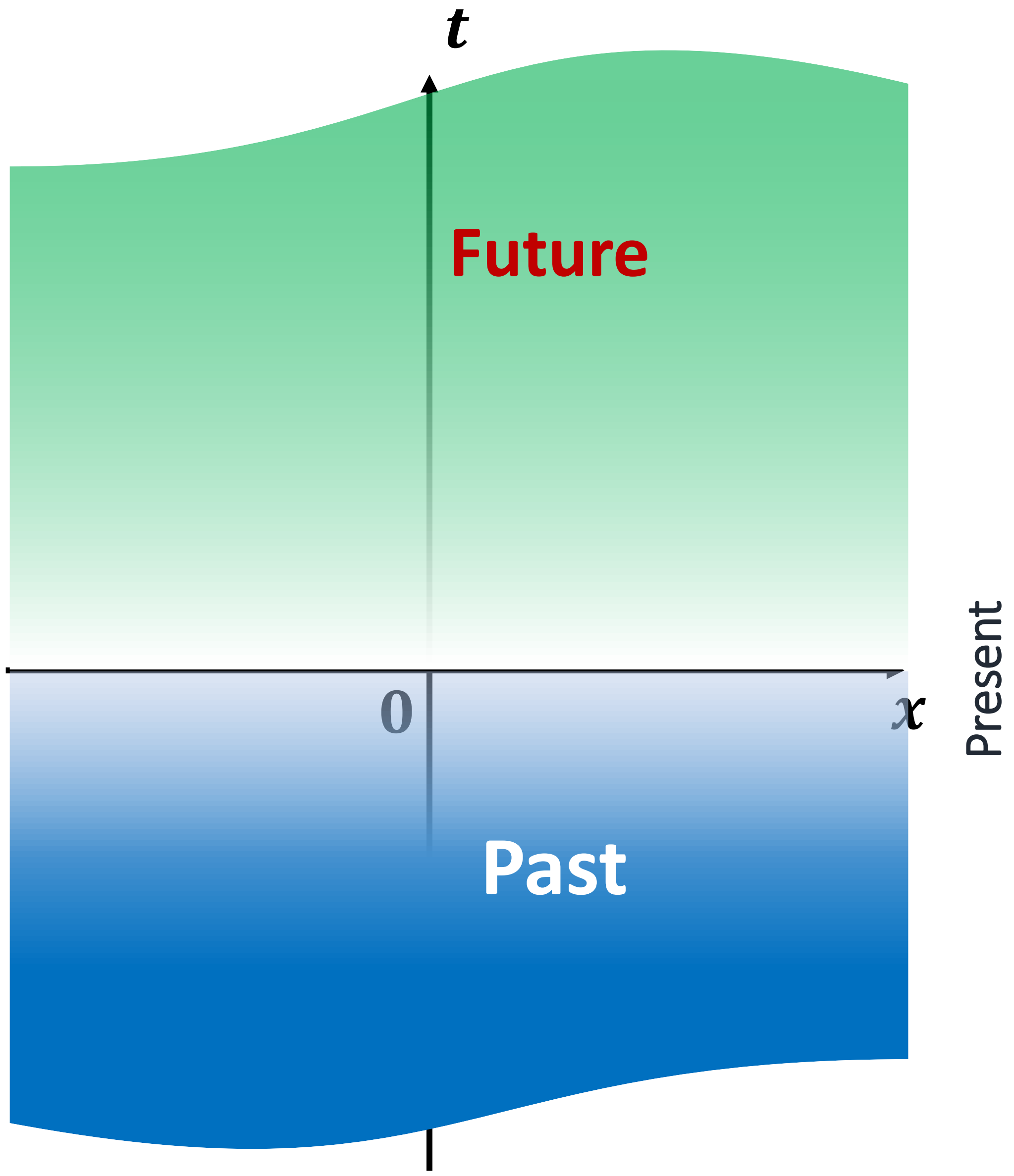
According to our perception of time, we define three times according to the chronological order:

- The past

- The present

- The future

This perception is easily translated by the time axis with a unique direction, usually used in classical physics.



Minkowski Space-Time and Quadri-vectors

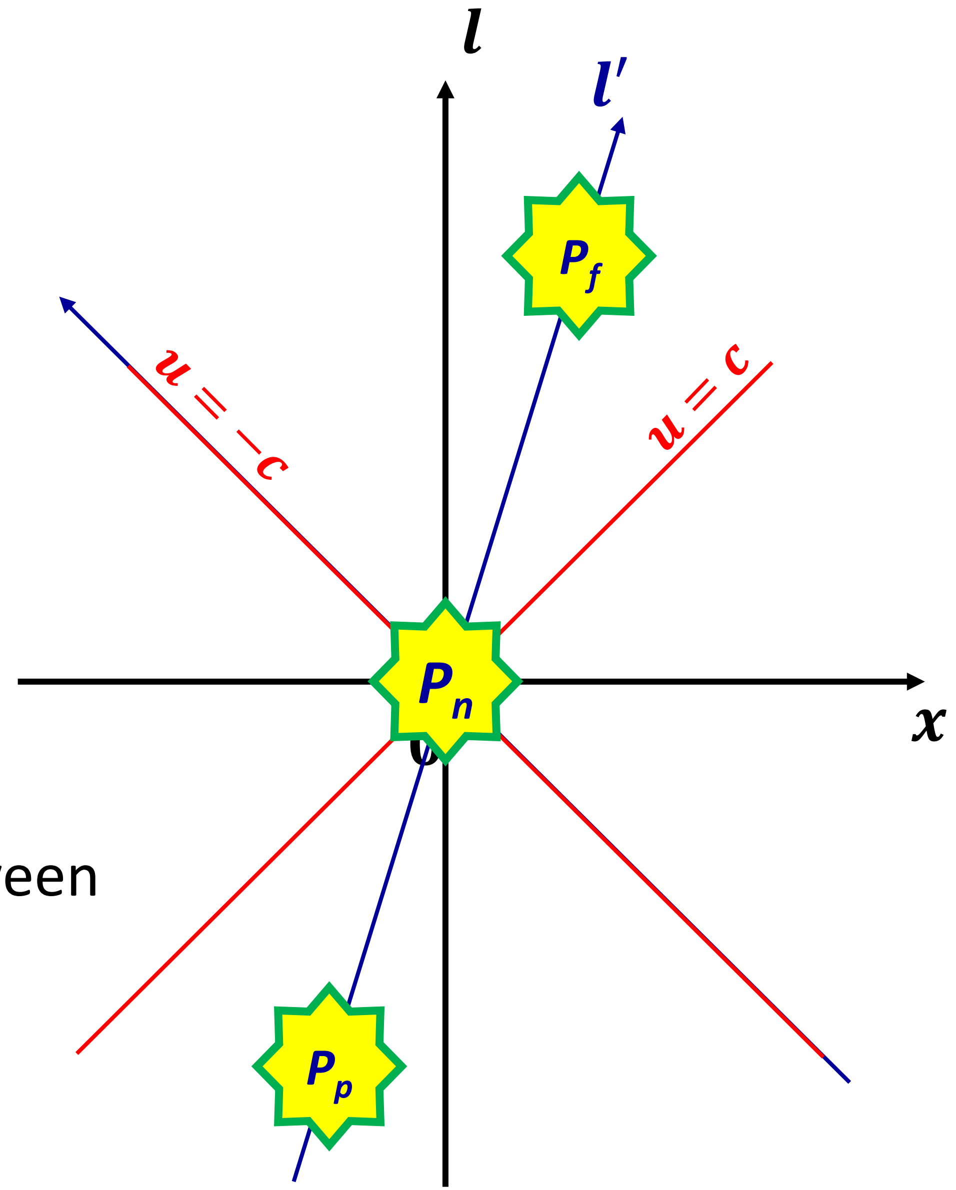
- **The light cone or the time order of events**

In relativistic physics, the order of events as seen by an observer O, could eventually change for another observer O' in function of its motion (velocity u) with respect of O.

However, the time-light axis l' will be always located at an angle $\varphi < 45°$ with respect to l.

Besides that, the time order is also respected between events (at the same position) on this axis :

$$P_{past}, P_{now} \text{ and } P_{future}$$



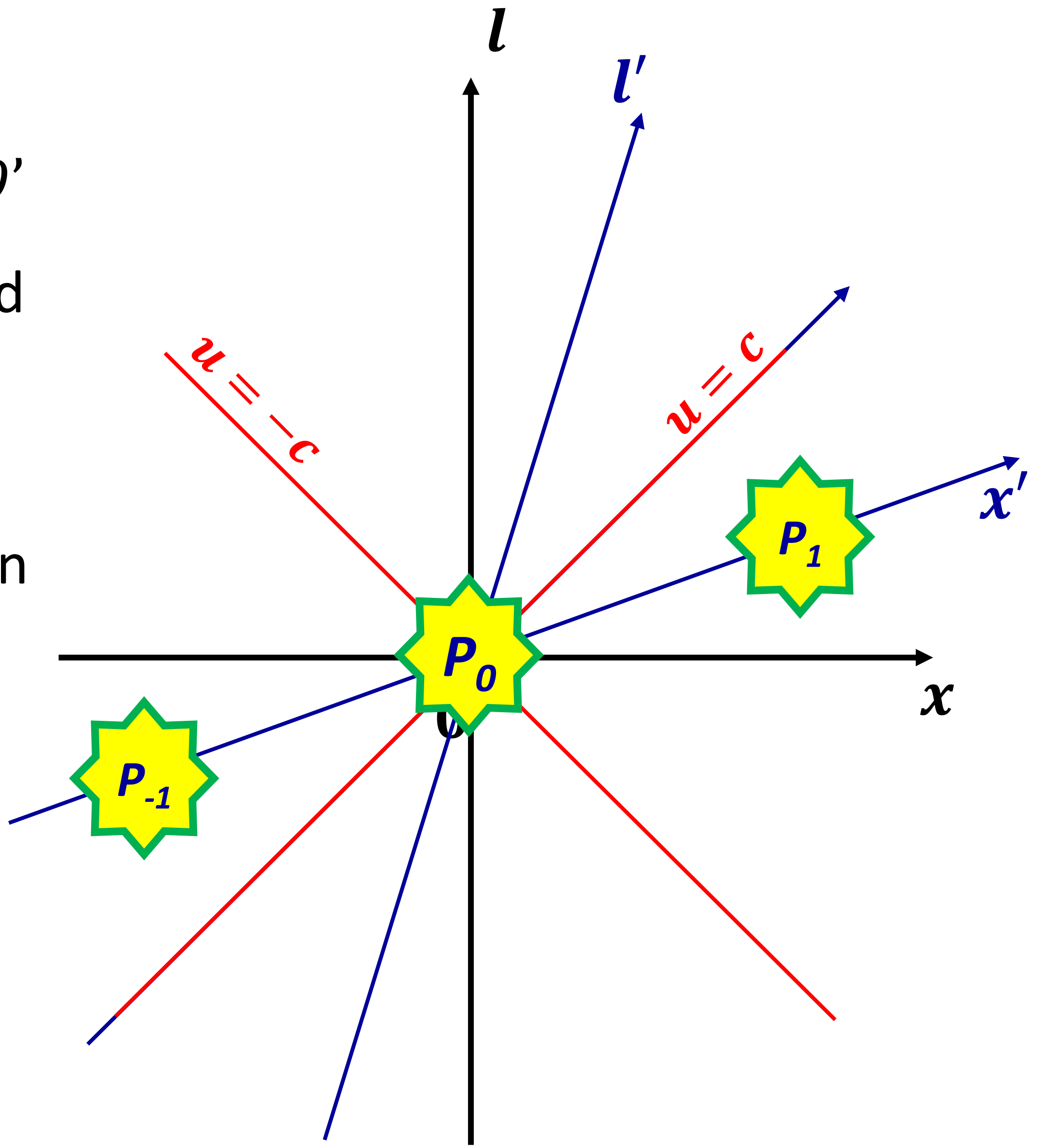
Minkowski Space-Time and Quadri-vectors

- **The light cone or the time order of events**

In the same way, the spatial axis of an observer O' moving with a velocity u, will be always oriented with an angle $\varphi < 45°$ with respect to x.

And the following events will always happen in simultaneous way but at different positions:

$$P_{-1}, P_0 \text{ and } P_1$$



Minkowski Space-Time and Quadri-vectors

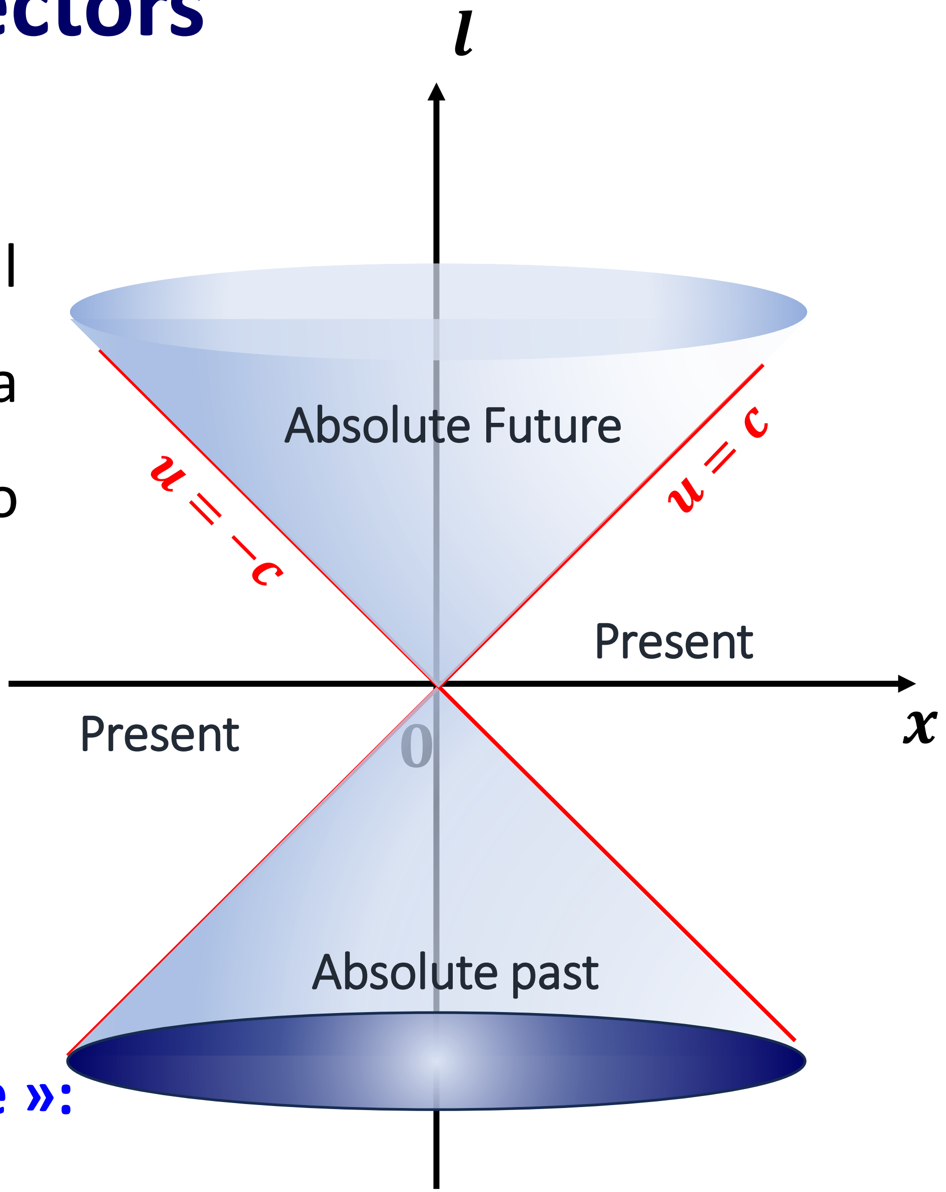
- **The light cone or the time order of events**

Therefore, in the Minkowski space describing the special relativity theory, it is possible for an observer O' with a velocity u, to perceive the order of events according to three zones defined below :

- Absolute past

- Present

- Absolute future



These three time zones are delimited by the « Light cone »:

$$(3D):\ x^2 + y^2 = c^2t^2 \rightarrow (4D): x^2 + y^2 + z^2 = c^2t^2$$