Khemis Miliana University – Djilali BOUNAAMA
Faculty of Material Sciences and Computer Science
Department of Physics

Advanced Electromagnetism

M1 Applied Physics

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Content

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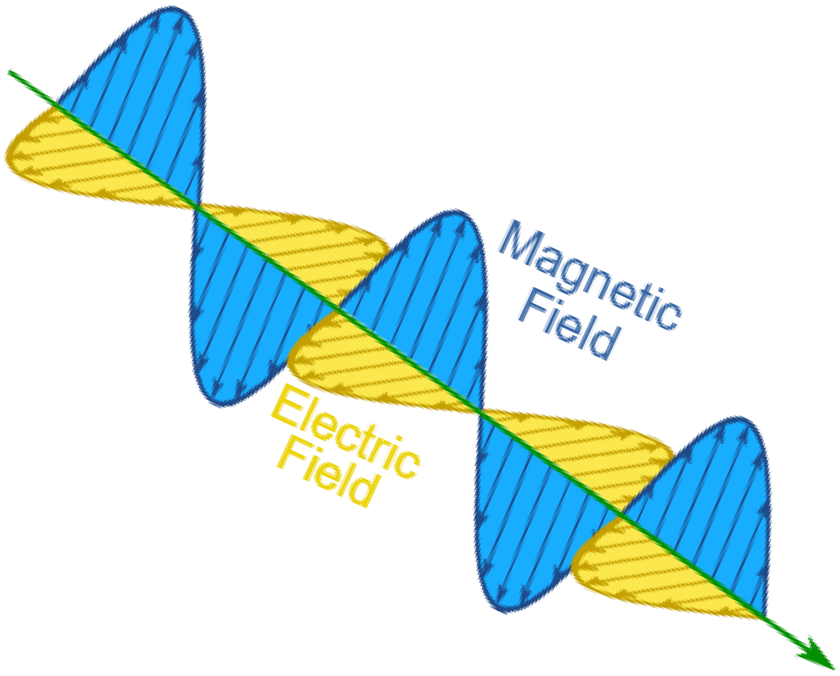
Chapter 03

Propagation of EM Waves in free space



I. Differential equation of the wave

We call the 2nd order differential equation of the following form (1D space):

$$\frac{\partial^2}{\partial x^2}F(x,t) - \alpha\frac{\partial^2}{\partial t^2}F(x,t) = 0$$

the wave's equation, and the function $F(x,t)$ verifying this equation (solution of the equation) is called the wave function.

The unit homogeneity implies that:

$$\alpha[s^2.m^{-2}] \rightarrow \frac{1}{\alpha}[m^2.s^{-2}] \equiv \left(v\left[\frac{m}{s}\right]\right)^2$$

This allows to rewrite the wave's equation:

$$\frac{\partial^2}{\partial x^2}F(x,t) - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}F(x,t) = 0$$

One can deduce that v represent the propagation velocity of the wave.

In 3D space, the wave's equation can be generalized:

$$\Delta F(x,y,z,t) - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}F(x,y,z,t) = 0$$

With: $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

In this case, the propagation velocity could be given as 3D vector:

$$\vec{v} = v_x\vec{i} + v_y\vec{j} + v_z\vec{k}$$

II. Solution of the wave's equation

Let's focus on the 1D space equation:

$$\frac{\partial^2}{\partial x^2}F(x,t) - \alpha\frac{\partial^2}{\partial t^2}F(x,t) = 0$$

We can recognize the difference of two squares identity: $(a^2 - b^2) = (a-b).(a+b)$ *in the differential operator:*

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right)F(x,t) = \left(\frac{\partial}{\partial x} - \frac{1}{v}\frac{\partial}{\partial t}\right).\left(\frac{\partial}{\partial x} + \frac{1}{v}\frac{\partial}{\partial t}\right)F(x) = 0$$

Consequently, a variable change could be performed here.

The following variable change is considered:

$$\begin{cases} X(x,t) = x - vt \\ Y(x,t) = x + vt \end{cases} \leftrightarrow \begin{cases} 2x = X + Y \\ 2vt = Y - X \end{cases}$$

Using the fact that:

$$\frac{\partial}{\partial X} = \frac{\partial x}{\partial X}\frac{\partial}{\partial x} + \frac{\partial t}{\partial X}\frac{\partial}{\partial t}$$

$$\frac{\partial}{\partial Y} = \frac{\partial x}{\partial Y}\frac{\partial}{\partial x} + \frac{\partial t}{\partial Y}\frac{\partial}{\partial t}$$

Which leads to:

$$\frac{\partial}{\partial x} - \frac{1}{v}\frac{\partial}{\partial t} = 2\frac{\partial}{\partial X}\ ; \frac{\partial}{\partial x} + \frac{1}{v}\frac{\partial}{\partial t} = 2\frac{\partial}{\partial Y}$$

Which leads to the new form of the differential equation:

$$\frac{\partial}{\partial X}\left(\frac{\partial}{\partial Y}\right)F(X,Y) = 0$$

Supporting a solution of the type:

$$F(X,Y) = A(X) + B(Y)$$

II. Solution of the wave's equation

The wave's function could be written with original variables x and t:

$$F(x,t) = A(x - vt) + B(x + vt)$$

Indicating that both solutions:

- A: *represent propagation in* $+v$ *direction*
- *B: represent propagation in* $-v$ *direction*

Besides that, A and B functions should be periodic functions to satisfy the 2nd order differential equation of the wave:

$$A(x,t) = a_1.sin(x - vt) + a_2.cos(x - vt)$$

$$B(x,t) = b_1.sin(x + vt) + b_2.cos(x + vt)$$

The coefficients a_i *and* b_i *could be determined by initial and boundary conditions.*

In the simplest case of 1D space, the propagating wave in the $+x$ *direction, then only the function* $A(x - vt)$ *is considered* $(b_1 = b_2 = 0)$.

Besides that, if we consider at $t = 0$ *and* $x = 0$ *position we have:* $A(0,0) = 0$, *We can deduce easily that the solution is of the form* $(a_2 = 0)$:

$$A(x,t) = a_1.sin(x - vt)$$

This corresponds to a sinusoidal function with an amplitude a_1.

III. Derivation of E.M wave's equations

Let's consider the general set of Maxwell's equations for a given medium characterized with an electric permittivity ε and magnetic permeability μ:

$$\begin{cases} \vec{\nabla}.\vec{E} = \dfrac{\rho}{\varepsilon} & (I) \\ \vec{\nabla} \wedge \vec{E} = -\dfrac{\partial \vec{B}}{\partial t} & (II) \\ \vec{\nabla}.\vec{B} = 0 & (III) \\ \vec{\nabla} \wedge \vec{B} = \mu \vec{J} + \mu\varepsilon \dfrac{\partial \vec{E}}{\partial t} & (IV) \end{cases}$$

By applying the following rule on (II) and (IV):

$$\vec{\nabla} \wedge (\vec{\nabla} \wedge \vec{A}) = \vec{\nabla}(\vec{\nabla}.\vec{A}) - \Delta\vec{A}$$

From (II) we get:

$$\vec{\nabla} \wedge (\vec{\nabla} \wedge \vec{E}) = \vec{\nabla}(\vec{\nabla}.\vec{E}) - \Delta\vec{E}$$

$$\vec{\nabla} \wedge \left(-\frac{\partial \vec{B}}{\partial t}\right) = \vec{\nabla}\left(\frac{\rho}{\varepsilon}\right) - \Delta\vec{E}$$

$$-\frac{\partial}{\partial t}(\vec{\nabla} \wedge \vec{B}) = \vec{\nabla}\left(\frac{\rho}{\varepsilon}\right) - \Delta\vec{E}$$

$$-\frac{\partial}{\partial t}\left(\mu\vec{J} + \mu\varepsilon\frac{\partial \vec{E}}{\partial t}\right) = \vec{\nabla}\left(\frac{\rho}{\varepsilon}\right) - \Delta\vec{E}$$

$$\Delta\vec{E} - \mu\varepsilon\frac{\partial^2 \vec{E}}{\partial t^2} - \mu\frac{\partial}{\partial t}\vec{J} = \vec{\nabla}\left(\frac{\rho}{\varepsilon}\right)$$

Since $\vec{j} = \sigma\vec{E}$***:***

$$\Delta\vec{E} - \mu\varepsilon\frac{\partial^2 \vec{E}}{\partial t^2} - \mu\sigma\frac{\partial}{\partial t}\vec{E} = \frac{1}{\varepsilon}\vec{\nabla}\rho \qquad (Eq.3.1)$$

III. Derivation of E.M wave's equations

From (IV) one can also derive the following equation in the same way:

$$\Delta\vec{B} - \mu\varepsilon\frac{\partial^2\vec{B}}{\partial t^2} - \mu\sigma\frac{\partial\vec{B}}{\partial t} = 0 \qquad (Eq.3.2)$$

Finally, we will get the following system of 2nd degree differential equations:

$$\begin{cases} \underbrace{\Delta\vec{E}}_{Spat.var.} - \underbrace{\mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2}}_{Propagation} - \underbrace{\mu\sigma\frac{\partial}{\partial t}\vec{E}}_{Dispersion} = \underbrace{\frac{1}{\varepsilon}\vec{\nabla}\rho}_{Source} \\ \underbrace{\Delta\vec{B}}_{Spat.var.} - \underbrace{\mu\varepsilon\frac{\partial^2\vec{B}}{\partial t^2}}_{Propagation} - \underbrace{\mu\sigma\frac{\partial\vec{B}}{\partial t}}_{Dispersion} = 0 \end{cases}$$

In the free space ($\rho = 0, \varepsilon = \varepsilon_0, \mu = \mu_0$):

$$\begin{cases} \Delta\vec{E} - \mu_0\varepsilon_0\frac{\partial^2\vec{E}}{\partial t^2} - \mu_0\sigma\frac{\partial\vec{E}}{\partial t} = 0 \qquad (Eq.3.3) \\ \Delta\vec{B} - \mu_0\varepsilon_0\frac{\partial^2\vec{B}}{\partial t^2} - \mu_0\sigma\frac{\partial\vec{B}}{\partial t} = 0 \qquad (Eq.3.4) \end{cases}$$

Which shows that we get a 2nd degree differential equations without constant terms (homogeneous equations).

It should be noticed that 1st degree terms :

$\mu_0\sigma\frac{\partial\vec{E}}{\partial t}$ and $\mu_0\sigma\frac{\partial\vec{B}}{\partial t}$ came from the presence of non-null current.

III. Derivation of E.M wave's equations

Indeed, considering free space as non conducting medium ($\sigma = 0$), both equations (3.3) and (3.4) will be reduced to:

$$\begin{cases} \Delta\vec{E} - \mu_0\varepsilon_0 \dfrac{\partial^2\vec{E}}{\partial t^2} = 0 & (Eq.3.5) \\ \Delta\vec{B} - \mu_0\varepsilon_0 \dfrac{\partial^2\vec{B}}{\partial t^2} = 0 & (Eq.3.6) \end{cases}$$

These equations are identical to general wave's equation (3D), and by identification we can find that propagation velocity:

$$v = \frac{1}{\sqrt{\mu\varepsilon}}$$

Application (5min):

Calculate $v_0 = \frac{1}{\sqrt{\mu_0\varepsilon_0}}$ in the void (Air), and comment your finding.

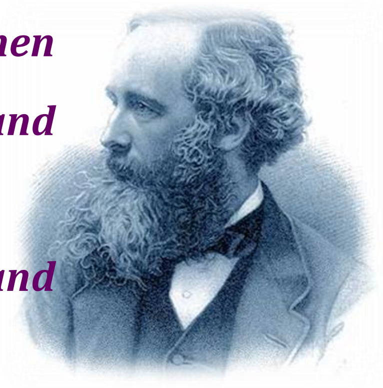
$$\varepsilon = \varepsilon_0 = 8.85 \times 10^{-12}[C^2.N^{-1}.m^{-2}]$$

$$\mu = \mu_0 = 4\pi \times 10^{-7}[N.A^{-2}]$$

We find:

$$v_0 = \frac{1}{\sqrt{\mu_0\varepsilon_0}} = 2.99 \times 10^8\left[\frac{m}{s}\right]$$

The first measurements of light speed by Bradley in 1729 ($3.01 \times 10^8[m/s]$), then Fizeau in 1849 ($3.15 \times 10^8[m/s]$), and Foucault in 1862 ($2.98 \times 10^8[m/s]$). Maxwell's treatise in Electricity and Magnetism was published in 1873!!!



III. Derivation of E.M wave's equations

The most important results of Maxwell's work was the linking between light and Electromagnetic fields:

"Light is electromagnetic wave propagating in the void with a speed:

$$c \cong 3 \times 10^8 [m/s]\text{"}$$

The differential equations (3.5) and (3.6) will support a periodic functions as solutions of the following form:

$$\vec{E}(\vec{r},t) = \vec{E}_0 . \cos\left(\vec{r}.\vec{k} - \omega t\right)$$

$$\vec{B}(\vec{r},t) = \vec{B}_0 . \cos\left(\vec{r}.\vec{k} - \omega t\right)$$

With $\vec{k}$ is the wave vector to be determined.

IV. The general solution of E.M equation wave

Let's go back to the first system of 2nd differential equations including 1st order time term ($\rho = 0$):

$$\begin{cases} \Delta\vec{E} - \mu_0\varepsilon_0 \dfrac{\partial^2\vec{E}}{\partial t^2} - \mu_0\sigma\dfrac{\partial\vec{E}}{\partial t} = 0 \qquad (Eq.3.3) \\ \Delta\vec{B} - \mu_0\varepsilon_0 \dfrac{\partial^2\vec{B}}{\partial t^2} - \mu_0\sigma\dfrac{\partial\vec{B}}{\partial t} = 0 \qquad (Eq.3.4) \end{cases}$$

<u>10min Test:</u> *We propose the following form as general solutions of (3.3) & (3.4). Replace them and deduce the new differential equations of space phasors $\tilde{e}(\vec{r})$ & $\tilde{b}(\vec{r})$:*

$$\tilde{E}(\vec{r}, t) = \tilde{e}(\vec{r}).e^{i\omega t} = \tilde{e}(x, y, z).e^{i\omega t}$$

$$\tilde{B}(\vec{r}, t) = \tilde{b}(\vec{r}).e^{i\omega t} = \tilde{b}(x, y, z).e^{i\omega t}$$

With: $e^{i\omega t} = \cos\omega t + i.\sin\omega t,\ i^2 = -1$

We need just to replace both solutions in equations (3.3) and (3.4):

$$\Delta\left(\tilde{e}(\vec{r}).e^{i\omega t}\right) - \mu\varepsilon\frac{\partial^2\left(\tilde{e}(\vec{r}).e^{i\omega t}\right)}{\partial t^2} - \mu\sigma\frac{\partial}{\partial t}\left(\tilde{e}(\vec{r}).e^{i\omega t}\right) = 0$$

$$\Delta\left(\tilde{b}(\vec{r}).e^{i\omega t}\right) - \mu\varepsilon\frac{\partial^2\left(\tilde{b}(\vec{r}).e^{i\omega t}\right)}{\partial t^2} - \mu\sigma\frac{\partial}{\partial t}\left(\tilde{b}(\vec{r}).e^{i\omega t}\right) = 0$$

This will give us the new space-differential equations:

$$e^{i\omega t}\Delta\tilde{e}(\vec{r}) + \mu\varepsilon\omega^2\tilde{e}(\vec{r})e^{i\omega t} - i\omega\mu\sigma\tilde{e}(\vec{r})e^{i\omega t} = 0$$

$$e^{i\omega t}\Delta\tilde{b}(\vec{r}) + \mu\varepsilon\omega^2\tilde{b}(\vec{r})e^{i\omega t} - i\omega\mu\sigma\tilde{b}(\vec{r})e^{i\omega t} = 0$$

To be reduced to (phasor's equations):

$$\Delta\tilde{e}(\vec{r}) + \left[\mu\varepsilon\omega^2 - i\omega\mu\sigma\right]\tilde{e}(\vec{r}) = 0$$

$$\Delta\tilde{b}(\vec{r}) + \left[\mu\varepsilon\omega^2 - i\omega\mu\sigma\right]\tilde{b}(\vec{r}) = 0$$

IV. The general solution of E.M equation wave

By introducing complex permittivity:

$$\varepsilon_c = \varepsilon - i\frac{\sigma}{\omega} = \varepsilon' - i\varepsilon'', \varepsilon' = \varepsilon, \varepsilon'' = \frac{\sigma}{\omega}$$

We got: $k^2 = \mu\varepsilon_c\omega^2 = \mu\omega^2\left[\varepsilon - i\frac{\sigma}{\omega\varepsilon}\right] = -\gamma^2 = (i\gamma)^2$

Finally, the 2^{nd} order space differential equations known as Helmholtz equation of E.M wave could be written :

$$\begin{cases} \Delta\tilde{e}(\vec{r}) + k^2\tilde{e}(\vec{r}) = 0 & (Eq.3.7) \\ \Delta\tilde{b}(\vec{r}) + k^2\tilde{b}(\vec{r}) = 0 & (Eq.3.8) \end{cases}$$

Consequently, solutions are of the form:

$$\tilde{e}(\vec{r}) = \vec{E}_0 e^{\pm ik(\vec{r}.\vec{u})} = \vec{E}_0 e^{\pm i(\vec{r}.\vec{k})}$$
$$\tilde{b}(\vec{r}) = \vec{B}_0 e^{\pm ik(\vec{r}.\vec{u})} = \vec{B}_0 e^{\pm i(\vec{r}.\vec{k})}$$

E_0 and B_0: maximal amplitudes.

Where: $k = \omega\sqrt{\mu\varepsilon}\sqrt{1 - i\,{}^{\sigma}/_{\omega\varepsilon}} = \alpha + i\beta$

is known as "Wave number".

And the parameter γ is called "propagation constant"

In the specific case of lossless medium:

$$\sigma = 0 \rightarrow \varepsilon'' = 0$$

The wave number is purely real and the propagation is done without loss of the strength of E.M wave, and we have:

$$k = \omega\sqrt{\mu\varepsilon} = \frac{\omega}{v} = \frac{2\pi}{vT} = \frac{2\pi}{\lambda}\left[\frac{rad}{m}\right]$$

IV. The general solution of E.M equation wave

Replacing now $\vec{e}(\vec{r})$ and $\vec{b}(\vec{r})$ in the general expression:

$$\tilde{\vec{E}}(\vec{r},t) = \tilde{\vec{e}}(\vec{r}).e^{i\omega t} = \vec{E}_0 e^{i(\omega t \pm \vec{r}.\vec{k})}$$

$$\tilde{\vec{B}}(\vec{r},t) = \tilde{\vec{b}}(\vec{r}).e^{i\omega t} = \vec{B}_0 e^{i(\omega t \pm \vec{r}.\vec{k})}$$

Since $\vec{E}_0$ and $\vec{B}_0$ are amplitudes at initial conditions they could be written:

$$\vec{E}_0 = \vec{E}(0,0) = E_0 \vec{u}_E = |E_0| e^{i\varphi_0} \vec{u}_E$$

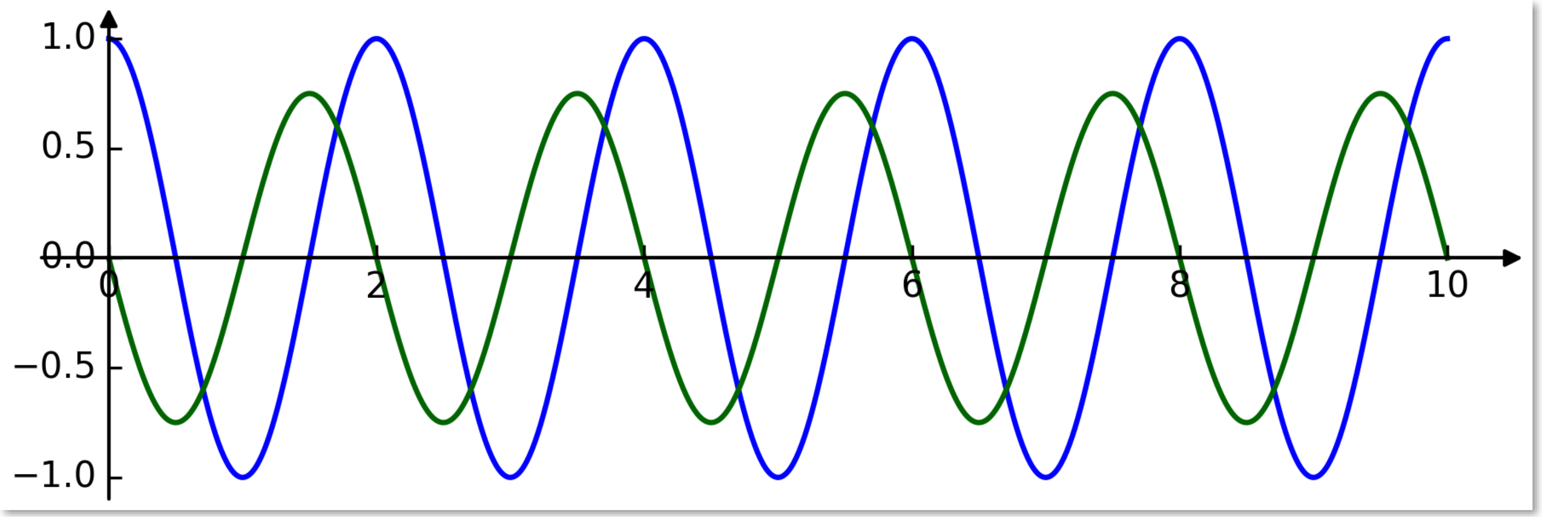
$$\vec{B}_0 = \vec{B}(0,0) = B_0 \vec{u}_B = |B_0| e^{i\varphi_0} \vec{u}_B$$

φ_0: initial phase of the wave

For instance, if we consider two waves represented by their electric fields, taken as the real part of complex phasors:

$$\vec{E}_1(\vec{r},t) = |E_{10}|.\Re e\left[e^{i(\omega t \pm \vec{r}.\vec{k})}\right]\vec{u};\ \varphi_1 = 0$$

$$\vec{E}_2(\vec{r},t) = |E_{20}|\Re e\left[e^{i(\omega t \pm \vec{r}.\vec{k} + \pi/2)}\right]\vec{u};\ \varphi_2 = \pi/2$$



V. Phasors Maxwell's equations

One of the important results of the previous solutions given in complex notation, is the new form of Maxwell equations. Indeed, let's take the following expressions of E.M fields:

$$\tilde{E}(\vec{r},t) = \tilde{e}(\vec{r}).e^{i\omega t}$$

$$\tilde{H}(\vec{r},t) = \tilde{h}(\vec{r}).e^{i\omega t}$$

When replaced in the Maxwell questions, taking in consideration that (similarly for $\tilde{H}(\vec{r},t)$) :

$$\frac{\partial \tilde{E}(\vec{r},t)}{\partial t} = \frac{\partial\left[\tilde{e}(\vec{r}).e^{i\omega t}\right]}{\partial t} = \tilde{e}(\vec{r})\frac{\partial e^{i\omega t}}{\partial t} = i\omega\tilde{e}(\vec{r}).e^{i\omega t}$$

$$\begin{cases} \vec{\nabla}.\tilde{e} = \dfrac{\tilde{\rho}}{\varepsilon} & (I) \\ \vec{\nabla}\wedge\tilde{e} = -i\omega\mu\tilde{h} & (II) \\ \vec{\nabla}.\tilde{h} = 0 & (III) \\ \vec{\nabla}\wedge\tilde{h} = \tilde{j} + i\omega\varepsilon\tilde{e} & (IV) \end{cases}$$

Which could be rewritten by taking $\tilde{j} = \sigma\tilde{e}$, we get in free space ($\rho = 0 \rightarrow \tilde{\rho} = 0$):

$$\begin{cases} \vec{\nabla}.\tilde{e} = 0 & (I) \\ \vec{\nabla}\wedge\tilde{e} = -i\omega\mu\tilde{h} & (II) \\ \vec{\nabla}.\tilde{h} = 0 & (III) \\ \vec{\nabla}\wedge\tilde{h} = i\omega\varepsilon_c\tilde{e} & (IV) \end{cases}$$

With: $\varepsilon_c = \varepsilon - i\frac{\sigma}{\omega}$ as introduced above.

VI. Spherical and Planar waves

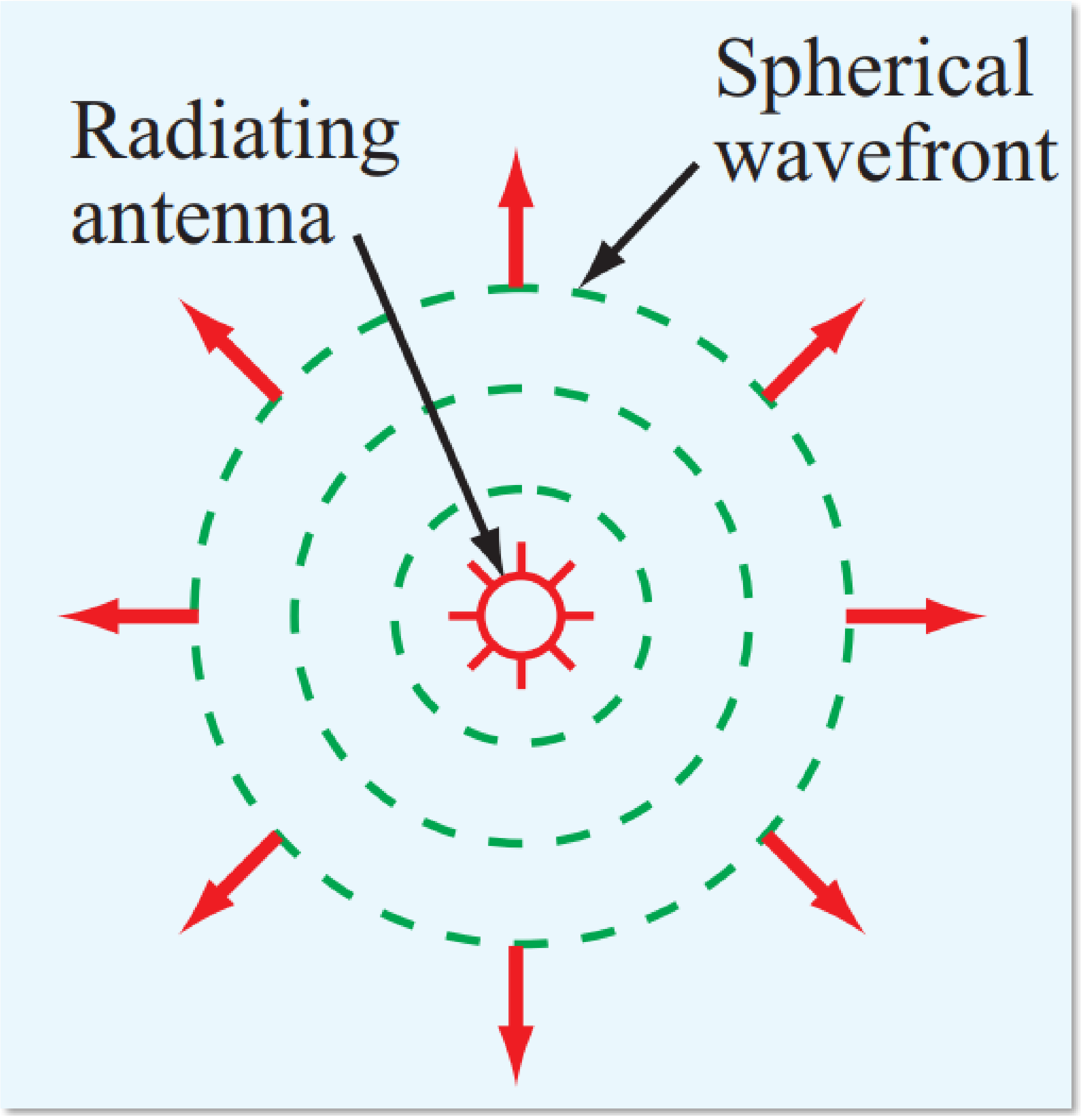
According to previous results, both electric and magnetic fields verifying differential equations are of the form:

$$\begin{cases} \tilde{\vec{E}}(\vec{r},t) = \vec{E}_0 e^{\pm i(\vec{r}.\vec{k})} e^{i\omega t} = \vec{E}_0 e^{i(\omega t \pm \vec{r}.\vec{k})} & (Eq.3.9) \\ \tilde{\vec{B}}(\vec{r},t) = \vec{B}_0 e^{\pm i(\vec{r}.\vec{k})} e^{i\omega t} = \vec{B}_0 e^{i(\omega t \pm \vec{r}.\vec{k})} & (Eq.3.10) \end{cases}$$

Along positive direction, physical solutions are:

$$\begin{cases} \vec{E}(\vec{r},t) = \vec{E}_0 . \Re e\left[e^{i(\omega t - \vec{r}.\vec{k})}\right] & (Eq.3.11) \\ \vec{B}(\vec{r},t) = \vec{B}_0 . \Re e\left[e^{i(\omega t - \vec{r}.\vec{k})}\right] & (Eq.3.12) \end{cases}$$

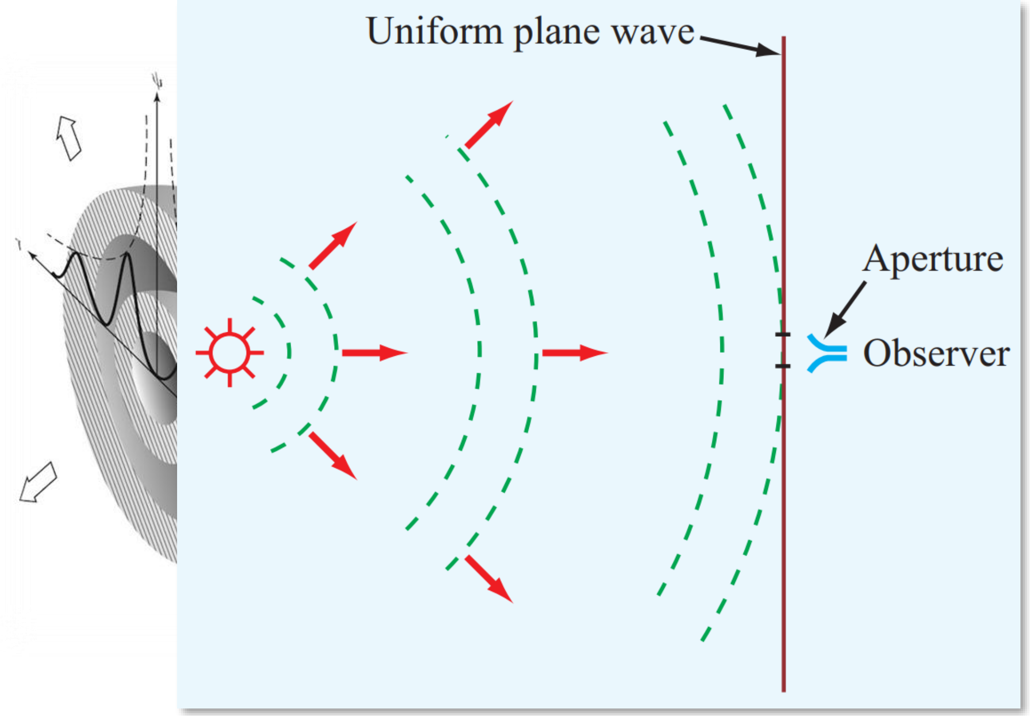
Such wave is propagating in all directions with the same intensities, therefore it constitutes a spherical wave.



VI. Spherical and Planar waves

A wave produced by a localized source, such as an antenna, expands outwardly in the form of a spherical wave. Even though an antenna may radiate more energy along some directions than along others, the spherical wave travels at the same speed in all directions.

To an observer very far away from the source, however, the wavefront of the spherical wave appears approximately planar, as if it were part of a uniform plane wave with identical properties at all points in the plane tangent to the wavefront. Plane waves are easily described using a Cartesian coordinate system, which is mathematically easier to work with than the spherical coordinate system needed to describe spherical waves.



VII. Uniform plane waves

It the case of plane waves, it is possible to choose an arbitrary cartesian direction to point the propagation direction along one of the XYZ axes. For instance, if we take the +z-direction, so one can write the wave number vector: $\vec{k} = k\vec{u}_z$

And the scalar product will reduce the spatial term to: $\vec{r}.\vec{k} = \left(x\vec{u}_x + y\vec{u}_y + z\vec{u}_z\right).k\vec{u}_z = kz$

Thus, the expression of electric field will be:

$$\tilde{\vec{E}}(\vec{r},t) = \tilde{e}(\vec{r}).e^{i\omega t} = |E_0|e^{i(\omega t - kz + \varphi_0)}\vec{u}_E$$

$\vec{u}_E = a\vec{u}_x + b\vec{u}_y + c\vec{u}_z$; *a, b, c are cosine directors*

When replaced in the first Maxwell equation a free space as propagation medium $(\rho = 0)$:

$$\vec{\nabla}.\vec{E} = \vec{\nabla}.\left(|E_0|e^{i(\omega t - \vec{r}.\vec{k} + \varphi_0)}\vec{u}_E\right) = 0$$

$$\leftrightarrow \underbrace{\partial_x e^{-i(kz)}}_{=0}\left(\vec{u}_x.\vec{u}_E\right) + \underbrace{\partial_y e^{-i(kz)}}_{=0}\left(\vec{u}_y.\vec{u}_E\right)$$

$$+ \underbrace{\partial_z e^{-i(kz)}}_{=-ike^{-i(kz)} \neq 0}\left(\vec{u}_z.\vec{u}_E\right) = 0 \rightarrow \vec{u}_z.\vec{u}_E = 0$$

Which means that $c = 0$:

$$\vec{u}_E = a\vec{u}_x + b\vec{u}_y$$

VII. Uniform plane waves

The previous result, will allow us to write the electric field with its XY components:

$$\tilde{E}(\vec{r},t) = |E_0|e^{i(\omega t - kz + \varphi_0)}\left(a\vec{u}_x + b\vec{u}_y\right)$$

Now let's use the second Maxwell equation: $\vec{\nabla} \wedge \vec{E} = -\frac{\partial \vec{B}}{\partial t} \leftrightarrow$

$$\vec{\nabla} \wedge \left(|E_0|e^{i(\omega t - kz + \varphi_0)}\left(a\vec{u}_x + b\vec{u}_y\right)\right) = -\frac{\partial\left(|B_0|e^{i(\omega t - kz + \varphi_0)}\vec{u}_B\right)}{\partial t} = -i\omega|B_0|e^{i(\omega t - kz + \varphi_0)}\vec{u}_B$$

Performing the curl on the left hand and simplifying similar terms will produce:

$$-ikE\left(-b\vec{u}_x + a\vec{u}_y\right) = -i\omega B\vec{u}_B \rightarrow \vec{u}_B = \frac{kE}{B}\left(-b\vec{u}_x + a\vec{u}_y\right)$$

Consequently, it will be easy to verify that $\vec{u}_B \perp \vec{u}_E$, *which implies that* $\vec{E}(\vec{r},t)$ *and* $\vec{B}(\vec{r},t)$ *are orthogonal.*

VII. Uniform plane waves

Therefore, the plane electromagnetic wave propagating in the +z-direction, could be represented by both electric and magnetic fields lying on XY plane, with a practical choice ($a = 1, b = 0$):

$$\tilde{E}(\vec{r},t) = |E_0|e^{i(\omega t - kz + \varphi_0)}\vec{u}_x$$

$$\tilde{B}(\vec{r},t) = |B_0|e^{i(\omega t - kz + \varphi_0)}\vec{u}_y$$

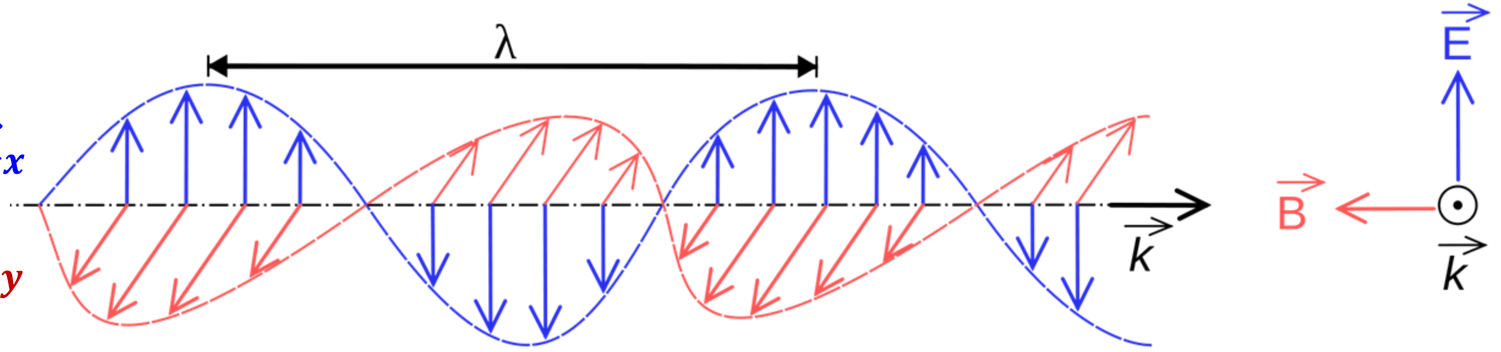
Taking the real part of each phasor:

$$\vec{E}(\vec{r},t) = E_0 cos(\omega t - kz + \varphi_0)\vec{u}_x$$

$$\vec{B}(\vec{r},t) = B_0 cos(\omega t - kz + \varphi_0)\vec{u}_y$$

Thus, the plane E.M wave propagating in a given direction, is represented by two orthogonal E.M fields lying on the perpendicular plan of the propagation direction given by the wave vector $\vec{k}$.

The vectors $\vec{E}, \vec{B}$ and $\vec{k}$ form a direct trihedral.



VIII. Relation between E and H: intrinsic impedance

By considering now that both E.M fields are lying on XY-plane and oriented along $\vec{u}_x$ and $\vec{u}_y$, respectively, the use of the second Maxwell equation will provide the following relation between $\vec{E}$ and $\vec{H}$ (or between $\vec{E}$ and $\vec{B}$), called the "intrinsic impedance" of the given medium of propagation:

$$-ikE = -i\omega\mu H \rightarrow \frac{E[V/m]}{H[A/m]} = \frac{\mu\omega}{k} = \eta[\Omega] = \frac{\mu\omega}{\omega\sqrt{\mu(\varepsilon' - i\varepsilon'')}} = |\eta|e^{i\theta} \leftrightarrow H = \frac{E}{\eta} = \frac{E}{|\eta|}e^{-i\theta}$$

<u>10min Test:</u> In the case of free space, where: $\mu = \mu_0, \varepsilon = \varepsilon_0, \varepsilon'' = 0$, Calculate η_0.

$\mu_0 = 4\pi \times 10^{-7} S.I\ ;\ \varepsilon_0 = 8.85 \times 10^{-12} S.I$

VIII. Relation between E and H: intrinsic impedance

By considering now that both E.M fields are lying on XY-plane and oriented along $\vec{u}_x$ and $\vec{u}_y$, respectively, the use of the second Maxwell equation will provide the following relation between $\vec{E}$ and $\vec{H}$ (or between $\vec{E}$ and $\vec{B}$), called the "intrinsic impedance" of the given medium of propagation:

$$-ikE = -i\omega\mu H \rightarrow \frac{E[V/m]}{H[A/m]} = \frac{\mu\omega}{k} = \eta[\Omega] = \frac{\mu\omega}{\omega\sqrt{\mu(\varepsilon' - i\varepsilon'')}} = |\eta|e^{i\theta} \leftrightarrow H = \frac{E}{\eta} = \frac{E}{|\eta|}e^{-i\theta}$$

<u>10min Test:</u> In the case of free space, where: $\mu = \mu_0, \varepsilon = \varepsilon_0, \varepsilon'' = 0$, Calculate η_0.

$\mu_0 = 4\pi \times 10^{-7} S.I \,;\, \varepsilon_0 = 8.85 \times 10^{-12} S.I$

The intrinsic impedance of free space:

$$k = \omega\sqrt{\mu\varepsilon} \rightarrow \eta_0 = \frac{\mu_0\omega}{\omega\sqrt{\mu_0\varepsilon_0}} = \sqrt{\frac{\mu_0}{\varepsilon_0}}$$

$$= \sqrt{\frac{4\pi \times 10^{-7}}{8.85 \times 10^{-12}}} \cong 377[\Omega] \equiv 120\pi[\Omega]$$

IX. Reminder: Poynting's vector

The Poynting's equation:

$$-\underbrace{\oint_S (\vec{E} \wedge \vec{H}).d\vec{S}}_{\substack{\text{total E.M power} \\ \text{flowing out V through S}}} = \underbrace{\int_V \frac{\partial}{\partial t}\left(\frac{1}{2}\vec{B}.\vec{H}\right).dv}_{\substack{\text{total energy stored} \\ \text{in magnetic field}}} + \underbrace{\int_V \frac{\partial}{\partial t}\left(\frac{1}{2}\vec{D}.\vec{E}\right)}_{\substack{\text{Total energy stored} \\ \text{in electric field}}} + \underbrace{\int_V \vec{E}.\vec{J}\,dv}_{\substack{\text{Ohmic power} \\ \text{dissipated over V}}}$$

This theorem gives the time rates of increase of energy stored within the volume V, or the instantaneous power going to increase the stored energy.

The cross product of $\vec{E}$ and $\vec{H}$ define the Poynting's vector, indicating the power density flowing in the direction of $\vec{\mathcal{P}}$ at a given point. (homonym "Poynting" and "pointing" is accidentally "True")

$$\vec{\mathcal{P}}[W.m^{-2}] = \vec{E} \wedge \vec{H}$$

The measured value of Poynting value is an average value over a specific time (period) and could be obtained using general phasors:

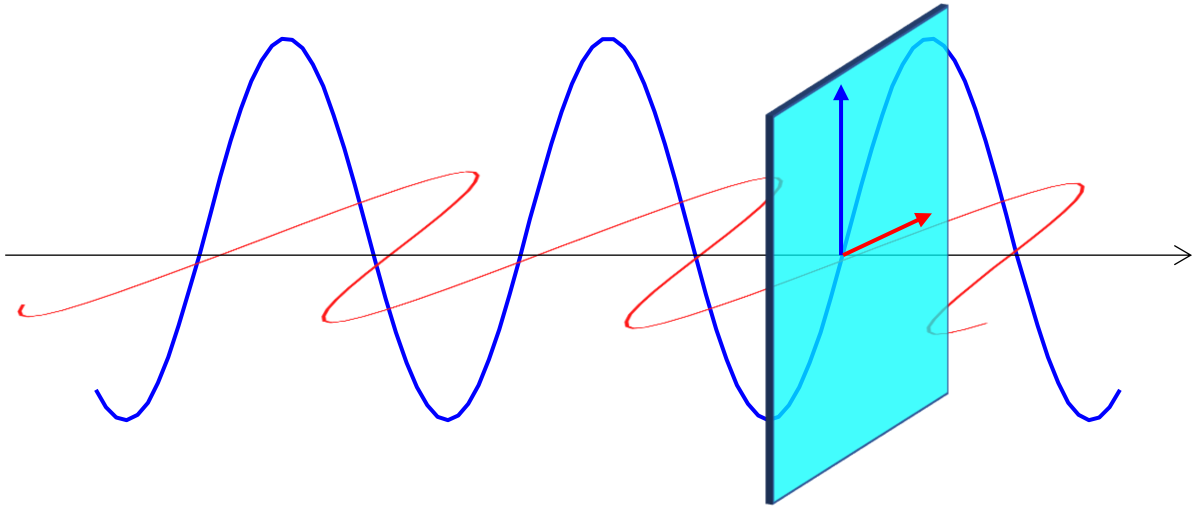
$$\langle\vec{\mathcal{P}}\rangle \equiv \bar{\mathcal{P}} = \frac{1}{2}\Re e\left[\tilde{E} \wedge \tilde{H}^*\right] \propto \frac{1}{2|\eta|}E_0^2 e^{-2\beta z}$$

With: $\tilde{H}^*$ is the conjugate of $\tilde{H}$

X. Polarization of E.M wave

Let's consider a non attenuated plane E.M wave given by its electric and magnetic fields lying on the plane corresponding to the wave front, normal to the incidence direction (using space phasors):

$$\vec{E}(z,t) = \tilde{E}_0 \,.\, e^{\omega t};\ \vec{H}(z,t) = \tilde{H}_0 \,.\, e^{i\omega t}$$



In general, the electric field (and magnetic field) did not keep the same orientation on the wave plane, and it could vary with time and traces a curve by the tip of the field vector on the plane.

In such situation, the electric field (similarly the magnetic field), could be divided into two components on the wave front plane (x-y in this case) propagating in +z-direction, :

$$\tilde{E}(z) = \tilde{E}_x(z)\vec{u}_x + \tilde{E}_y(z)\vec{u}_y$$

And we can set:

$$\tilde{E}_x(z) = E_{x0}e^{-ikz}; \tilde{E}_y(z) = E_{y0}e^{-ikz}$$

X. Polarization of E.M wave

Both initial amplitudes E_{x0} and E_{y0} are in general complex numbers and could be written in exponential form:

$$E_{x0} = a_x e^{i\varphi_x}; E_{y0} = a_y e^{i\varphi_y}$$

With: $a_x = |E_{x0}| > 0; a_y = |E_{y0}| > 0$

Consequently, we can rewrite $\tilde{E}(z)$:

$$\tilde{E}(z) = a_x e^{-ikz} e^{i\varphi_x} \vec{u}_x + a_y e^{-ikz} e^{i\varphi_y} \vec{u}_y \rightarrow \tilde{E}(z) = e^{-ikz} e^{i\varphi_x}\left(a_x \vec{u}_x + a_y e^{i\varphi} \vec{u}_y\right)$$

With: $\varphi = \varphi_y - \varphi_x$ *called the phase difference between* $\tilde{E}_y(z)$ *and* $\tilde{E}_x(z)$

For the sake of simplicity, we can choose to take $\varphi_x = 0 \rightarrow \varphi = \varphi_y$: $\tilde{E}(z) = e^{-ikz}\left(a_x \vec{u}_x + a_y e^{i\varphi} \vec{u}_y\right)$

Taking the real part of the phasor, we will get the instantaneous electric field:

$$\vec{E}(z,t) = a_x . cos(\omega t - kz)\, \vec{u}_x + a_y . cos(\omega t - kz + \varphi)\, \vec{u}_x$$

X. Polarization of E.M wave

The specific cases of the E.M wave polarization could be discussed upon the values of phase difference φ, by analyzing the amplitude of $\vec{E}(z,t)$ and its direction:

The amplitude is given by:

$$|\vec{E}(z,t)| = \left[E_x^2(z,t) + E_y^2(z,t)\right]^{1/2}$$

$$= \left[a_x^2 cos^2(\omega t - kz) + a_y^2 cos^2(\omega t - kz + \varphi)\right]^{1/2}$$

The direction is dictated by the inclination angle:

$$\psi(z,t) = tan^{-1}\left(\frac{E_y(z,t)}{E_x(z,t)}\right)$$

<u>a. Linear polarization $\varphi = 0 \; or \; \pi$:</u>

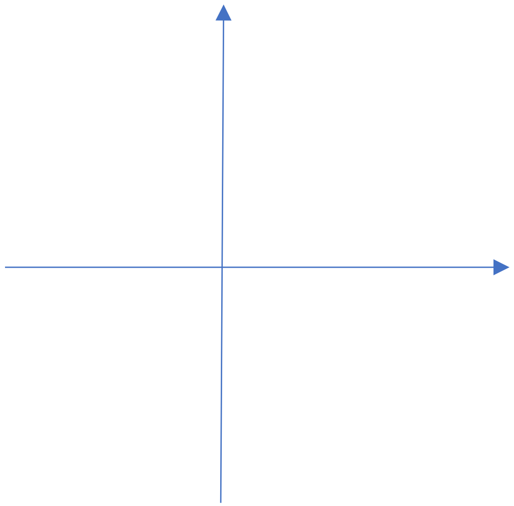
For $\varphi = 0$ (in-phase):

$$\vec{E}(z,t) = cos(\omega t - kz + \varphi)\left(a_x.\vec{u}_x + a_y.\vec{u}_x\right)$$

$$|\vec{E}(z,t)| = \left[a_x^2 + a_y^2\right]^{1/2} |cos(\omega t - kz)|$$

$$\psi(z,t) = tan^{-1}\left(\frac{a_y}{a_x}\right)$$

The amplitude is indeed function of z and t, whereas the direction is not (fixed direction).



X. Polarization of E.M wave

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The amplitude is given by:

$$\left|\vec{E}(z,t)\right| = \left[E_x^2(z,t) + E_y^2(z,t)\right]^{1/2}$$

$$= \left[a_x^2 cos^2(\omega t - kz) + a_y^2 cos^2(\omega t - kz + \varphi)\right]^{1/2}$$

The direction is dictated by the inclination angle:

$$\psi(z,t) = tan^{-1}\left(\frac{E_y(z,t)}{E_x(z,t)}\right)$$

<u>a. Linear polarization $\varphi = 0 \; or \; \pi$:</u>

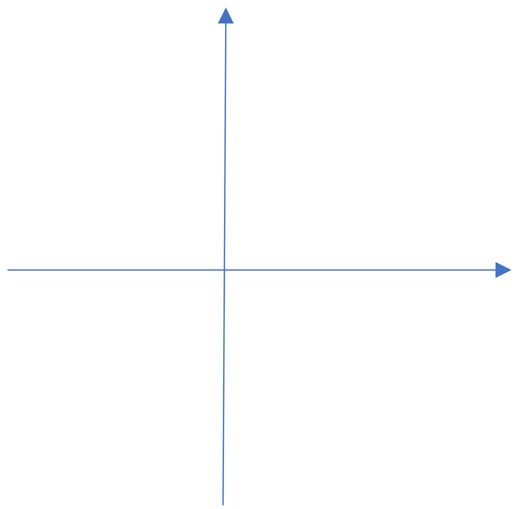
For $\varphi = \pi$ (out-phase):

$$\vec{E}(z,t) = cos(\omega t - kz + \varphi)\left(a_x.\vec{u}_x - a_y.\vec{u}_x\right)$$

$$\left|\vec{E}(z,t)\right| = \left[a_x^2 + a_y^2\right]^{1/2}\left|cos(\omega t - kz)\right|$$

$$\psi(z,t) = tan^{-1}\left(\frac{-a_y}{a_x}\right)$$

The amplitude is indeed function of z and t, whereas the direction is not (fixed direction).



X. Polarization of E.M wave

The specific cases of the E.M wave polarization could be discussed upon the values of phase difference φ, by analyzing the amplitude of $\vec{E}(z,t)$ and its direction:

The amplitude is given by:

$$\left|\vec{E}(z,t)\right| = \left[E_x^2(z,t) + E_y^2(z,t)\right]^{1/2}$$

$$= \left[a_x^2 cos^2(\omega t - kz) + a_y^2 cos^2(\omega t - kz + \varphi)\right]^{1/2}$$

The direction is dictated by the inclination angle:

$$\psi(z,t) = tan^{-1}\left(\frac{E_y(z,t)}{E_x(z,t)}\right)$$

<u>b. Circular polarization</u> $\varphi = \pm {}^{\pi}/_{2}, a_x = a_y = a$

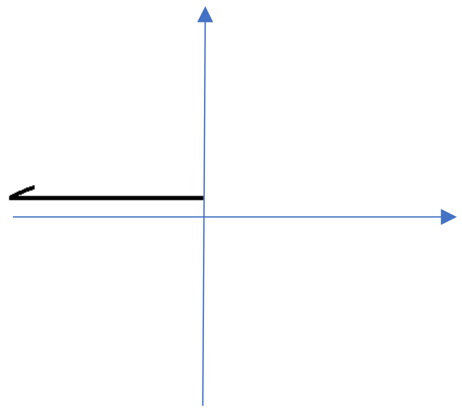
For $\varphi = {}^{\pi}/_{2}$ (Left Circular Polarization):

$$\vec{E}(z,t) = a(cos(\omega t - kz)\vec{u}_x - sin(\omega t - kz)\vec{u}_x)$$

$$\left|\vec{E}(z,t)\right| = a$$

$$\psi = tan^{-1}\left(\frac{-a.sin(\omega t - kz)}{a.cos(\omega t - kz)}\right) = -(\omega t - kz)$$

The direction is tracing a circular movement in counter-clockwise direction.



X. Polarization of E.M wave

The specific cases of the E.M wave polarization could be discussed upon the values of phase difference φ, by analyzing the amplitude of $\vec{E}(z,t)$ and its direction:

The amplitude is given by:

$$\left|\vec{E}(z,t)\right| = \left[E_x^2(z,t) + E_y^2(z,t)\right]^{1/2}$$

$$= \left[a_x^2 cos^2(\omega t - kz) + a_y^2 cos^2(\omega t - kz + \varphi)\right]^{1/2}$$

The direction is dictated by the inclination angle:

$$\psi(z,t) = tan^{-1}\left(\frac{E_y(z,t)}{E_x(z,t)}\right)$$

<u>b. Circular polarization</u> $\varphi = \pm \pi/2\,, a_x = a_y = a$

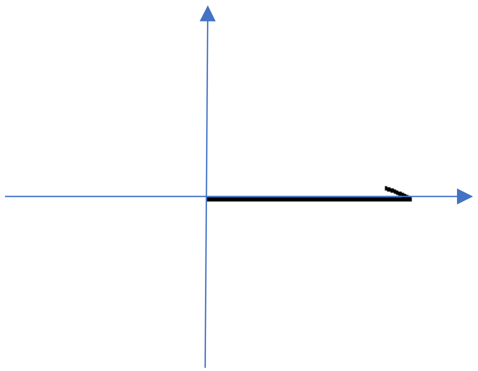
For $\varphi = -\pi/2$ (Right Circular Polarization):

$$\vec{E}(z,t) = a(cos(\omega t - kz)\vec{u}_x + sin(\omega t - kz)\vec{u}_x)$$

$$\left|\vec{E}(z,t)\right| = a$$

$$\psi = tan^{-1}\left(\frac{a.sin(\omega t - kz)}{a.cos(\omega t - kz)}\right) = (\omega t - kz)$$

The direction is tracing a circular movement in counter-clockwise direction.



X. Polarization of E.M wave

The specific cases of the E.M wave polarization could be discussed upon the values of phase difference $\boldsymbol{\varphi}$, by analyzing the amplitude of $\vec{E}(z,t)$ and its direction:

The amplitude is given by:

$$\left|\vec{E}(z,t)\right| = \left[E_x^2(z,t) + E_y^2(z,t)\right]^{1/2}$$

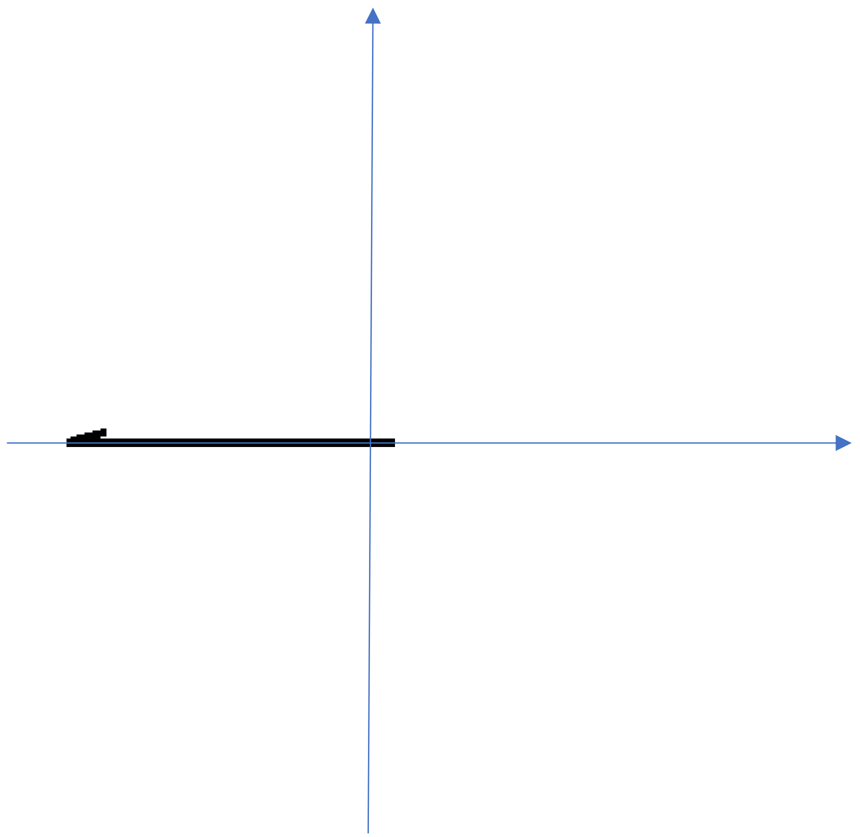
$$= \left[a_x^2 cos^2(\omega t - kz) + a_y^2 cos^2(\omega t - kz + \varphi)\right]^{1/2}$$

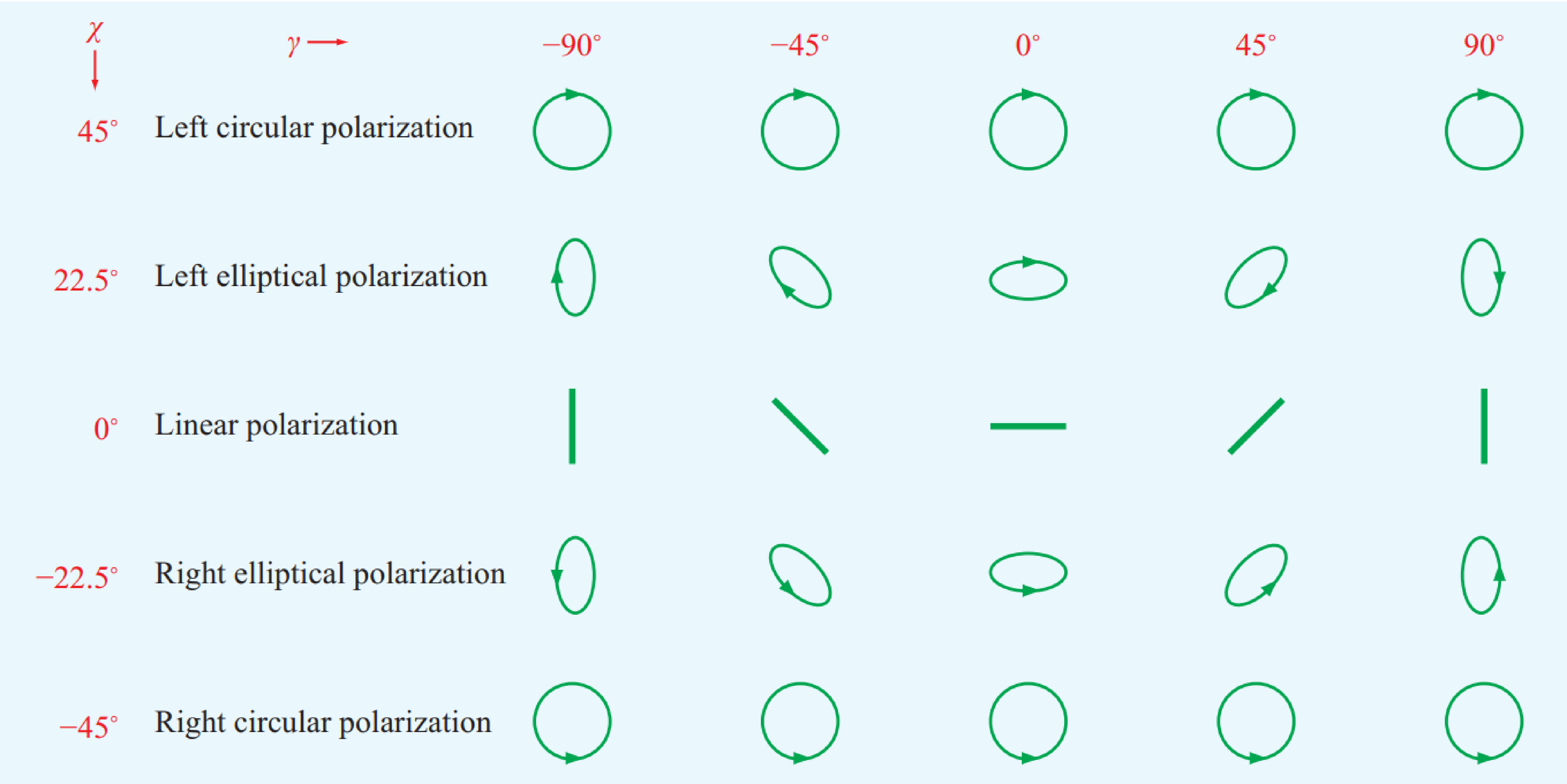
The direction is dictated by the inclination angle:

$$\psi(z,t) = tan^{-1}\left(\frac{E_y(z,t)}{E_x(z,t)}\right)$$

b. Elliptical polarization

$$0 < \varphi < {}^{\pi}/_{2}\,, a_x \neq a_y$$





χ
γ
−90°
−45°
0°
45°
90°
45°
Left circular polarization
22.5°
Left elliptical polarization
0°
Linear polarization
−22.5°
Right elliptical polarization
−45°
Right circular polarization