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Advanced Electromagnetism

M1 Applied Physics

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2024 / 2025

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Chapter 02

Electromagnetic potential and energy

Reminder 1

Electrostatic and Magnetostatic equations:

This set of four equations could be expressed as a double set of decoupled equations since no explicit relationships exist between electric and magnetic fields:

$$\begin{cases} \vec{\nabla}.\vec{E} = \dfrac{\rho}{\varepsilon_0} \\ \vec{\nabla} \wedge \vec{E} = 0 \end{cases} ; \begin{cases} \vec{\nabla}.\vec{B} = 0 \\ \vec{\nabla} \wedge \vec{B} = \mu_0 \vec{j} \end{cases}$$

This allows us to study electricity and magnetism as two distinct and separate phenomena as long as the spatial distributions of charge and current flow remain constant in time.

I. Scalar electric potential (Electrostatic)

We should recall that for any scalar f we have:

$$\vec{\nabla} \wedge (\vec{\nabla} f) = 0$$

If we take the 2nd E.M equation:

$$\vec{\nabla} \wedge \vec{E} = 0$$

It is easy to conclude that electric field should be derived from a scalar function, which is the scalar electric potential $(f \equiv V)$

$$\vec{E} = \vec{\nabla} f \rightarrow \vec{\nabla} \wedge \vec{E} = \vec{\nabla} \wedge (\vec{\nabla} f) = 0$$

By convention we consider that the variation of electric field is the opposite direction of the gradient of electrical potential V:

$$\vec{E} = -\vec{\nabla} V$$

By definition, we can write the total differentiation of electrical potential as:

$$dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz$$

With the definition:

$$\vec{E} = -\vec{\nabla} V \leftrightarrow \frac{\partial V}{\partial x} = -E_x ; \frac{\partial V}{\partial y} = -E_y ; \frac{\partial V}{\partial z} = -E_z$$

This allows us to write the other relation between electrical field and scalar potential:

$$dV = -\vec{E} . d\vec{l} \rightarrow V = -\int_C \vec{E} . d\vec{l}$$

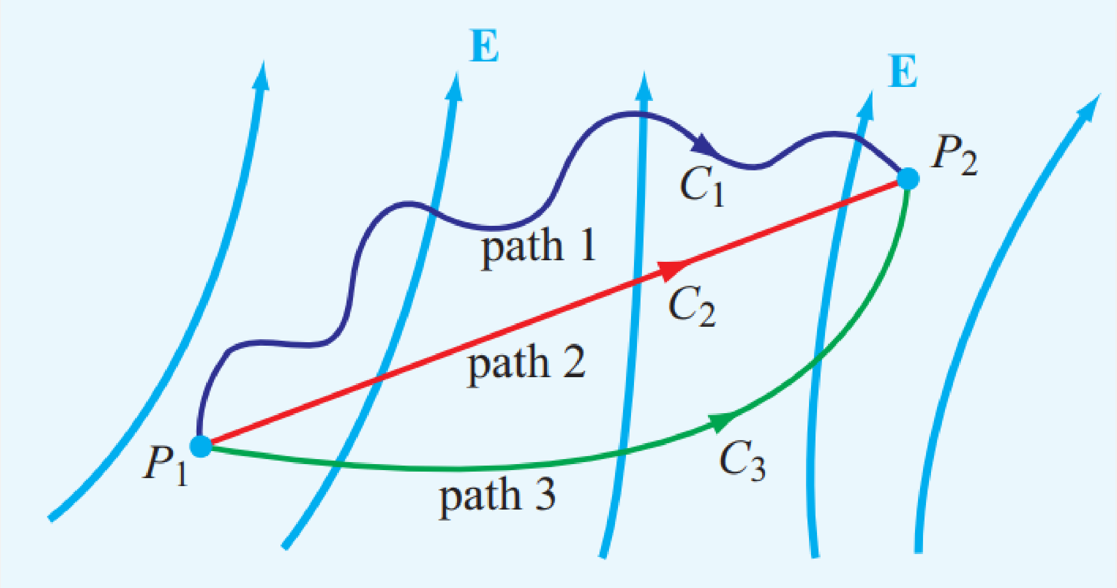
Which gives the potential as the circulation of the electrical field.

I. Scalar electric potential (Electrostatic)

To be more accurate, the integral of the circulation on a given path should start from a given point and end at another one.

This leads us to define The potential difference corresponding to moving a point charge from point *P_1 to point P_2, which is obtained by integrating the last expression along any path between them:*

$$\Delta V = V_{21} = V_2 - V_1 = \int_{P_1}^{P_2} dV = -\int_{P_1}^{P_2} \vec{E}.d\vec{l}$$



Let's recall here once again, that this integral did not depend on the taken path, but only on the starting and ending points. That what make the electrostatic field conservative.

II. Gauss's law and Poisson's equation

Now, using both equations:

$$\vec{E} = -\vec{\nabla} V \ldots\ldots\ldots(1)$$

$$\vec{\nabla}.\vec{E} = \frac{\rho}{\varepsilon_0} \ldots\ldots\ldots(2)$$

And, by replacing (1) into (2), we obtain:

$$\vec{\nabla}.\vec{E} = -\vec{\nabla}.(\vec{\nabla} V) = -\nabla^2 V = -\Delta V = \frac{\rho}{\varepsilon_0}$$

Which could be rewritten as:

$$\Delta V = -\frac{\rho}{\varepsilon_0} \leftrightarrow \Delta V + \frac{\rho}{\varepsilon_0} = 0$$

This a second degree differential equation with source term is known as *"Poisson's equation".*

The special case of absence of electric charges in the free space, the Poisson's equation will be reduced to homogeneous differential equation:

$$\Delta V = \nabla^2 V = 0$$

Known as *"Laplace's equation".*

In rectangular coordinates:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

In cylindrical coordinates:

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \varphi^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

In spherical coordinates:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2.sin\theta}\frac{\partial}{\partial \theta}\left(sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2.sin^2\theta}\frac{\partial^2 V}{\partial \varphi^2} = 0$$

III. Magnetostatic vector potential

Again, let's recall here that for any given vector $\vec{A}$, we always get:

$$\vec{\nabla}.\left(\vec{\nabla} \wedge \vec{A}\right) = 0$$

By taking now the 3rd E.M equation:

$$\vec{\nabla}.\vec{B} = 0$$

It becomes easy to see that to verify this equation, the magnetic field $\vec{B}$ needs to be always a curl of primary vector $\vec{A}$, known as vector magnetic potential, in similar way to the electrostatic field derived from scalar potential.

As a result the magnetic field could be written as:

$$\vec{B} = \vec{\nabla} \wedge \vec{A}$$

In such way, we could always verify that:

$$\vec{\nabla}.\vec{B} = \vec{\nabla}.\left(\vec{\nabla} \wedge \vec{A}\right) = 0; \ \forall \vec{A}$$

Since the magnetic field unit in S.I is Tesla :

$$1[T] = 1\left[Weber.m^{-2}\right] = 1\left[Wb.m^{-2}\right]$$

Consequently the S.I unit for the vector magnetic potential will be : $Wb.m^{-1} \equiv \frac{Wb}{m}$

IV. Vector Poisson's equation

In the same way as we achieve it for electric field, let's exploit both equations:

$$\vec{B} = \vec{\nabla} \wedge \vec{A} \ldots\ldots\ldots (3)$$

$$\vec{\nabla} \wedge \vec{B} = \mu_0 \vec{j} \ldots\ldots\ldots (4)$$

And replace (3) into (4):

$$\vec{\nabla} \wedge \vec{B} = \vec{\nabla} \wedge \left(\vec{\nabla} \wedge \vec{A}\right) = \mu_0 \vec{j}$$

We know already (from vector calculus) that for any vector $\vec{A}$, the Laplacian of $\vec{A}$ obeys the vector identity given by:

$$\nabla^2 \vec{A} = \vec{\nabla}\left(\vec{\nabla}.\vec{A}\right) - \vec{\nabla} \wedge \left(\vec{\nabla} \wedge \vec{A}\right)$$

This implies:

$$\vec{\nabla}\left(\vec{\nabla}.\vec{A}\right) - \nabla^2 \vec{A} = \mu_0 \vec{j}$$

An appropriate and simplest choice about the term $\vec{\nabla}.\vec{A}$ is to take (Coulomb gauge):

$$\vec{\nabla}.\vec{A} = 0$$

To avoid any conflicting with the requirement of equation (3).

Using this choice leads to the *"Vector Poisson's equation"*:

$$\Delta \vec{A} = -\mu_0 \vec{j}$$

Which is very similar to the Poisson's equation for the scalar electric potential:

$$\Delta V = -\frac{\rho}{\varepsilon_0}$$

IV. Vector Poisson's equation

Using the definition for $\nabla^2\vec{A}$, the vector Poisson's equation can be decomposed into three scalar Poisson's equations

$$\begin{cases} \dfrac{\partial^2 A_x}{\partial x^2} = -\mu_0 J_{Sx} \\ \dfrac{\partial^2 A_x}{\partial y^2} = -\mu_0 J_{Sy} \\ \dfrac{\partial^2 A_x}{\partial z^2} = -\mu_0 J_{Sz} \end{cases}$$

As for Poisson's equation for scalar potential, it is possible to get back into vector potential components:

$$A_x = \frac{\mu_0}{4\pi}\int_{S\prime} \frac{J_{Sx}}{r} dS$$

Similar solutions could be found for the remain components y and z:

Volume density:

$$\vec{A} = \frac{\mu_0}{4\pi}\int_{v\prime} \frac{\vec{J}_V}{r} dv$$

Surface density:

$$\vec{A} = \frac{\mu_0}{4\pi}\int_{S\prime} \frac{\vec{J}_S}{r} dS$$

Linear density:

$$\vec{A} = \frac{\mu_0}{4\pi}\int_{l\prime} \frac{I}{r} d\vec{l}$$

The vector magnetic potential provides a 3rd approach for computing the magnetic field due to current-carrying conductors in addition to the methods suggested by Biot-Savart and Ampère law.

Reminder 2

Maxwell's equations:

The electromagnetism now are well described by the set of Maxwell's equations:

$$
\begin{cases}
\vec{\nabla}.\vec{E} = \dfrac{\rho}{\varepsilon_0} & I(Maxwell - Gauss\ law) \\
\vec{\nabla} \wedge \vec{E} = -\dfrac{\partial \vec{B}}{\partial t} & II(Maxwell - Faraday\ Law) \\
\vec{\nabla}.\vec{B} = 0 & III(Gauss\ Law\ for\ magnetism) \\
\vec{\nabla} \wedge \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \dfrac{\partial \vec{E}}{\partial t} & IV(Maxwell - Ampere\ Law)
\end{cases}
$$

These equations are also known as Maxwell equations for time-varying fields $\vec{E}(t)$ and $\vec{B}(t)$.

V. Electromagnetic potential

Both Faraday's and Ampère's laws revealed two aspects of the link between time-varying electric and magnetic fields. Let's now examine the implications of this interconnection on the electric scalar potential V and the vector magnetic potential A.

Indeed, we have already saw that in electrostatic, that the 2nd equation; $\vec{\nabla}\wedge\vec{E} = 0$; implies $\vec{E} = -\vec{\nabla}V$

While in dynamic case, (considering that $\vec{B} = \vec{\nabla} \wedge \vec{A}$) the 2nd equation (Maxwell's equation) implies:

$$\vec{\nabla}\wedge\vec{E} = -\frac{\partial\vec{B}}{\partial t} = -\frac{\partial(\vec{\nabla}\wedge\vec{A})}{\partial t} = -\vec{\nabla}\wedge\left(\frac{\partial\vec{A}}{\partial t}\right)$$

$$\vec{\nabla}\wedge\left(\vec{E} + \frac{\partial\vec{A}}{\partial t}\right) = 0 \quad \ldots\ldots (5)$$

If we define a new electrical field $\overrightarrow{E'}$ such as:

$$\overrightarrow{E'} = \vec{E} + \frac{\partial\vec{A}}{\partial t} \ldots\ldots (6)$$

Which is verifying the following equation:

$$\vec{\nabla}\wedge\overrightarrow{E'} = 0$$

Recalling the 2nd equation of Electrostatic. This means that the new electric field is a conservative field and could be derived from a scalar electric field as follows:

$$\overrightarrow{E'} = -\vec{\nabla}V \quad \ldots\ldots (7)$$

By substituting (6) in (7) we get:

$$\vec{E} = -\vec{\nabla}V - \frac{\partial\vec{A}}{\partial t}$$

V. Electromagnetic potential

Finally both electromagnetic fields $(\vec{E}, \vec{B})$ in dynamics case (time-dependent system), could be obtained from scalar and vector potential, forming an electromagnetic potential:

$$\begin{cases} \vec{E} = -\vec{\nabla}V - \dfrac{\partial \vec{A}}{\partial t} \\ \vec{B} = \vec{\nabla} \wedge \vec{A} \end{cases}$$

Besides that, both potential verify Poisson's equations:

$$\begin{cases} \Delta V = -\dfrac{\rho}{\varepsilon_0} \\ \Delta \vec{A} = -\mu_0 \vec{J} \end{cases}$$

Where, in general case of volume distribution charge and density current, the calculation of both potential is given through volume integrals:

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|} dV'$$

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}')}{|\vec{r} - \vec{r}'|} dV'$$

The physics will be unchanged under gauge transformations of both scalar and vector potentials:

$V \rightarrow V + \phi_0$ **and** $\vec{A} \rightarrow \vec{A} + \vec{\nabla}\psi_0$

Where : ϕ_0 **is a constant, and** ψ_0 **is a scalar function, since:**

$$-\vec{\nabla}(V + \phi_0) = -\vec{\nabla}V = \vec{E}; (\vec{\nabla}\phi_0 = 0)$$

$$\vec{\nabla} \wedge (\vec{A} + \vec{\nabla}\psi_0) = \vec{\nabla} \wedge \vec{A} = \vec{B};\ (\vec{\nabla} \wedge (\vec{\nabla}\psi_0) = 0)$$

VI. Poynting's theorem and E.M energy

In order to find the power flow associated with a time-dependent electromagnetic field, it was necessary to develop a power theorem for the electromagnetic field known as the Poynting theorem. It was originally postulated in 1884 by an English physicist, John H. Poynting.

The development begins with the fourth Maxwell's equation, in which we assume that the medium may be conductive:

$$\vec{\nabla} \wedge \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

Next, we take the scalar product of both sides with $\vec{E}$:

$$\vec{E}.(\vec{\nabla} \wedge \vec{H}) = \vec{E}.\vec{J} + \vec{E}.\frac{\partial \vec{D}}{\partial t}$$

Using the following vectors identity:

$$\vec{\nabla}.(\vec{E} \wedge \vec{H}) = \vec{H}.\vec{\nabla} \wedge \vec{E} - \vec{E}.\vec{\nabla} \wedge \vec{H}$$

Using the latter equation in the left side of IV Maxwell's equation:

$$\vec{H}.(\vec{\nabla} \wedge \vec{E}) - \vec{\nabla}.(\vec{E} \wedge \vec{H}) = \vec{E}.\vec{J} + \vec{E}.\frac{\partial \vec{D}}{\partial t}$$

$$\vec{H}.\left(-\frac{\partial \vec{B}}{\partial t}\right) - \vec{\nabla}.(\vec{E} \wedge \vec{H}) = \vec{E}.\vec{J} + \vec{E}.\frac{\partial \vec{D}}{\partial t}$$

VI. Poynting's theorem and E.M energy

Make few adjustments about derivatives, since we know that: $\vec{D} = \varepsilon\vec{E};\ \vec{B} = \mu\vec{H}$,

we can write:

$$\vec{E}.\frac{\partial\vec{D}}{\partial t} = \varepsilon\vec{E}.\frac{\partial\vec{E}}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{2}\vec{D}.\vec{E}\right)$$

$$\vec{H}.\frac{\partial\vec{B}}{\partial t} = \mu\vec{H}.\frac{\partial\vec{H}}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{2}\vec{B}.\vec{H}\right)$$

We get:

$$-\vec{\nabla}.\left(\vec{E}\wedge\vec{H}\right) = \frac{\partial}{\partial t}\left(\frac{1}{2}\vec{B}.\vec{H}\right) + \frac{\partial}{\partial t}\left(\frac{1}{2}\vec{D}.\vec{E}\right) + \vec{E}.\vec{J}$$

Integrated over given volume V:

$$-\int_V \vec{\nabla}.\left(\vec{E}\wedge\vec{H}\right).dv = \int_V \frac{\partial}{\partial t}\left(\frac{1}{2}\vec{B}.\vec{H}\right).dv + \int_V \frac{\partial}{\partial t}\left(\frac{1}{2}\vec{D}.\vec{E}\right) + \int_V \vec{E}.\vec{J}\,dv$$

The trick is to pass from volume integral to a surface one:

$$\int_V \vec{\nabla}.\vec{A}dv = \oint_S \vec{A}.d\vec{S}$$

VI. Poynting's theorem and E.M energy

The new form of Poynting's equation:

$$-\underbrace{\oint_S (\vec{E}\wedge\vec{H}).d\vec{S}}_{\substack{total\ E.M\ power \\ flowing\ out\ V\ through\ S}} = \underbrace{\int_V \frac{\partial}{\partial t}\left(\frac{1}{2}\vec{B}.\vec{H}\right).dv}_{\substack{total\ energy\ stored \\ in\ magnetic\ field}} + \underbrace{\int_V \frac{\partial}{\partial t}\left(\frac{1}{2}\vec{D}.\vec{E}\right)}_{\substack{Total\ energy\ stored \\ in\ electric\ field}} + \underbrace{\int_V \vec{E}.\vec{J}\,dv}_{\substack{Ohmic\ power \\ dissipated\ over\ V}}$$

This theorem gives the time rates of increase of energy stored within the volume V, or the instantaneous power going to increase the stored energy.

The cross product of $\vec{E}$ and $\vec{H}$ define the Poynting's vector, indicating the power density flowing in the direction of $\vec{\mathcal{P}}$ at a given point. (homonym "Poynting" and "pointing" is accidentally "True")

$$\vec{\mathcal{P}}[W.m^{-2}] = \vec{E}[V.m^{-1}]\wedge\vec{H}[A.m^{-1}]$$