Serie of Exercises

Chapter 01: Maxwell's equations

Exercise 01:

1. Demonstrate that for any vector:
$$\vec{A} = A_x\vec{\imath} + A_y\vec{\jmath} + A_z\vec{k}$$
We always get the following result:
$$\vec{\nabla}.\left(\vec{\nabla}\wedge\vec{A}\right) = 0$$
2. Do the same for any scalar function f:
$$\vec{\nabla}\wedge\left(\vec{\nabla}f\right) = 0$$

Exercise 02:

Find the electric field at point P with Cartesian coordinates $(0, 0, h)$ due to a circular disk of radius a and uniform charge density σ residing in the x–y plane (Fig. 1). Also, evaluate E due to an infinite sheet of charge density σ by letting $a \rightarrow \infty$.

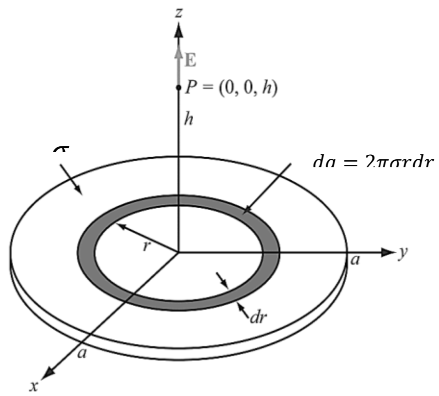

Figure 1. Electric field created by a charged disk

Exercise 03:

Two infinite lines, each carrying a uniform charge density λ , reside in free space parallel to the z-axis at $x = 1$ and $x = -1$. Determine $\vec{E}$ at an arbitrary point along the y axis

Exercise 04:

A circular loop of radius r carries a steady current I. Determine the magnetic field $\vec{B}$ at a point on the axis of the loop.

Exercise 05:

Two parallel wires of length $l_1 = l_2 = l$, separated by a distance $d \ll l$ carrying each one steady current I_1 and I_2, respectively. Both currents are flowing in the same direction (Fig. 3).

(a) By using Ampere law, determine the corresponding magnetic field induced by each wire at the location of the other wire.

(b) What is the Laplace force applied on each wire by the other one.

(c) Deduce the forces by unit length

(d) Suppose that the conductor carrying the current I_2 is rotated so that it is parallel to the x-axis. What would $\vec{F}_2$be in this case?

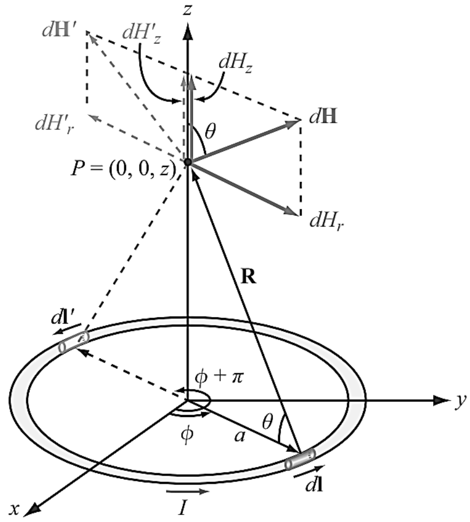

Figure 2. Circular loop carrying a current I

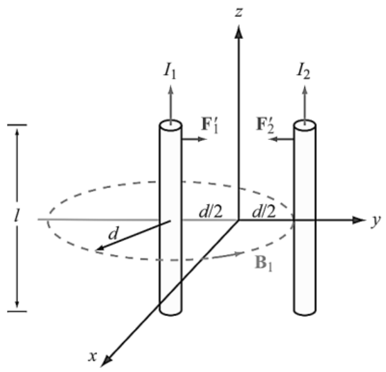

Figure 3. Parallel wires carrying steady currents

Exercise 06:

A long (practically infinite) straight wire of radius carries a steady current I that is uniformly distributed over its cross section (Fig. 4). Determine the magnetic field H a distance r from the wire axis for (a) $r \leq a$ (inside the wire) and (b) $r \geq a$ (outside
the wire).

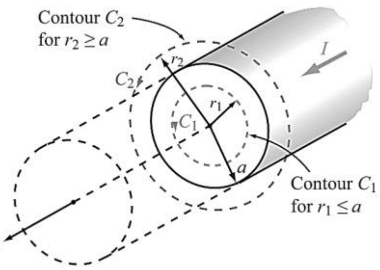

Figure 4. Infinitely long wire of radius a carrying a uniform current I along the +z direction

Exercise 07:

An inductor is formed by winding N turns of a thin conducting wire into a circular loop of radius a. The inductor loop is in the $x-y$ plane with its center at the origin and connected to a resistor R, as shown in Fig. 5. In the presence of a magnetic field $B = B_0(2\vec{j} + 3\vec{k})\sin\omega t$, where ω is the angular frequency, find:

(a) the magnetic flux linking a single turn of the inductor,
(b) the transformer V_{emf} given that $N = 10$, $B_0 = 0.2\,T, a = 10\,cm$, and $\omega = 103\,rad/s$,
(c) the polarity of V_{emf}^{tr} at $t = 0$, and
(d) the induced current in the circuit for $R = 1\,k\Omega$ (assume the wire resistance to be much smaller than R).

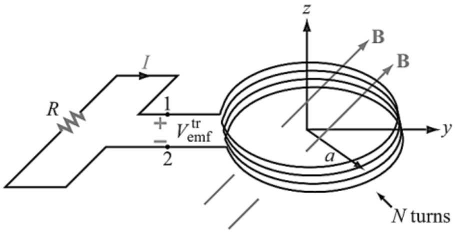

Figure 5. Circular loop with N turns in the x-y plane.

Exercise 08:

Determine voltages V_1 and V_2 across the 2Ω and 4Ω resistors shown in Fig. 6. The loop is located in the $x-y$ plane, its area is $4m^2$, the magnetic flux density is $B = -0.3t\vec{k}[T]$, and the internal resistance of the wire may be ignored.

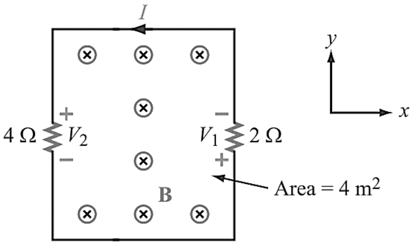

Figure 6. Circuit under magnetic flux

Exercise 09:

A rectangular conducting loop (with an area $S = l.w$) is rotated around its longest axis in the presence of a homogeneous magnetic field along $+z$-direction as shown in Fig. 7. The rotation is expressed by the angle α and ruled by the angular frequency ω:

$\alpha = \omega t + \alpha_0$,

where α_0 is the initial angle corresponding to $t = 0$.

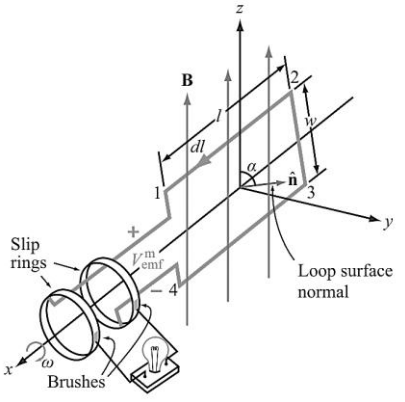

Figure 1. Electromagnetic generator principle

(a) express the magnetic flux for any instant t as a function of magnetic field amplitude B, the surface enclosed by the loop shape S, the angle α made between the $\vec{B}$ and the surface vector $\vec{n}$

(b) deduce the emf produced at the ends of the rectangular loop in each instant t

Exercise 10:

The Ohm law in its point form, relates between the current density $\vec{j}[A.m^{-2}]$ and the applied electric field $\vec{E}[V.m^{-1}]$, through the conductivity $\sigma[S.m^{-1}]$ of the medium in which the charges flow: $\vec{j} = \sigma\vec{E}$.

(a) Use the charge continuity equation: $\vec{\nabla}.\vec{j} + \frac{\partial\rho}{\partial t} = 0$ and 1st Maxwell's equation to establish the differential equation of charge density variation in such a medium,

(b) Solve this time-equation by considering the boundary conditions: $\rho(t = 0) = \rho_0$, and define the characteristic time τ of ρ variation (Free-charge dissipation)

(c) Evaluate and comment τ for both cases:

(1) Copper: $\sigma = 5.8 \times 10^7 [S.m^{-1}]$; $\varepsilon \approx \varepsilon_0$

(2) Mica: $\sigma = 10^{-15}[S.m^{-1}]$; $\varepsilon = 6\varepsilon_0$

Exercise 11:

The conduction current flowing through a cross-section $S = 1mm^2$ of a wire with a conductivity $\sigma = 2 \times 10^7\ [S.m^{-1}]$ and permittivity $\varepsilon = \varepsilon_0$, is given by:

$$I_c = 2\sin\omega t\ \ [mA]$$

If $\omega = 10^9\ [rad.s^{-1}]$, find the displacement current $\vec{G} \equiv \vec{j}_d = \varepsilon\frac{\partial\vec{E}}{\partial t}$.

Exercise 12:

Considering equivalent situation like in Exercise 10, evaluate the displacement current for a poor conductor with $\sigma = 100\ [S.m^{-1}]$ and permittivity $\varepsilon = 4\varepsilon_0$. What should be the angular frequency to get equivalent amplitudes $\|J_c\| = \|J_d\|$?