Khemis Miliana University – Djilali BOUNAAMA
Faculty of Material Sciences and Computer Science
Department of Physics

Advanced Electromagnetism

M1 Applied Physics

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1. Maxwell's Equations
2. Electromagnetic potential and energy
3. Propagation of EM Waves in free space
4. Reflection and diffraction of EM waves
5. Wave guide and antennas

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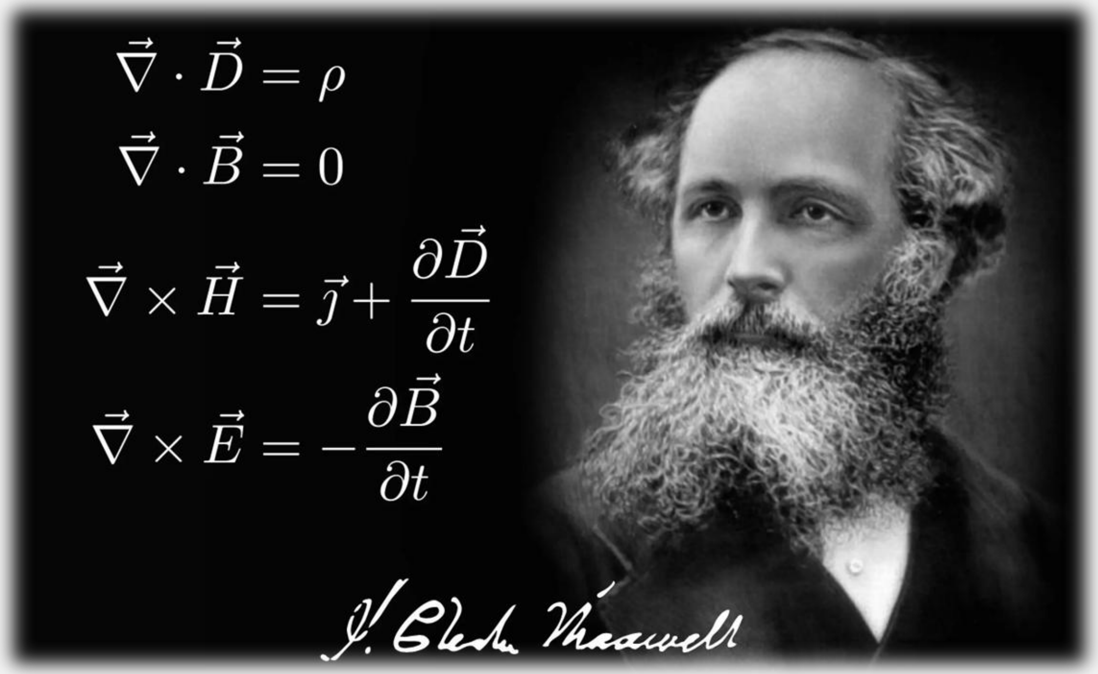
Chapter 01

Maxwell's Equations



I. Class Reminder

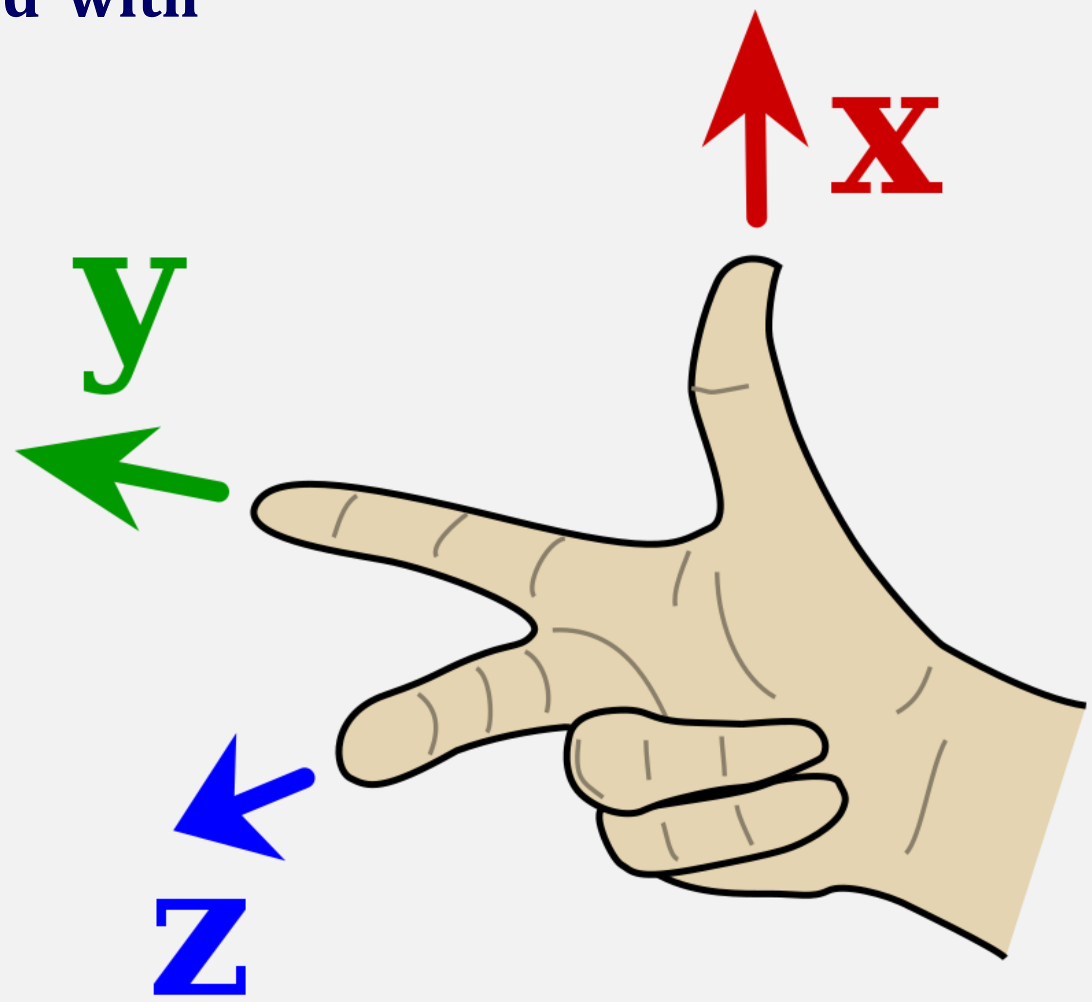
The cartesian frame system is defined with the orthonormal basis:

$\vec{i}.\vec{i} = \vec{j}.\vec{j} = \vec{k}.\vec{k} = 1$

$\vec{i}.\vec{j} = \vec{j}.\vec{k} = \vec{k}.\vec{i} = 0$

$\vec{i} \wedge \vec{i} = \vec{j} \wedge \vec{j} = \vec{k} \wedge \vec{k} = 0$

$\vec{i} \wedge \vec{j} = \vec{k};\ \vec{j} \wedge \vec{k} = \vec{i};\ \vec{k} \wedge \vec{i} = \vec{j}$



I. Class Reminder

❑ Line element:

- *Cartesian coordinates:*

$$d\vec{l} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$

- *Cylindrical coordinates:*

$$d\vec{l} = d\rho\vec{u}_\rho + \rho d\varphi\vec{u}_\varphi + dz\vec{k}$$

- *Spherical coordinates:*

$$d\vec{l} = dr\vec{u}_r + r.\sin\theta d\varphi\vec{u}_\varphi + rd\theta\vec{u}_\varphi$$

❑ Surface element:

- *Cartesian coordinates:*

$$OZ:\ dS = dx.dy$$

$$OY:\ dS = dx.dz$$

$$OX:\ dS = dy.dz$$

- *Cylindrical coordinates:*

$$radial:\ dS = \rho d\varphi dz$$

$$axial:\ dS = \rho d\varphi d\rho$$

- *Spherical coordinates:*

$$radial: dS = r^2 \sin\theta d\varphi d\theta$$

I. Class Reminder

For a given variable x, we recall that partial derivation noted:

$$\frac{\partial}{\partial x} = \partial_x = \left.\frac{d}{dx}\right|_{y=z=t=Cte}$$

We define the vector operator Nabla:

$$\vec{\nabla} = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$$

In such case, when applied on a given vector $\vec{A}$, we obtain the divergent of $\vec{A}$ (Scalar):

$$\vec{\nabla}.\vec{A} = div.\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Similarly, the curl of $\vec{A}$ (Vector) is given by:

$$\vec{\nabla} \wedge \vec{A} = rot\vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ A_x & A_y & A_z \end{vmatrix}$$

When Nabla operator is applied on scalar function $f(x, y, z)$ it gives the gradient of f:

$$\vec{\nabla}f = \overrightarrow{grad}f = \vec{i}\frac{\partial f}{\partial x} + \vec{j}\frac{\partial f}{\partial y} + \vec{k}\frac{\partial f}{\partial z}$$

The Nabla operator could be applied twice on the same operand (scalar or vector function):

- *The scalar function*

$$(\vec{\nabla}.\vec{\nabla})f = \nabla^2 f = \Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

This squared operator is called Laplacian.

I. Class Reminder

- ***The vector function:***

$$\nabla^2\vec{A} = \nabla^2 A_x\vec{i} + \nabla^2 A_y\vec{j} + \nabla^2 A_z\vec{k}$$

Also, given as follows:

$$\nabla^2\vec{A} = \vec{\nabla}(\vec{\nabla}.\vec{A}) - \vec{\nabla}\wedge(\vec{\nabla}\wedge\vec{A})$$

Besides that, it is possible to demonstrate that alternate application of Nabla with dot and cross product on scalar or vector function will give always null result:

- $\vec{\nabla}.(\vec{\nabla}\wedge\vec{A}) = 0, \forall\vec{A}$
- $\vec{\nabla}\wedge(\vec{\nabla}f) = 0, \forall f$

❖ Useful integral rules:

For a given vector field $\vec{A}$, we have:

- **Stokes ' theorem (1D → 2D)**

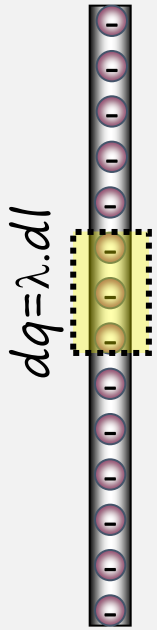
$$\oint_C \vec{A}.d\vec{l} = \iint_S (\vec{\nabla}\wedge\vec{A}).d\vec{S}$$

- **Divergence theorem/Ostrogradsky's theorem (2D → 3D)**

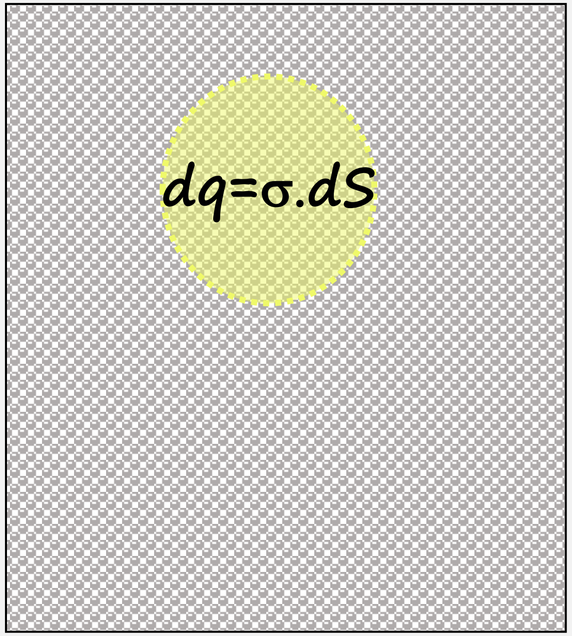
$$\oiint_S \vec{A}.d\vec{S} = \iiint_V (\vec{\nabla}.\vec{A})\,dV$$

I. Class Reminder

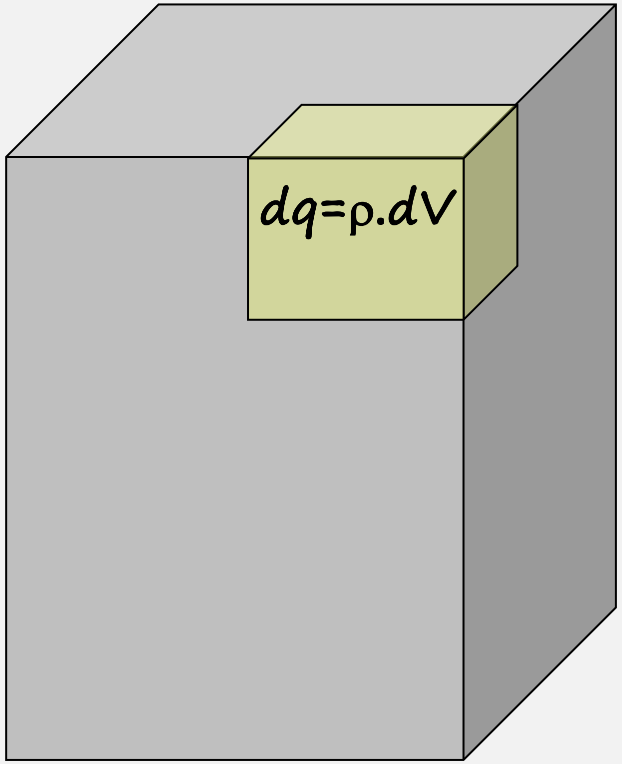
Continuous distribution of electrical charge



$$\lambda = \frac{dq}{dl}\,[C/m]$$



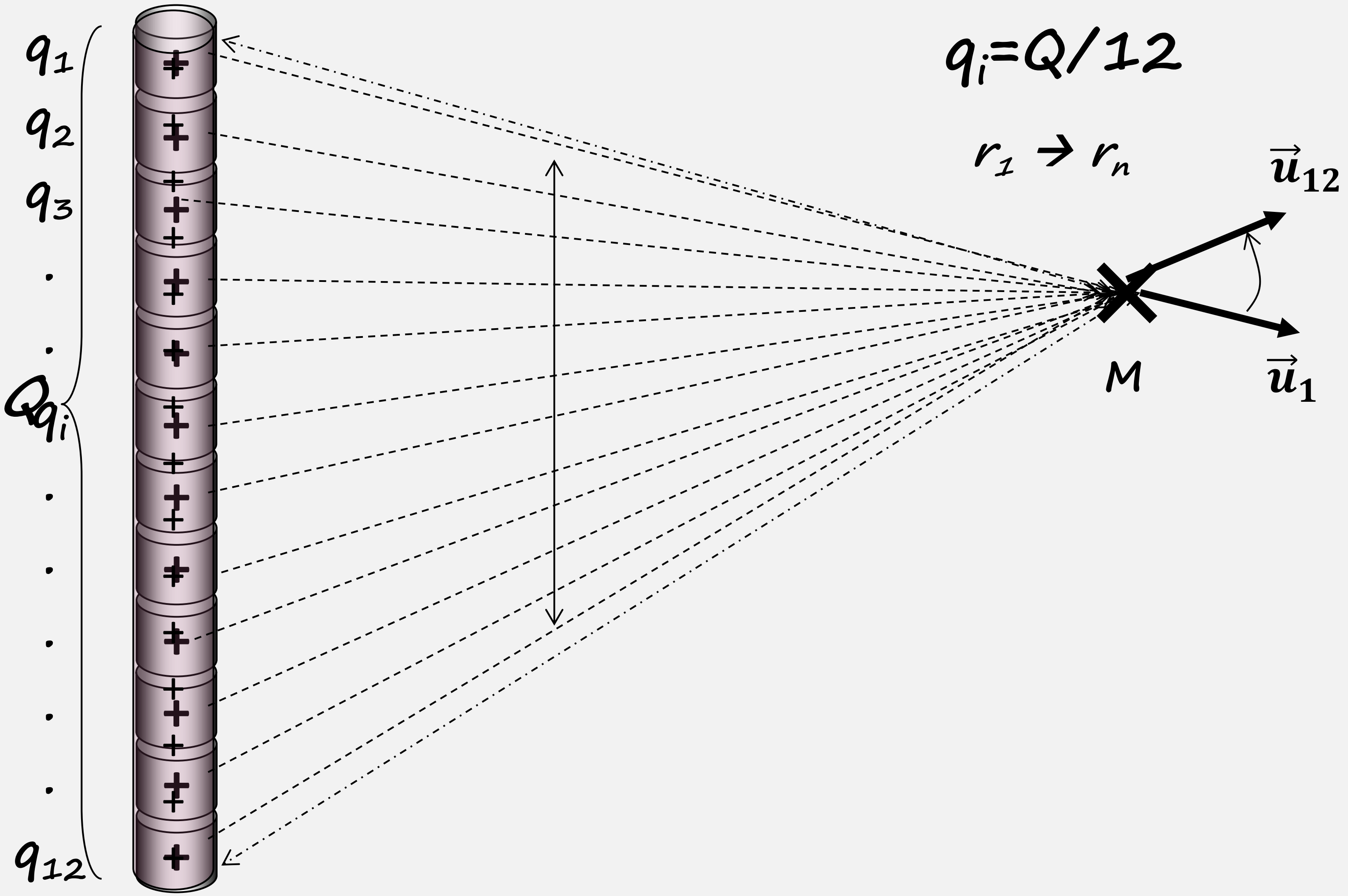
$$\sigma = \frac{dq}{dS}\,[C/m^2]$$



$$\rho = \frac{dq}{dV}\,[C/m^3]$$

I. Class Reminder

Electric Field of a charge distribution



$$\vec{e}_1 = K.\frac{q_1}{{r_1}^2}.\vec{u}_1$$

$$\vec{e}_2 = K.\frac{q_2}{{r_2}^2}.\vec{u}_2$$

$$\vec{e}_3 = K.\frac{q_3}{{r_3}^2}.\vec{u}_3$$

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$$\vec{e}_{12} = K.\frac{q_{12}}{{r_{12}}^2}.\vec{u}_{12}$$

I. Class Reminder

$$\vec{E} = \vec{e}_1 + \vec{e}_2 + \ldots + \vec{e}_{12} = \sum_{i=1}^{12} \vec{e}_i$$

$$\vec{E} = k.q \sum_{i=1}^{12} \frac{1}{r_i^2}.\vec{u}_i$$

N : large number

q_i ***: infinitesimal*** $\rightarrow q_i = dQ$

$$\vec{e}_i = d\vec{E} = K.\frac{dQ}{r_i^2}.\vec{u}_i$$

$$\vec{E} = \int_A^B d\vec{E} = K.\int_Q \frac{dQ}{r^2}.\vec{u}$$

we need to know:

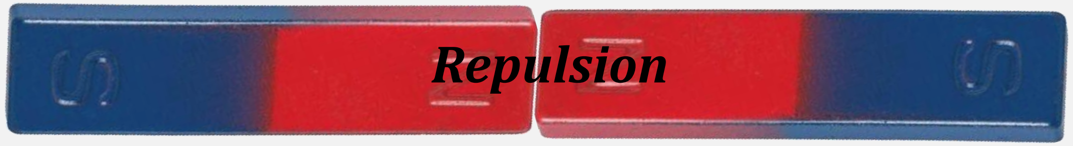
- The charge distribution: λ, σ or ρ;

- The geometry of the system;

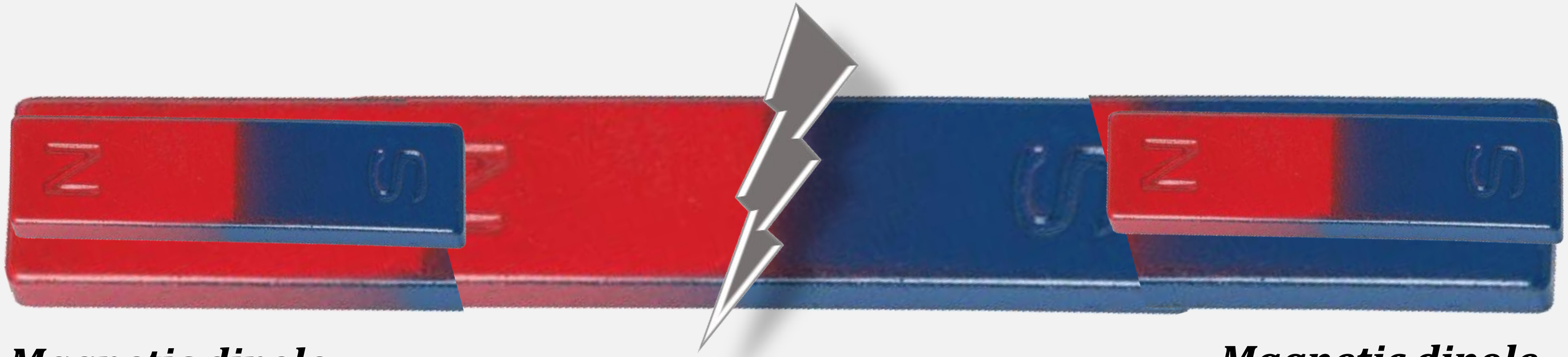
- And exploit the symmetry if it exists (rectangular, cylindrical, spherical)

I. Class Reminder

Attraction



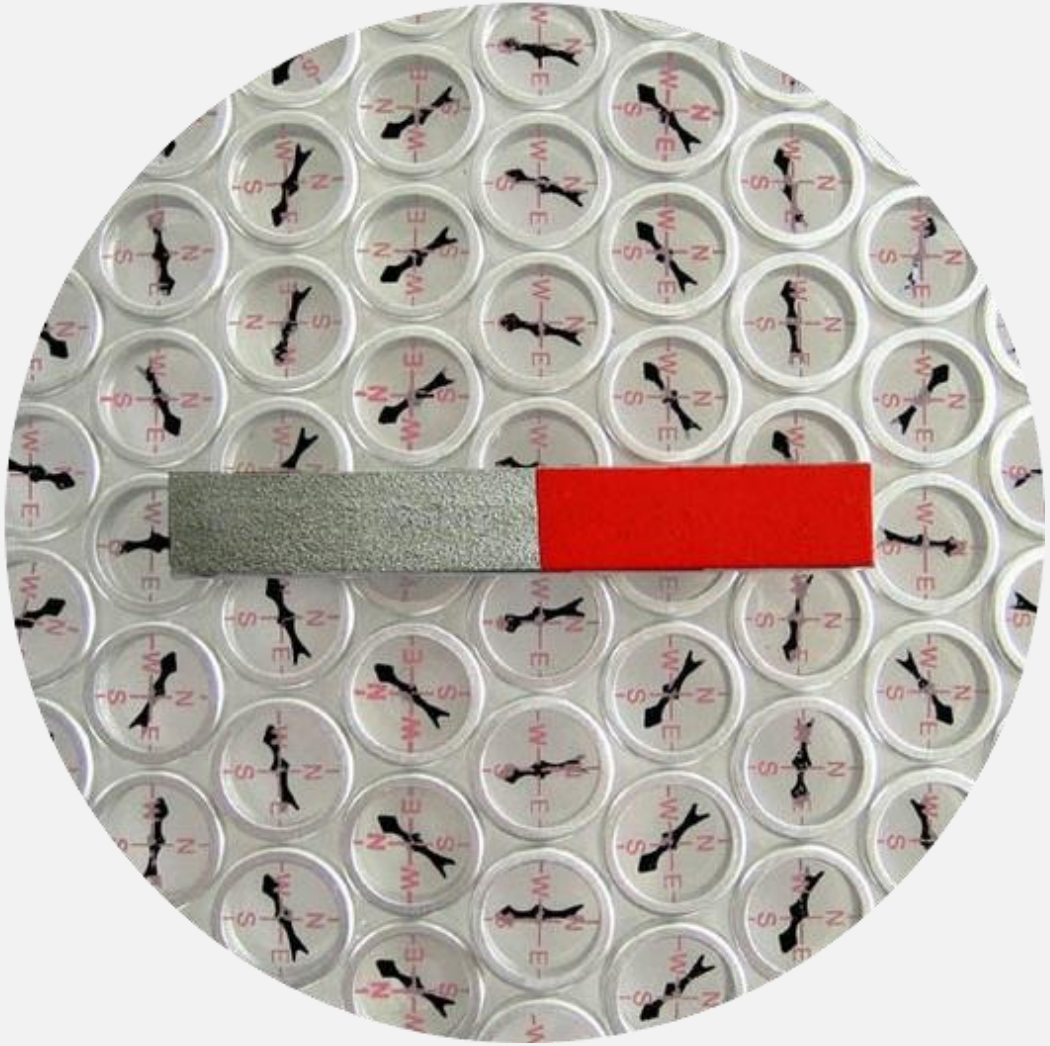
Repulsion



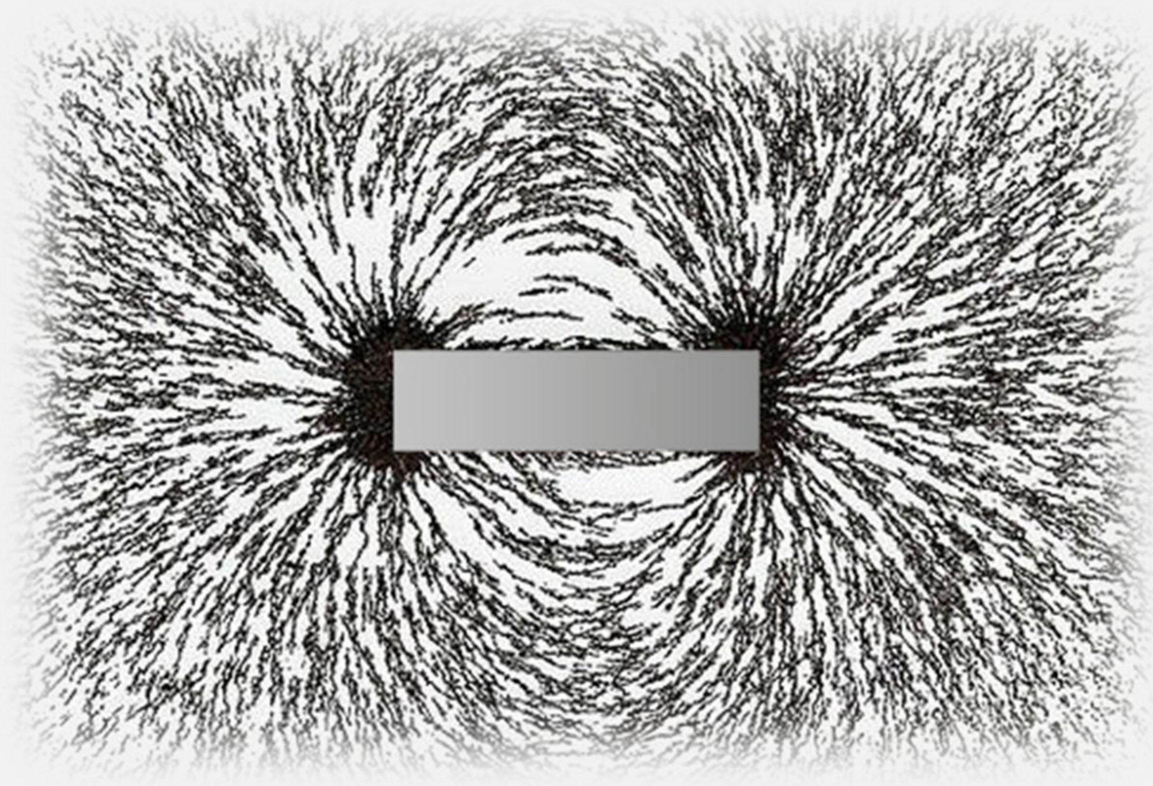
Magnetic dipole

Magnetic dipole

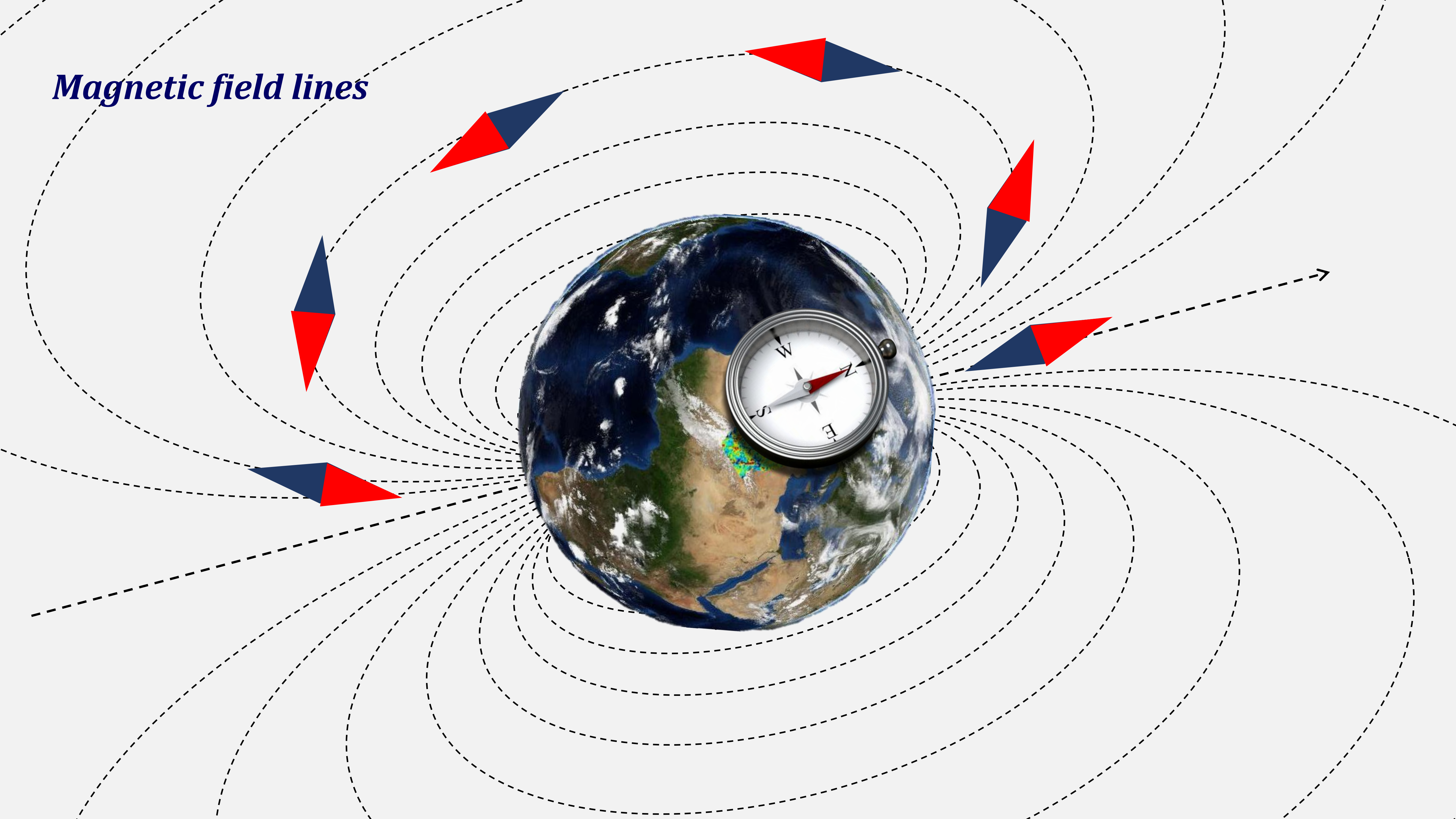
I. Class Reminder



Magnetic field lines



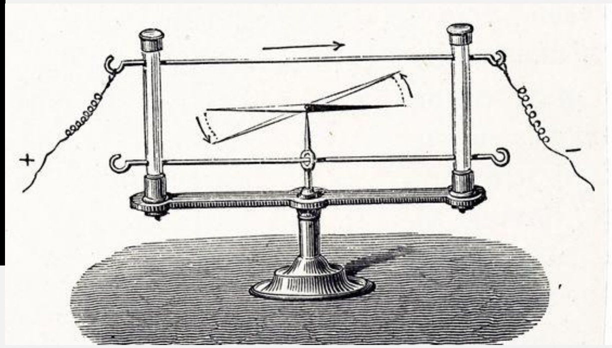
Magnetic field lines



I. Class Reminder

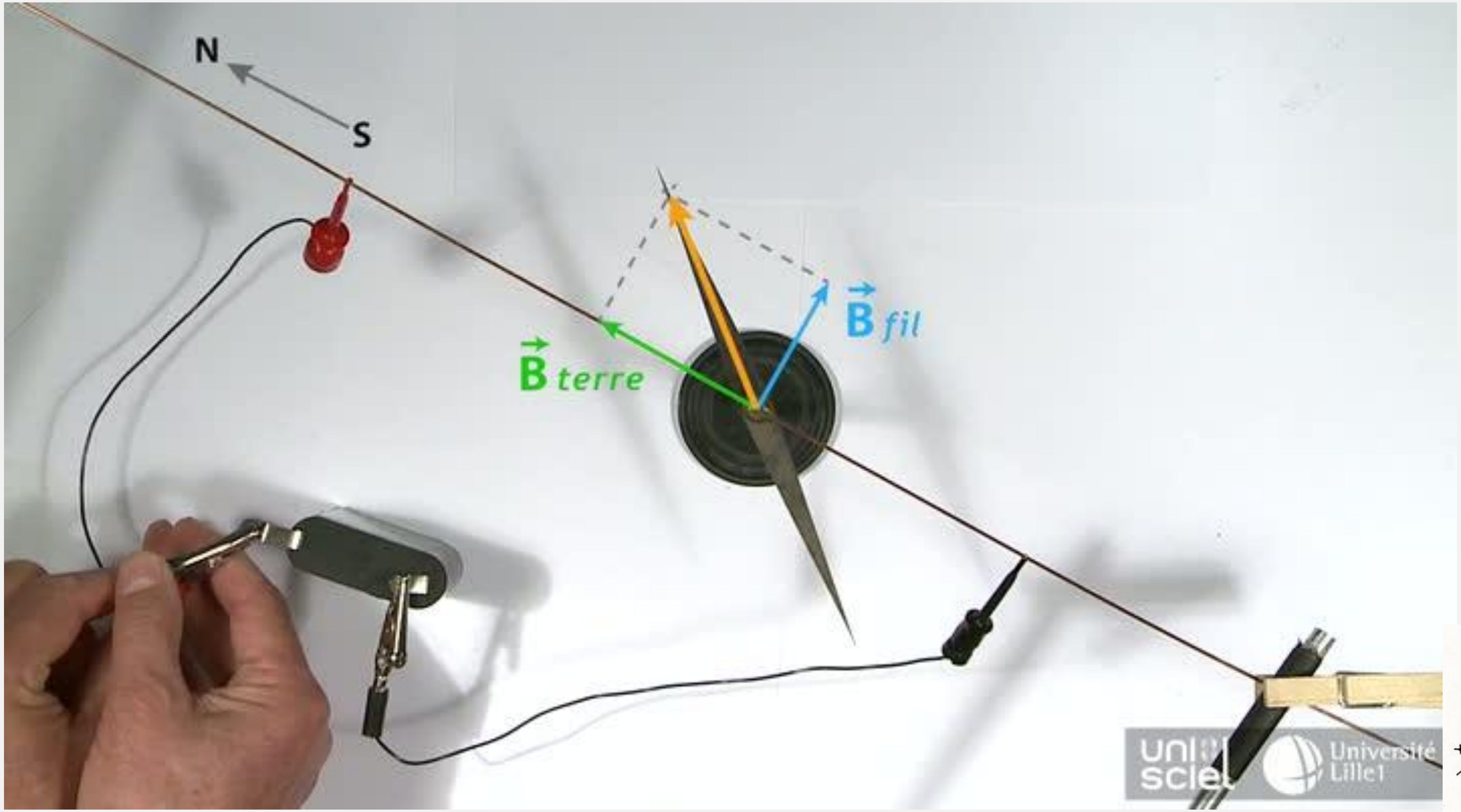
ØRSTED experiment (1819)

Hans Christian Ørsted (1777-1852), Danemark

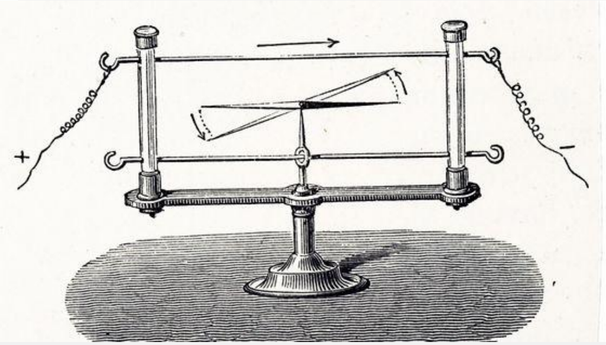


I. Class Reminder

ØRSTED experiment (1819)



Hans Christian Ørsted (1777-1852), Danemark



I. Class Reminder

The Ørsted's law

Ørsted found that, for a straight wire traversed by a steady direct current (DC):

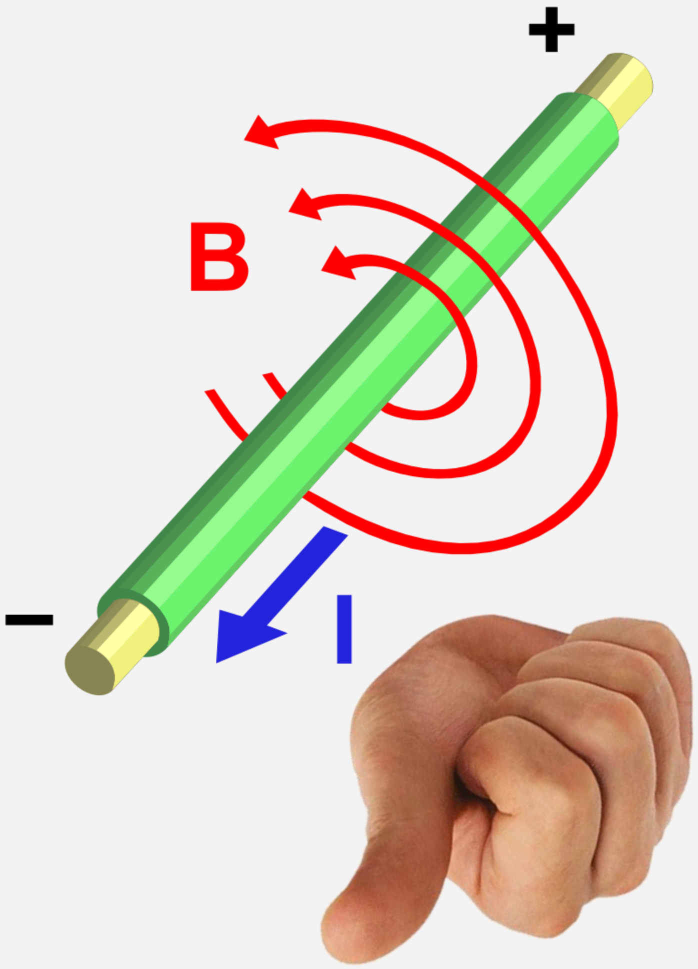
- The magnetic field lines encircle the current-carrying wire and they lie in a plane perpendicular to the wire;

- If the direction of the current is reversed, the direction of the magnetic field reverses;

- The strength of the field: $B \propto I$

- The strength of the field : $B \propto {}^{1}/_{r^2}$

- the direction of the field lines: thumb rule



Hans Christian Ørsted (1777-1852), Danemark

Phenomenology: Electricity ↔ Magnetism

I. Class Reminder

All the cited observations allowed Both two French scientists to deduce the mathematical formulation of an elementary magnetic field induced in a point P by an element $d\vec{l}$ (located at O) of the wire crossed by the electrical current intensity I:

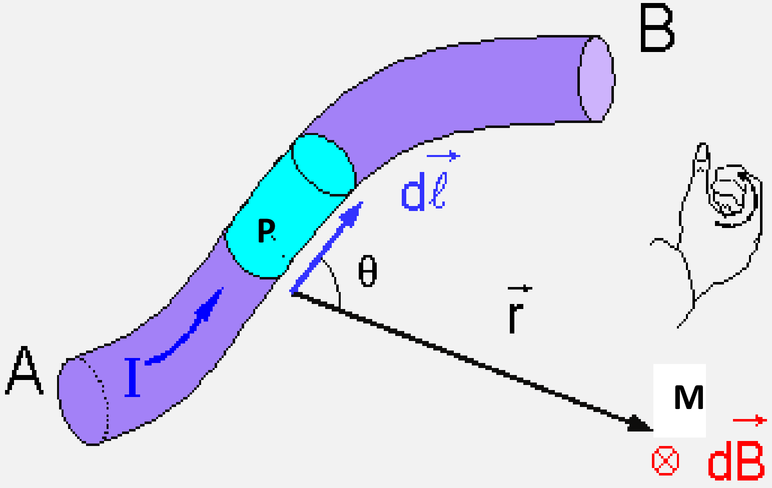
$$d\vec{B}(M) = \frac{\mu_0}{4\pi} I d\vec{l} \wedge \frac{\overrightarrow{PM}}{\left\|\overrightarrow{PM}\right\|^3} = \frac{\mu_0}{4\pi}\frac{I}{r^2} d\vec{l} \wedge \vec{u}$$

The magnetic field, could be then obtained via the integral form:

$$\vec{B}(M) = \int_{M\in(C)} d\vec{B}_P(M) = \int_{M\in(C)} \frac{\mu_0}{4\pi} . \frac{I.d\vec{l} \wedge \vec{u}_{PM}}{r^2}$$

"which is not an easy calculation to do!!!"

Magnetic permeability: $\mu_0 = 4\pi \times 10^{-7} [H.m^{-1}]$



I. Class Reminder

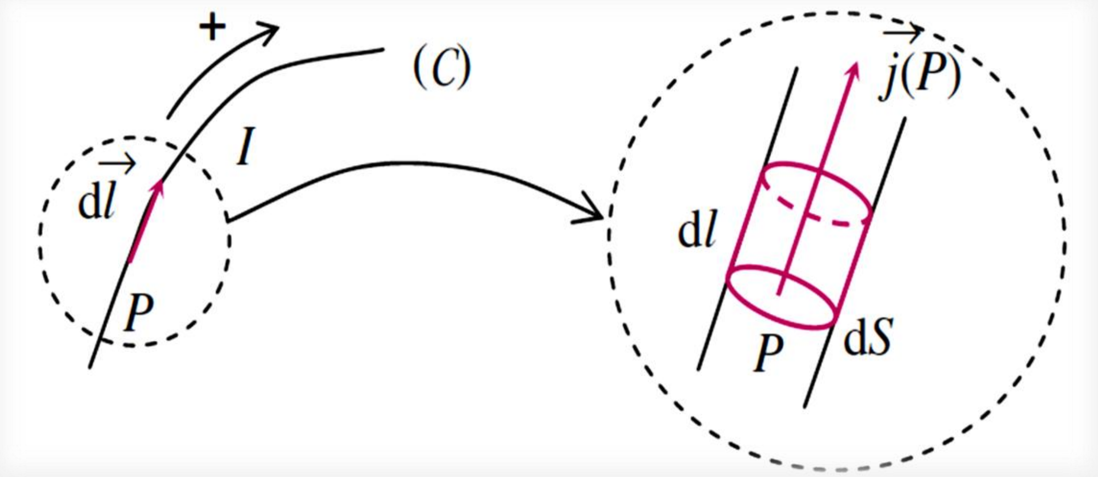
If the current density is known, it will be more convenient to calculate the magnetic field using the density instead of the current intensity:

$$I = \vec{J}.\overrightarrow{dS} \rightarrow I.d\vec{l} = J(P).dS.d\vec{l} = \overrightarrow{J(P)}.dV$$

Thus, the Biot-Savart law becomes in the case of volume density:

$$\vec{B}(M) = \int_{P\in(C)} d\vec{B}_P(M) = \iiint_V \frac{\mu_0}{4\pi}.\frac{\overrightarrow{J(P)}.dV \wedge \vec{u}_{PM}}{PM^2}$$

$$= \iiint_V \frac{\mu_0}{4\pi}.\frac{\overrightarrow{J(P)} \wedge \vec{u}_{PM}}{PM^2}.dV$$



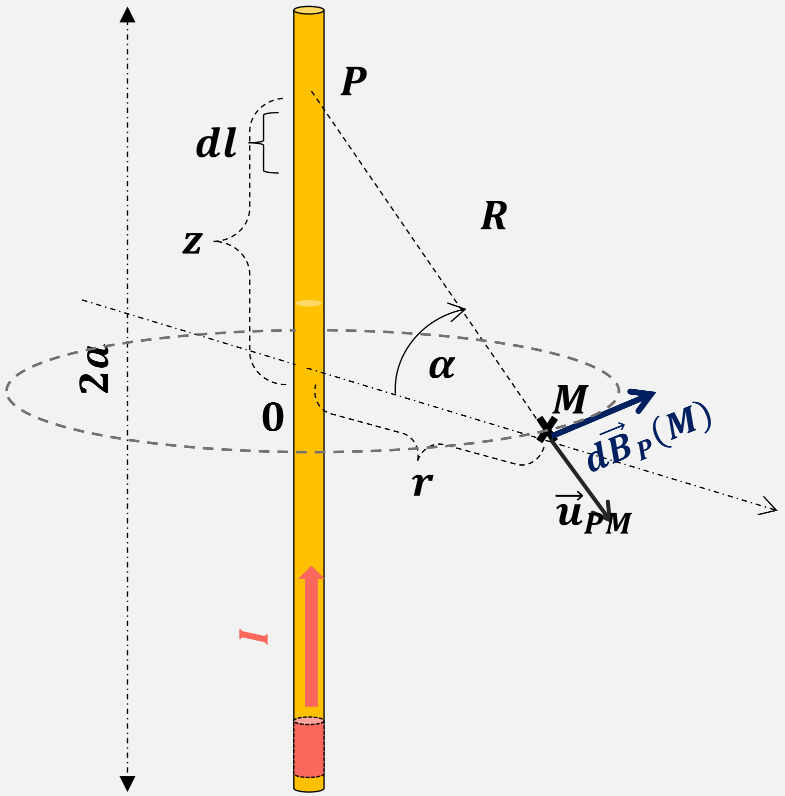
Besides that, the Biot-Savart law becomes in the case of surface density:

$$\vec{B}(M) = \int_{P\in(C)} d\vec{B}_P(M) = \iint_S \frac{\mu_0}{4\pi}.\frac{\overrightarrow{J_S(P)}.dS \wedge \vec{u}_{PM}}{PM^2}$$

$$= \iint_S \frac{\mu_0}{4\pi}.\frac{\overrightarrow{J_S(P)} \wedge \vec{u}_{PM}}{PM^2}.dS$$

Example 01:

In this exercise, we will calculate the magnetic field $\vec{B}(M)$ induced by a straight wire with a length $l = 2a$, crossed by a steady direct current I. We will examine the case $l \to \infty$



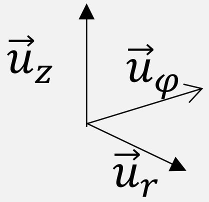
$$PM \equiv R$$

$$OM \equiv r$$

$$OP \equiv z$$

$$\cos\alpha = r/R$$

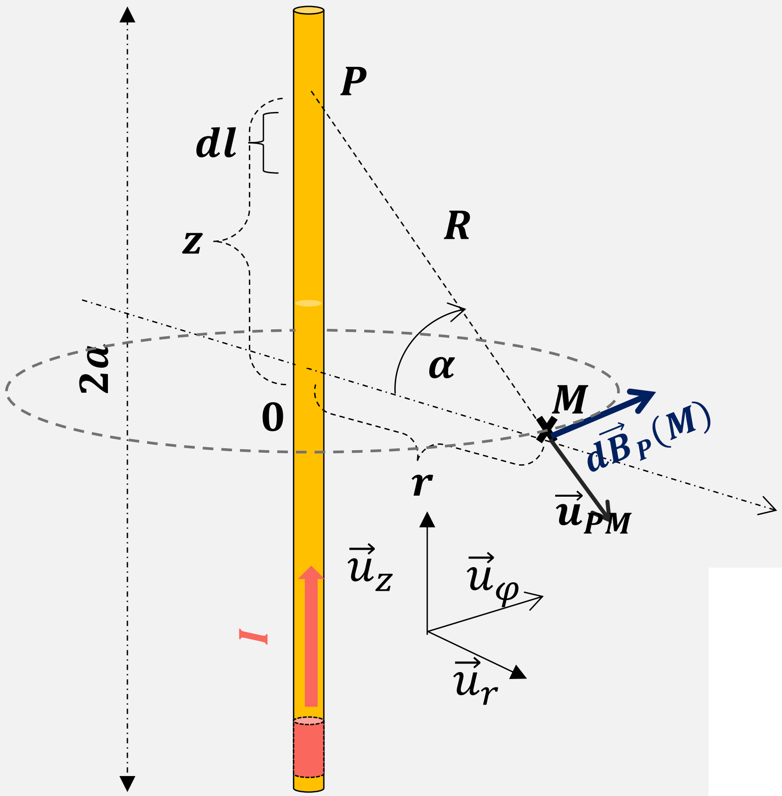
$$dl \equiv dz$$



$$d\vec{B}_P(M) = \frac{\mu_0}{4\pi}\frac{I.\,d\vec{l}\wedge\vec{u}_{PM}}{PM^2}$$

Solution:

Due to the cylindrical symmetry of the problem, the only non-zero component of $\vec{B}(M)$ is the azimuthal one $\overrightarrow{B_\varphi}(M)$:



$$d\vec{B}_P(M) = \frac{\mu_0}{4\pi}\frac{I.\,d\vec{l}\wedge\vec{u}_{PM}}{PM^2} = \frac{\mu_0}{4\pi}\frac{I.\,dl.\,\vec{u}_z\wedge\vec{u}_{PM}}{R^2}$$

With: $\vec{u}_{PM} = \cos\alpha.\,\vec{u}_r - \sin\alpha.\,\vec{u}_z$

$$\cos\alpha = \frac{r}{R} = \frac{r}{\sqrt{r^2+z^2}}$$

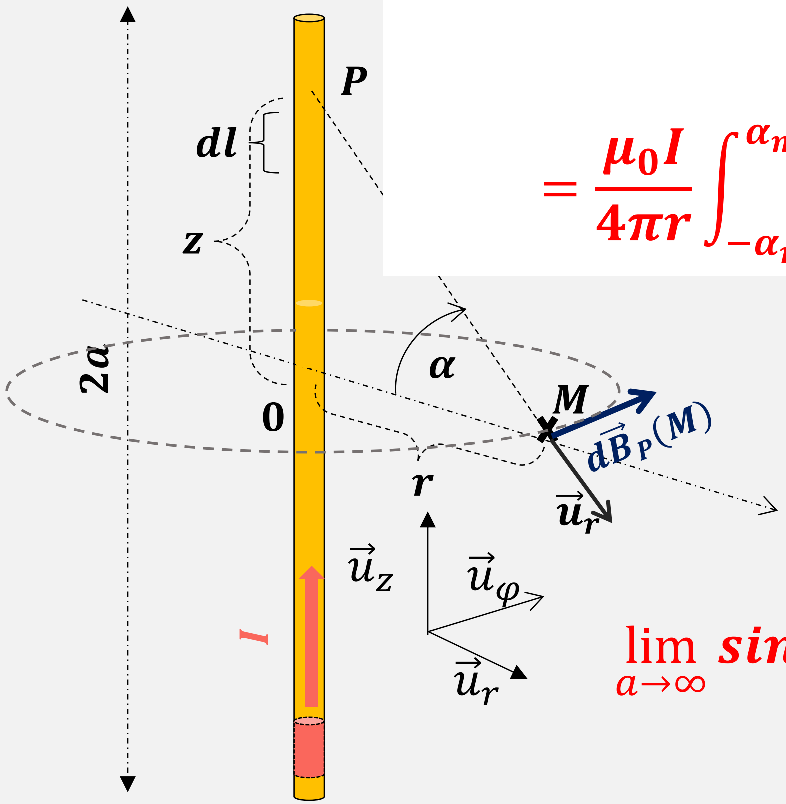
$$\frac{z}{r} = \tan\alpha \rightarrow dl \equiv dz = \frac{r\,d\alpha}{\cos^2\alpha}$$

$$d\vec{B}_P(M) = \frac{\mu_0}{4\pi}\frac{I.\,\vec{u}_z\wedge\vec{u}_r}{R^2}dz.\cos\alpha = \frac{\mu_0 I}{4\pi}\frac{\cos\alpha.\,dz}{R^2}\vec{u}_\varphi$$

$$= \frac{\mu_0 I}{4\pi}\cos\alpha\frac{\cos^2\alpha}{r}\frac{d\alpha}{\cos^2\alpha}\vec{u}_\varphi$$

Solution:

$$d\vec{B}_P(M) = \frac{\mu_0}{4\pi}\frac{I.\vec{u}_z\wedge\vec{u}_r}{R^2}\cos\alpha = \frac{\mu_0 I}{4\pi}\frac{\cos\alpha.r.dz}{R^2}\vec{u}_\varphi = \frac{\mu_0 I}{4\pi}\cos\alpha\frac{\cos^2\alpha}{r}\frac{d\alpha}{\cos^2\alpha}\vec{u}_\varphi$$



$$\vec{B}_P(M) = \int d\vec{B}_P(M) = \frac{\mu_0 I}{4\pi}\int\frac{\cos\alpha.d\alpha}{r}\vec{u}_\varphi$$

$$= \frac{\mu_0 I}{4\pi r}\int_{-\alpha_{max}}^{\alpha_{max}}\cos\alpha.d\alpha\,\vec{u}_\varphi = \frac{\mu_0 I}{4\pi r}[2\sin\alpha_{max}] = \frac{\mu_0 I}{2\pi r}[\sin\alpha_{max}]$$

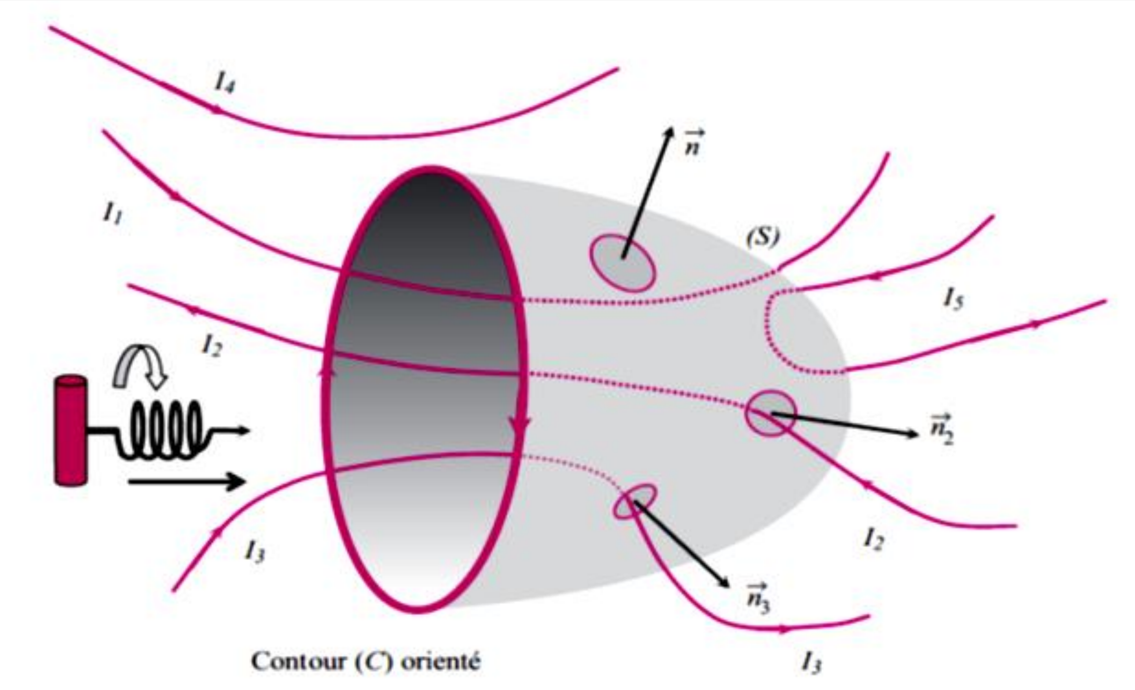
$$\sin\alpha_{max} = \frac{a}{\sqrt{a^2+r^2}}$$

$$\lim_{a\to\infty}\sin\alpha_{max} = 1 \rightarrow \lim_{a\to\infty}\frac{\mu_0 I}{2\pi r}[\sin\alpha_{max}] \rightarrow \vec{B}_P(M) = \frac{\mu_0 I}{2\pi r}\vec{u}_\varphi$$

I. Class Reminder

The Ampere theorem states that the magnetic field circulation through a closed path enclosing several currents I_k is directly proportional to the sum of these currents $\sum_k I_k$:

$$\oint \vec{B}.d\vec{l} = \mu_0 \sum_k I_k$$



André-Marie Ampère
1775-1836 (France)

I. Class Reminder

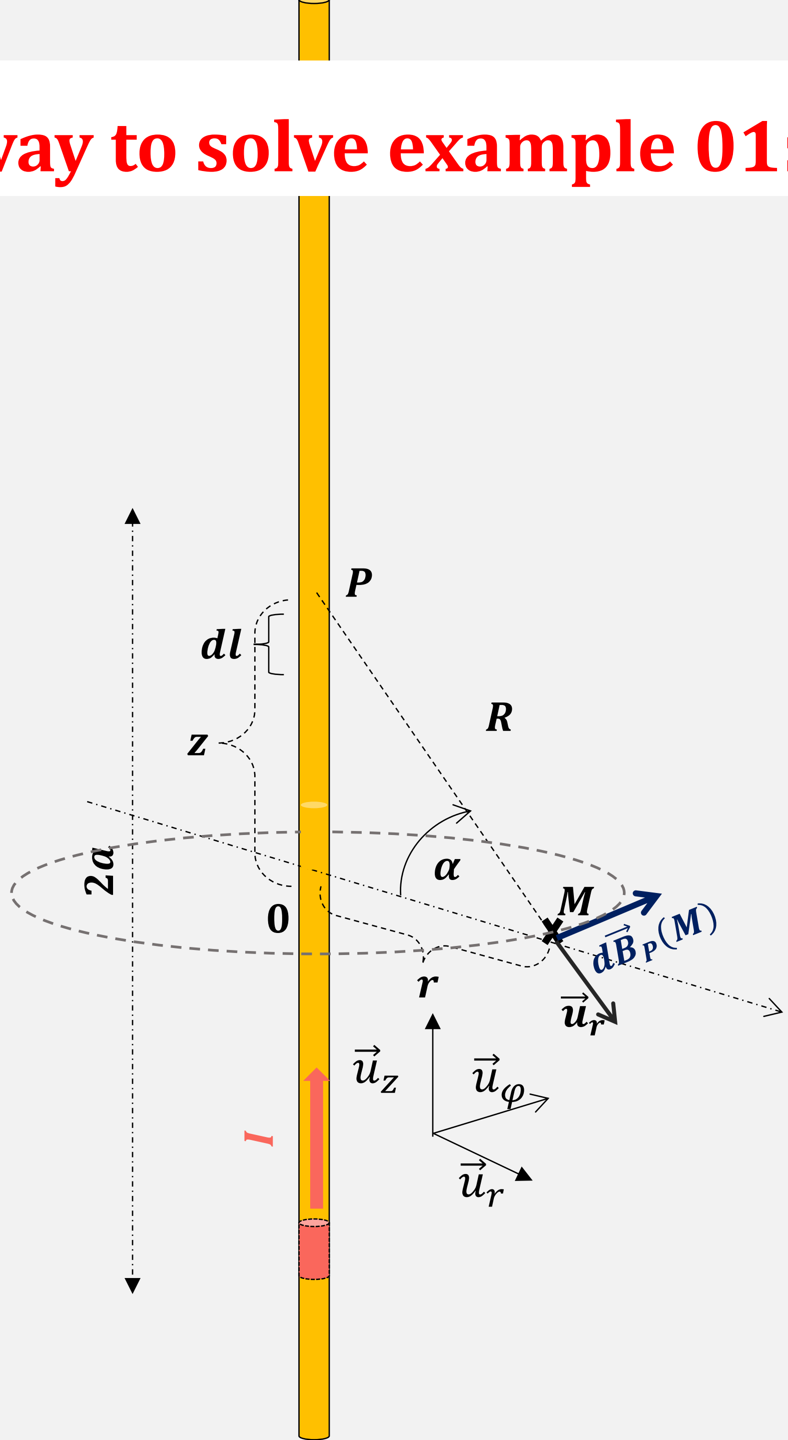
In the case of colinear straight currents, we obtain a uniform magnetic field parallel to the contour given by the Ampere law:

$$\oint \vec{B}.d\vec{l} = \vec{B}\oint d\vec{l} = B.L = \mu_0 \sum_k I_k \rightarrow B = \frac{\mu_0 \sum_k I_k}{L}$$

Application: *by using this law to calculate again the magnetic field induced by the straight wire traversed by steady direct current I, in a given point M located at the radial distance r from the wire.*

André-Marie Ampère
1775-1836 (France)

Another way to solve example 01:



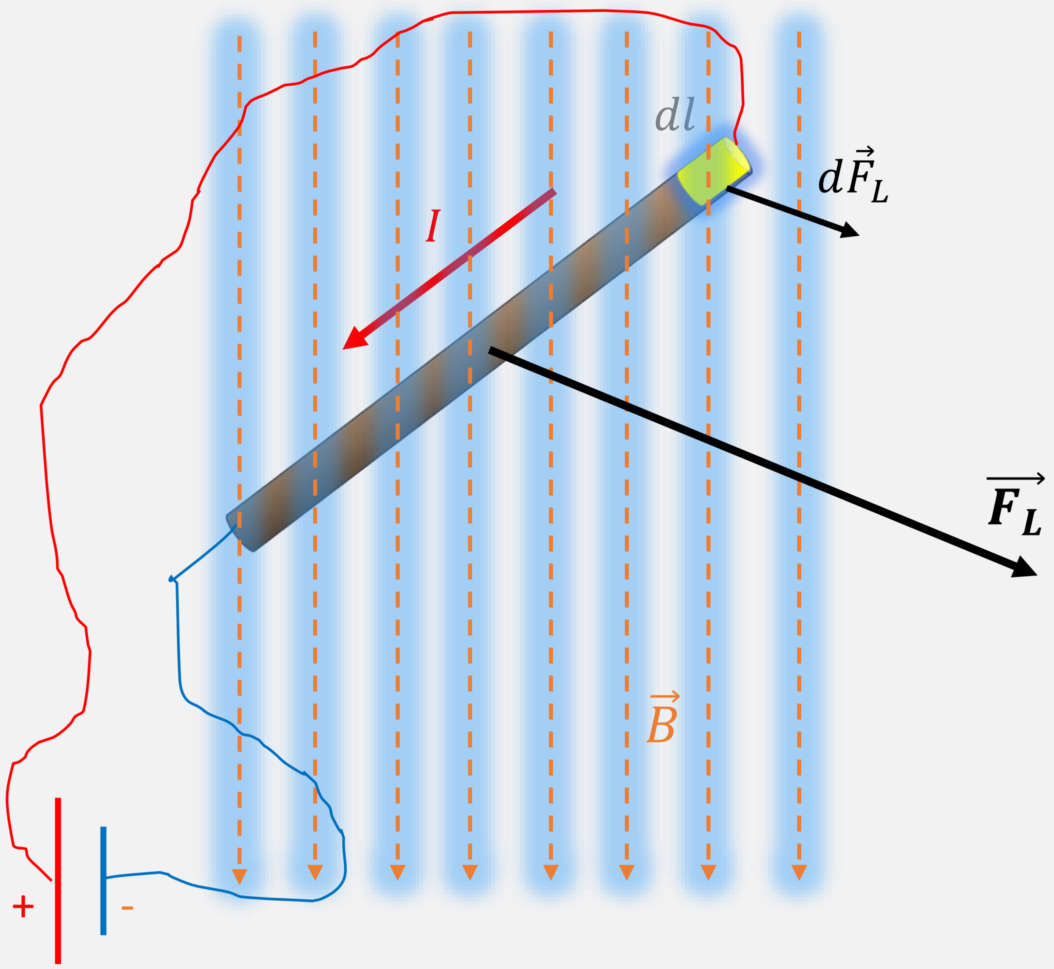
$$\int \vec{B}_P \,.\, d\vec{l} = \mu_0 \sum_k I_k$$

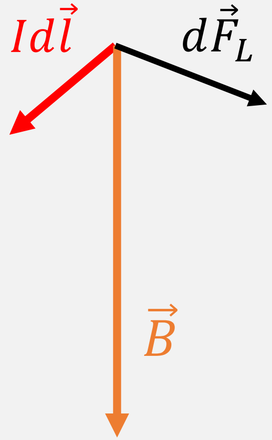
$$\rightarrow B_P \int_0^{2\pi} r d\,\varphi \vec{u}_\varphi . \vec{u}_\varphi = B_P 2\pi r = \mu_0 I$$

$$\rightarrow B_P = \frac{\mu_0 I}{2\pi r}$$

I. Class Reminder

Laplace force

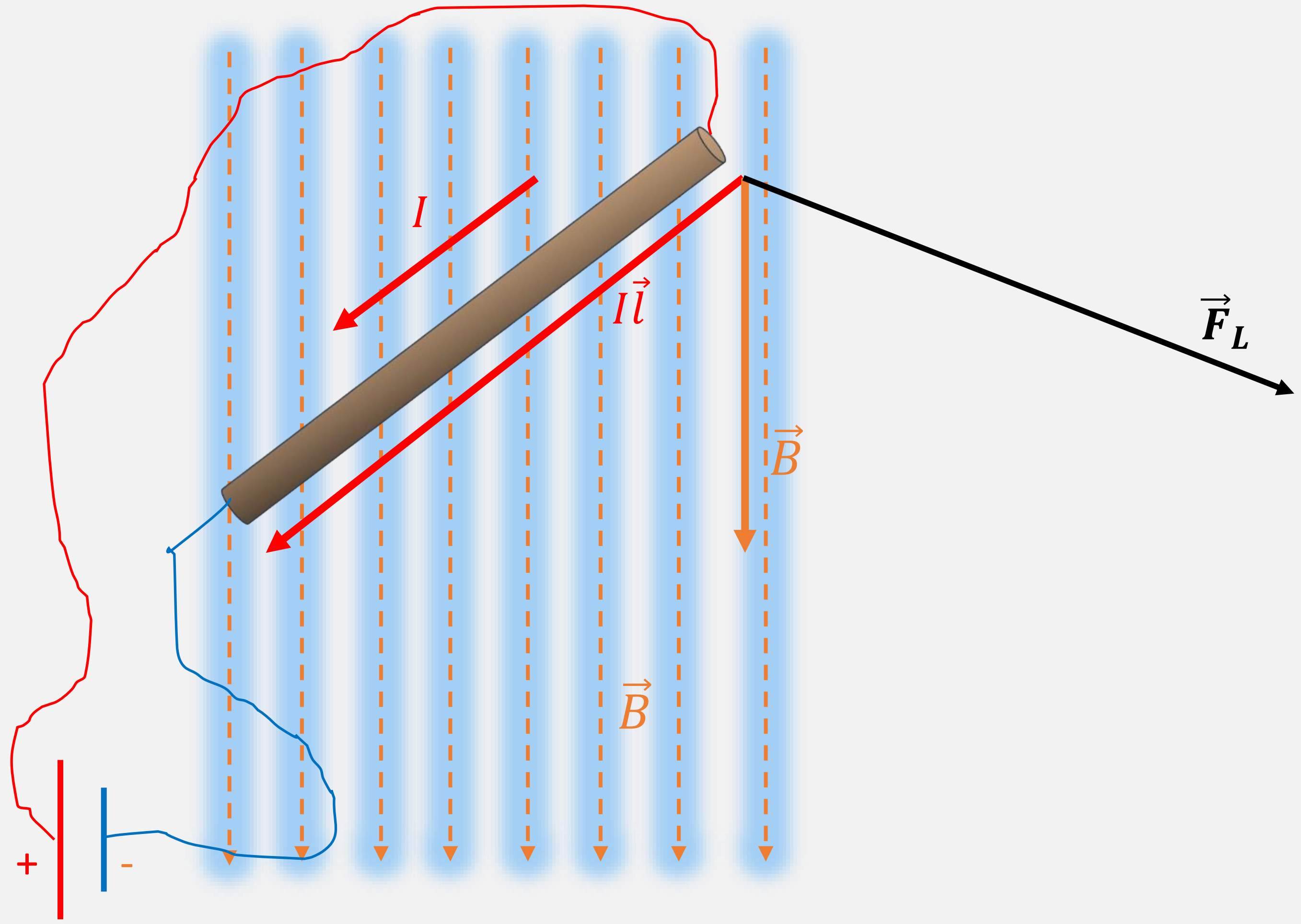




Pierre-Simon de Laplace

(1749-1827) France

I. Class Reminder



Pierre-Simon de Laplace

(1749-1827) France

I. Class Reminder

When a conductor carrying a direct current intensity *I*, is put near a magnetic field, a mechanical force is applied on the wire and it tends to displace him in a perpendicular direction on both magnetic field and current flow. This force known as Laplace force is given by:

$$d\vec{F}_L = I.\overrightarrow{dl}\wedge\vec{B} \rightarrow \vec{F}_L = \int I.\overrightarrow{dl}\wedge\vec{B}$$

Pierre-Simon de Laplace

(1749-1827) France

In the case of uniform magnetic field, it is possible de perform the integration to obtain the force expression:

$$\vec{F}_L = I.\vec{l}\wedge\vec{B} = I.l.B.\sin\theta\,\vec{u}$$

$$\vec{F}_L\Big|_{max} = I.\vec{l}\wedge\vec{B} = I.l.B.\sin\frac{\pi}{2}\,\vec{u} = I.l.B\vec{u}$$

I. Class Reminder

In the presence of electric field, any charged particle will feel an applied electrical force given by: $\vec{f}_e = q\vec{E}$

Similarly, if the same charged particle is animated with a celerity $\vec{v}$ **in presence of a magnetic field, it will feel a magnetic force known as Lorentz force:** $\vec{f}_m = q.\vec{v}\wedge\vec{B}$

In the case, where both fields are present, we get the general electromagnetic Lorentz force:

$$\vec{f}_L = \vec{f}_e + \vec{f}_m = \vec{f}_{E.M} = q.\vec{E} + q.\vec{v}\wedge\vec{B} = q.\left(\vec{E} + \vec{v}\wedge\vec{B}\right)$$

Hendrik LORENTZ

(1853-1928) Netherland

I. Class Reminder

Deriving Laplace force from magnetic Lorentz force:

If we consider a density n of charged particles animated with an average celerity $\vec{v}$ crossing a wire section S in presence of a magnetic field $\vec{B}$, where each individual particle will feel the force: $\vec{f}_m = q.\vec{v}\wedge\vec{B}$

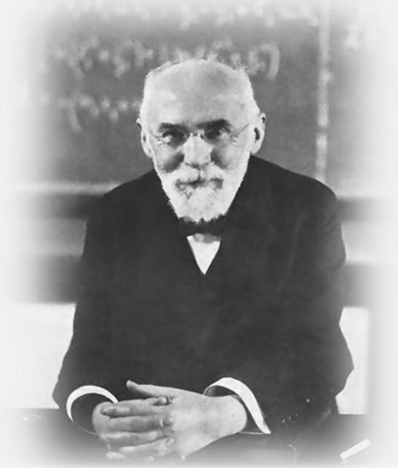
Over an elementary distance dl, an elementary volume $dV = S.dl$ will represent a number of charges: $N = n.S.dl$

This will constitute an element of macroscopic force:

$$d\vec{F}_m = n.S.dl.q.\vec{v}\wedge\vec{B} = (q.n.S.v)d\vec{l}\wedge\vec{B} = Id\vec{l}\wedge\vec{B}$$

By definition, we have: $I = q.n.S.v$

P-S. Laplace



H. LORENTZ

I. Class Reminder

In physics the term magnetic field points usually to the physical value measured in Tesla: $\vec{B}[T]$, While the physical value : $\vec{H}[A/m] = \mu\vec{B}[T]$ is defined as *"magnetic excitation"*.

Where $\mu = \mu_r\mu_0$ points to the magnetic permeability of the given media where $\vec{B}$ is present.

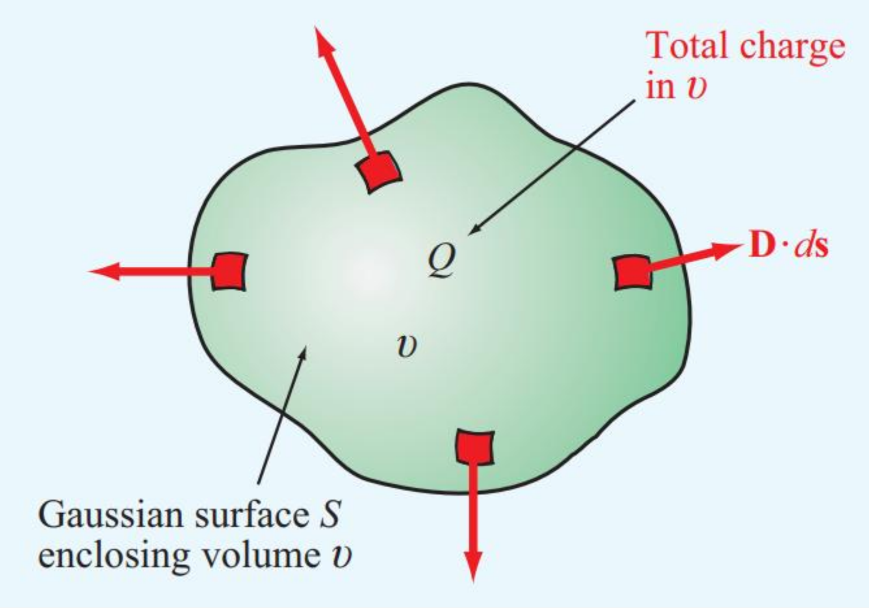
In engineering, $\vec{B}[T]$ is called the magnetic induction While : $\vec{H}[A/m] = \mu\vec{B}[T]$ is defined as "magnetic field".

Relative and absolute magnetic permeability for some media

Medium	μ_r	$\mu[H.m^{-1}]$
Vaccum	1.00000000	$1.25663062 \times 10^{-6}$
Air	1.00000037	$1.25663753 \times 10^{-6}$
Water	0.999992	1.256627×10^{-6}
Wood	1.00000043	$1.25663760 \times 10^{-6}$
Concrete	1.00000000	$1.25663062 \times 10^{-6}$
Iron	2×10^5	2.5×10^{-1}

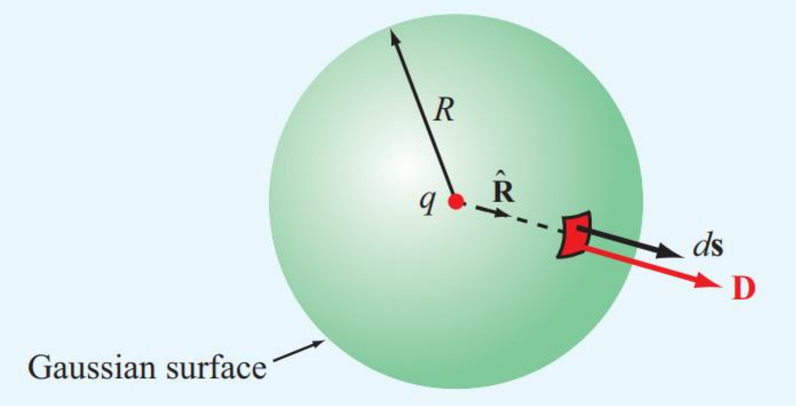
II. Equations of Electrostatics

The Gauss law states that for any enclosed charge inside a surface S', one can find the electric field resulting from this charge by calculating its flux:



$$\oiint_S \vec{E}\,.\,d\vec{S} = \frac{Q}{\varepsilon_0} \leftrightarrow \oiint_{S'} \vec{D}\,.\,d\vec{S} = Q$$

Where: $\vec{D} = \varepsilon_0 \vec{E}$ is electric flux density $\left[C.m^{-2}\right]$



A good choice of the Gaussian surface will conduct to a simple calculation of the electric field generated by the point charge q:

$$\oiint_S \vec{E}\,.\,d\vec{S} = \frac{q}{\varepsilon_0} \leftrightarrow \oiint_{S'} E_r\, r^2 sin\theta d\theta d\varphi = E_r r^2 \int_0^{\pi} sin\theta d\theta \int_0^{2\pi} d\varphi = \frac{q}{\varepsilon_0}$$

$$E_r r^2 4\pi = \frac{q}{\varepsilon_0} \rightarrow E_r = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}$$

II. Equations of Electrostatics

The Gauss law for a number of discrete charges could also obtained as a generalization of the previous law, when the surface S is enclosing N charges Q_i:

$$\oiint_S \vec{E}\,.\,d\vec{S} = \frac{1}{\varepsilon_0}\sum_{i=1}^{N} Q_i$$

And, for a continuous distributions of charge, we get:

$$\oiint_S \vec{E}\,.\,d\vec{S} = \frac{1}{\varepsilon_0}\int dq$$

- Linear distribution: $\oiint_S \vec{E}\,.\,d\vec{S} = \frac{1}{\varepsilon_0}\int_{l'} \lambda dl$
- surface distribution: $\oiint_S \vec{E}\,.\,d\vec{S} = \frac{1}{\varepsilon_0}\int_{S'} \sigma dS$
- volume distribution: $\oiint_S \vec{E}\,.\,d\vec{S} = \frac{1}{\varepsilon_0}\int_{v'} \rho dv$

It is interesting to see that for any vector field $\vec{A}$, the divergence theorem allows us to convert a surface integral into a volume integral:

$$\oiint_S \vec{A}.\,d\vec{S} = \int_V (\vec{\nabla}.\vec{A}).\,dv$$

Thus, it is possible to rewrite the left-hand term of the Gauss law with the volume distribution case:

$$\oiint_S \vec{E}\,.\,d\vec{S} = \int_{v'} (\vec{\nabla}.\vec{E}).\,dv = \frac{1}{\varepsilon_0}\int_{v'} \rho dv$$

By identification, we get the differential form of Gauss law (divergent of $\vec{E}$):

$$\vec{\nabla}.\vec{E} = \frac{\rho}{\varepsilon_0}$$

II. Equations of Electrostatics

Let's take the law relating electric potential to the electric field between two measurement points:

$$\Delta V = V_{21} = V_2 - V_1 = \int_{P_1}^{P_2} dV = -\int_{P_1}^{P_2} \vec{E}.d\vec{l}$$

We should note that:

- We have : $V_1 \rightarrow P_1$ *and* $V_2 \rightarrow P_2$
- *Kirchhoff law: For* $P_1 \equiv P_2$

$$\Delta V = 0 \leftrightarrow \int_{P_1}^{P_1} \vec{E}.d\vec{l} = \oint_C \vec{E}.d\vec{l} = 0$$

- *If* $P_1 \rightarrow \infty \leftrightarrow V_1 = 0 \rightarrow V = -\int_{\infty}^{P} \vec{E}.d\vec{l}$
- *The null potential is a referential value (not absolute) , and it is called "ground"*

If we use the Stokes's theorem to convert a surface integral into a closed line integral:

$$\int_S (\vec{\nabla} \wedge \vec{A}).d\vec{S} = \oint_C \vec{A}.d\vec{l}$$

Where C is a closed contour on which S is lying.

Thus, we can obtain the differential form from this integral expression:

$$\vec{\nabla} \wedge \vec{E} = 0$$

Any vector field verifying that its line integral along any closed path is zero, is called conservative or irrotational field.

Hence, the electrostatic field $\vec{E}$ is conservative.

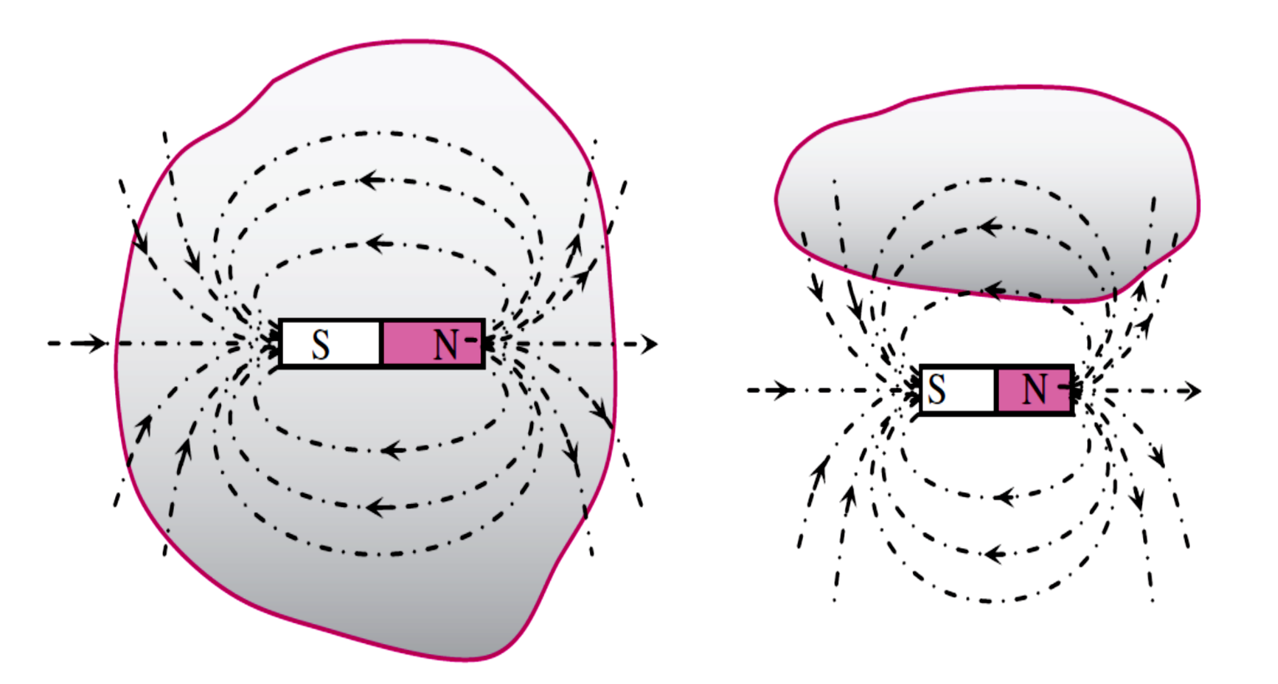
III. Equations of Magnetostatics

Gauss law for magnetism

By considering the flux of magnetic field lines through a given surface enclosing totally or partially these lines, it comes intuitively, due to the nature of the magnetic dipole (permanent or induced) that the same amount of field lines will enter and then exit from that surface. This will imply:

$$\oiint_S \vec{B} \, . \, d\vec{S} = 0$$

This is the equivalent Gauss law for magnetic field in its integral form.



The differential form will be deduced in similar way by using the conversion of surface integral to a volume one:

$$\int_{v'} (\vec{\nabla} . \vec{B}) . dv = 0 \rightarrow \vec{\nabla} . \vec{B} = 0$$

III. Equations of Magnetostatics

If now we rewrite the Ampere law of magnetic field induced by a set of currents:

$$\oint_C \vec{B}.d\vec{l} = \mu_0 \sum_k I_k$$

Passing from summation to integral and using surface density of current:

$$\oint_C \vec{B}.d\vec{l} = \mu_0 \iint_{S'} \vec{j}.d\vec{S}$$

If we use the Stokes's theorem to convert a line integral into a surface integral:

$$\int_S (\vec{\nabla}\wedge\vec{B}).d\vec{S} = \oint_C \vec{B}.d\vec{l}$$

We get by identification:

$$\vec{\nabla}\wedge\vec{B} = \mu_0\vec{j}$$

According to previous sections we could gather all the differential and integral equations of both electric and magnetic fields:

$$\begin{cases} \vec{\nabla}.\vec{E} = \dfrac{\rho}{\varepsilon_0} \\ \vec{\nabla}\wedge\vec{E} = 0 \\ \vec{\nabla}.\vec{B} = 0 \\ \vec{\nabla}\wedge\vec{B} = \mu_0\vec{j} \end{cases} \leftrightarrow \begin{cases} \vec{\nabla}.\vec{D} = \rho \\ \vec{\nabla}\wedge\vec{D} = 0 \\ \vec{\nabla}.\vec{H} = 0 \\ \vec{\nabla}\wedge\vec{H} = \vec{j} \end{cases} \leftrightarrow \begin{cases} \oint \vec{D}.d\vec{S} = Q \\ \oint_C \vec{D}.d\vec{l} = 0 \\ \oint \vec{H}.d\vec{S} = 0 \\ \oint_C \vec{H}.d\vec{l} = I \end{cases}$$

Where: $\vec{D} = \varepsilon\vec{E};\ \vec{B} = \mu\vec{H}$

in the case of free space: $\varepsilon = \varepsilon_0, \mu = \mu_0$

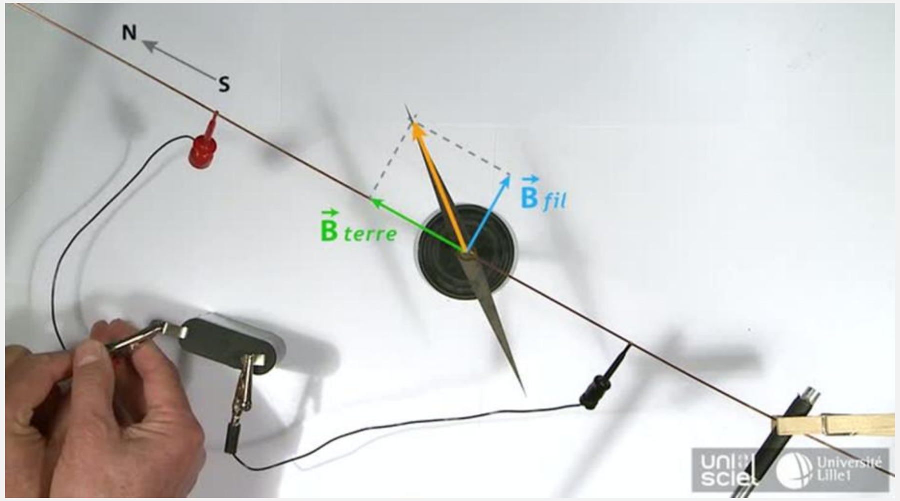
IV. Equations of Electrodynamics

Faraday's law

This set of four equations could be expressed as a double set of decoupled equations since no explicit relationships exist between electric and magnetic fields:

$$\begin{cases} \vec{\nabla}.\vec{E} = \dfrac{\rho}{\varepsilon_0} \\ \vec{\nabla} \wedge \vec{E} = 0 \end{cases} ; \begin{cases} \vec{\nabla}.\vec{B} = 0 \\ \vec{\nabla} \wedge \vec{B} = \mu_0 \vec{j} \end{cases}$$

This allows us to study electricity and magnetism as two distinct and separate phenomena as long as the spatial distributions of charge and current flow remain constant in time.



H. C. Ørsted 1819 experiment

Michael Faraday 1791- 1867, UK

Faraday hypothesized that if a current produces a magnetic field, then the converse should also be true: A magnetic field should produce a current in a wire.

To test his hypothesis, he conducted numerous experiments in his laboratory in London over a period of about 10 years (1821-1831)

IV. Equations of Electrodynamics

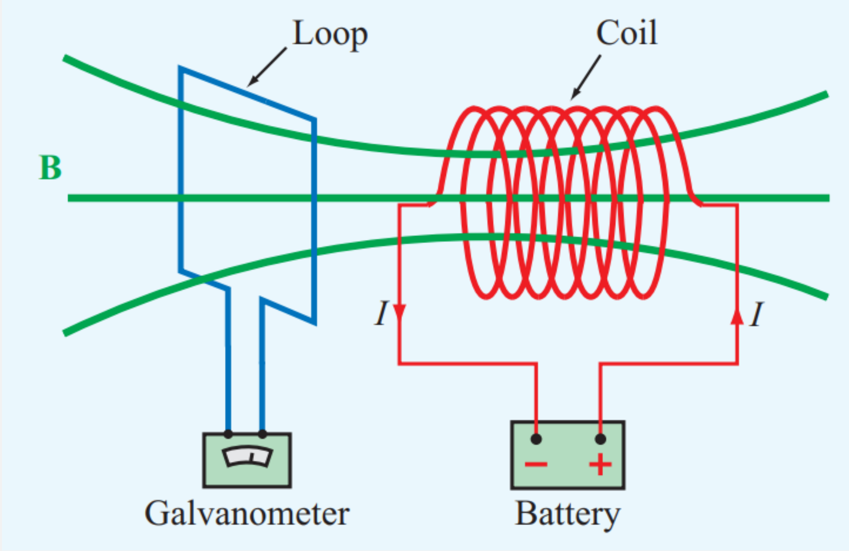
- **The principle of the experiments conducted by Faraday in his lab in London, consist to place a conducting loop (sensor) connected to a galvanometer (predecessor measurement device of voltmeter and amperemeter) next to a conducting coil connected to a battery (electro-magnet). This latter will produce a magnetic field when switch is on, with field lines going through the sensor loop.**

Michael Faraday
1791- 1867, UK

- **When the switch is turned on and the coil is crossed by a steady current, a constant magnetic flux is passing through the measurement loop:**

$$\Phi[Wb] = \int_S \vec{B}.d\vec{S}$$

But no current was detected by the measurement loop. Even repeated many times, but without a success to detect any current produced by magnetic field as Faraday hypothesized.



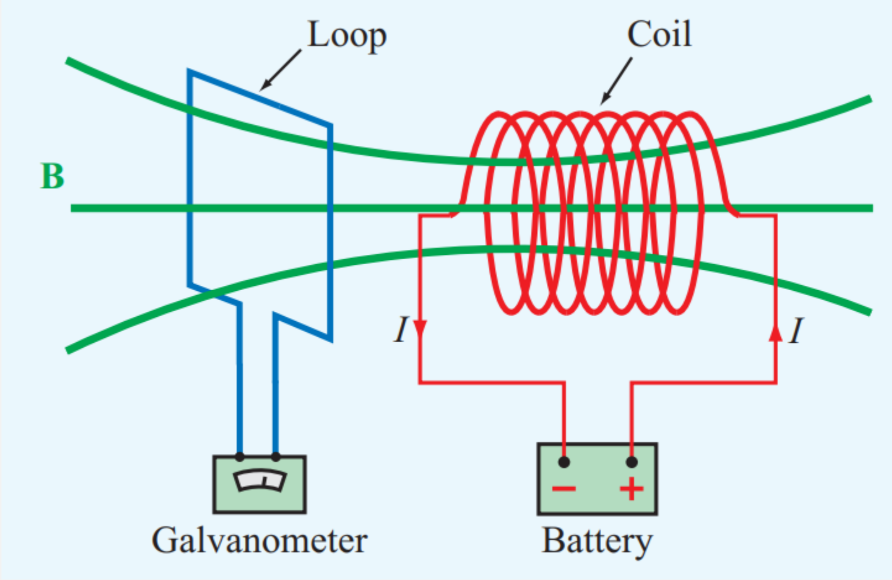
IV. Equations of Electrodynamics

Faraday's law

- After many attempts, Faraday noticed that the galvanometer needle showed a momentary deflection, indicating the presence of a current for a very short period, during the switching on or off of the coil circuit connected to the battery.
- *Consequently, Faraday deduced that the induced current in the loop appeared only when the magnetic flux crossing the loop area changes*

Michael Faraday
1791- 1867, UK

- He also remarked, that the direction of the current in the loop depends wether the flux is increasing (battery being connected) or decreasing (battery being disconnected).
- Besides that, Faraday noticed that if the loop is turning or moving either closer to or away from the inducing coil. Which is an equivalent change of the magnetic flux against the loop *(relative movement).*



IV. Equations of Electrodynamics

- **As an important consequence, when the galvanometer detects the flow of current through the loop, a voltage has been induced across the terminals of the galvanometer. Faraday called this voltage *"electromotive force" (emf)*, V_{emf}, *and the whole phenomenon is called "Electromagnetic induction".***

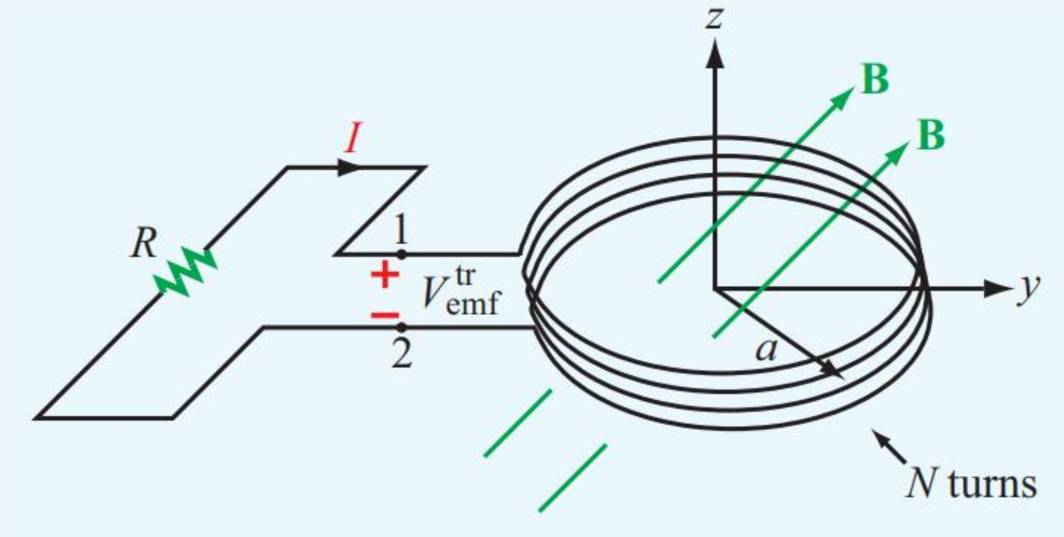
Michael Faraday
1791- 1867, UK

- ***This electromotive voltage is related to the magnetic flux variation by the simple law (Faraday's law):***

$$V_{emf} = -\frac{d\Phi}{dt} = -\frac{d}{dt}\int_S \vec{B}.d\vec{S}$$

- **For a closed conducting loop of N turns, the law could be generalized to :**

$$V_{emf} = -N\frac{d\Phi}{dt} = -N\frac{d}{dt}\int_S \vec{B}.d\vec{S}$$



IV. Equations of Electrodynamics

Michael Faraday
1791- 1867, UK

Accordingly, an EMF can be generated in a closed conducting loop under any of the following conditions:

- A time-varying magnetic field linking a stationary loop; the induced emf is then called the *"transformer emf"* V_{emf}^{tr}
- A moving loop with a time-varying area (relative to the normal component of B) in a static field B; the induced emf is then called *"the motional emf"*, V_{emf}^{m}
- A moving loop in a time-varying field $\vec{B}$

The total emf is given by:

$$V_{emf} = V_{emf}^{tr} + V_{emf}^{m}$$

With $V_{emf}^{m} = 0$ *if the loop is stationary, and* $V_{emf}^{tr} = 0$ *if* $\vec{B}$ *is static*

IV. Equations of Electrodynamics

Let's examine the case of a conducting loop with unique turn (steady S) existing in variable magnetic field $\vec{B}(t)$. In this situation, the former law of Faraday:

Michael Faraday
1791- 1867, UK

$$V_{emf} = -\frac{d\Phi}{dt} = -\frac{d}{dt}\int_S \vec{B}.d\vec{S} = -\int_S \frac{\partial \vec{B}}{\partial t}.d\vec{S}$$

At the same time, according to integral law of electric field with electric potential:

$$V_{emf} = -\oint_C \vec{E}.d\vec{l}$$

By comparison, and using the Stokes's theorem, we can write:

$$\oint_C \vec{E}.d\vec{l} = \int_S (\vec{\nabla} \wedge \vec{E}).d\vec{S} \equiv -\int_S \frac{\partial \vec{B}}{\partial t}.d\vec{S}$$

To obtain the Faraday's law in its differential form:

$$\vec{\nabla} \wedge \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

The Electric field is no more conservative in this case !!!

IV. Equations of Electrodynamics

When we have a varying magnetic field, the four equations of electrodynamics are given by:

$$\begin{cases} \vec{\nabla}.\vec{E} = \dfrac{\rho}{\varepsilon_0} & (Gauss's\ law) \\ \vec{\nabla} \wedge \vec{E} = -\dfrac{\partial \vec{B}}{\partial t} & (Faraday's\ Law) \\ \vec{\nabla}.\vec{B} = 0 & (Gauss\ Law\ for\ magnetism) \\ \vec{\nabla} \wedge \vec{B} = \mu_0\, \vec{j} & (Ampere's\ Law) \end{cases}$$

This also assumes that the magnetic field is induced by a time-varying current $I(t)$.

Charge-Current continuity relation:

In time-varying case, it is possible to connect the charge density ρ to the current $\vec{j}$. This is done by considering the definition of an electric current:

$$I = -\frac{dQ(t)}{dt} = -\frac{d}{dt}\int_V \rho.dV \qquad (5)$$

The sign (-) is introduced here to relate the conventional sense of the current with the variation amount of elementary charged particles (electrons).

Let's now consider the current density:

$$I = \oint_S \vec{j}.d\vec{S} \qquad (6)$$

IV. Equations of Electrodynamics

Charge-Current continuity relation:

To compare both equations (5) and (6), we need only to change the surface integral into volume integral by using divergent theorem for eq. 6:

$$I = \oint_S \vec{J}.d\vec{S} = \int_V (\vec{\nabla}.\vec{J})\, dV$$

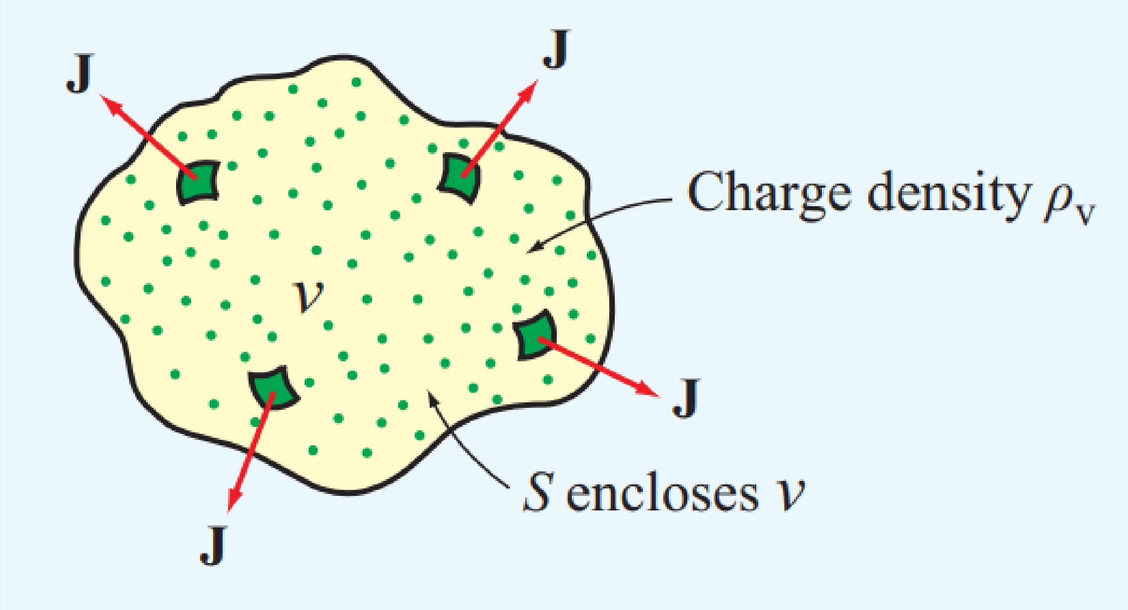
Now when compared to eq. 5:

$$I = -\frac{dQ(t)}{dt} = -\frac{d}{dt}\int_V \rho.dV = -\int_V \frac{\partial \rho}{\partial t}.dV$$

It comes that:

$$\vec{\nabla}.\vec{J} = -\frac{\partial \rho}{\partial t} \leftrightarrow \vec{\nabla}.\vec{J} + \frac{\partial \rho}{\partial t} = 0$$

Known as "Charge continuity equation"



In the case of time-conservative charge density:

$$\rho \neq \rho(t)$$

We get : $\vec{\nabla}.\vec{J} = 0$

It means that the net current flowing out of the volume is zero, or equivalently that, the incoming flow into V is equal to the outcoming one.

IV. Equations of Electrodynamics

Charge-Current continuity relation:

From the previous result, of constant flow:

$$\vec{\nabla}.\vec{J} = 0$$

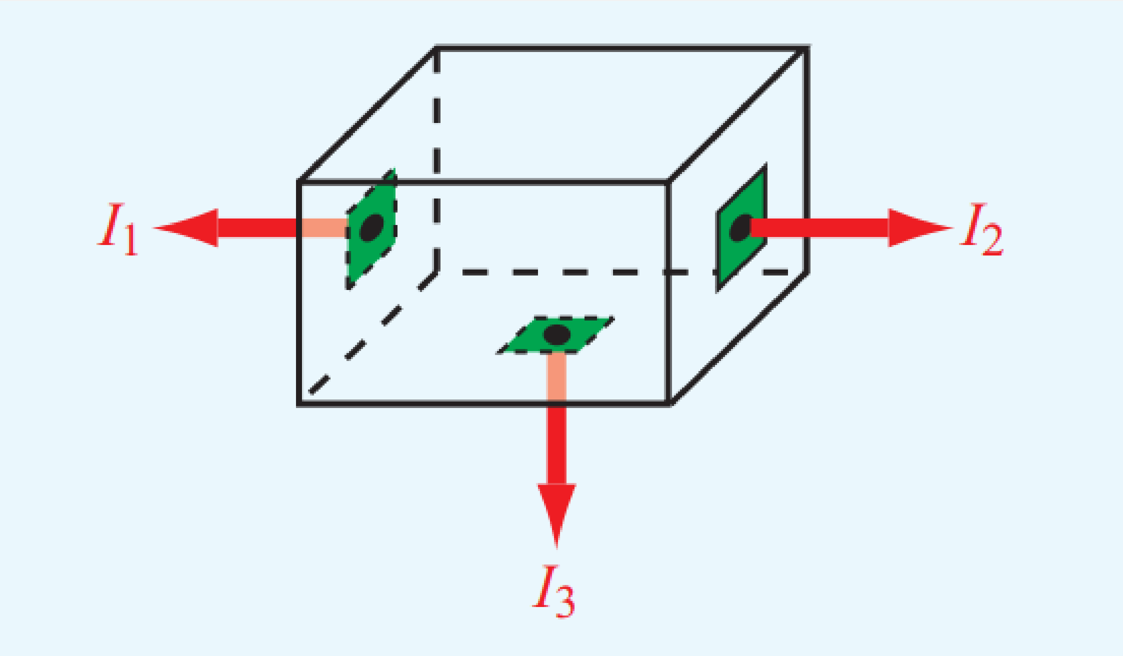
We can return to the integral form to find that:

$$\vec{\nabla}.\vec{J} = 0 \rightarrow \int_S \vec{j}.d\vec{S} = 0$$

Known as "Kirchhoff's current law".

The discrete form of this law is encountered in circuits analysis as *"nodes law"*:

$$\sum_n I_n = 0$$



It will be only sufficient to consider the junction of connected conducting wires as a volume enclosed into a surface and different currents are flow to/from it.

V. Maxwell's Equations

$$\vec{\nabla}.(\vec{\nabla}\wedge\vec{V}) = 0, \forall \vec{V}$$
$$\vec{\nabla}\wedge(\vec{\nabla}f) = 0, \forall f$$

$$\vec{\nabla}.\vec{j} + \frac{\partial\rho}{\partial t} = 0$$

Maxwell correction

Let's consider again the electrodynamics set of equations:

$$\begin{cases} \vec{\nabla}.\vec{E} = \dfrac{\rho}{\varepsilon_0} & I(Gauss's\ law) \\ \vec{\nabla}\wedge\vec{E} = -\dfrac{\partial\vec{B}}{\partial t} & II(Faraday's\ Law) \\ \vec{\nabla}.\vec{B} = 0 & III(Gauss\ Law\ for\ magnetism) \\ \vec{\nabla}\wedge\vec{B} = \mu_0\vec{J} & IV(Ampere's\ Law) \end{cases}$$

When applying the divergent of equations II and IV, we will find:

$$\underbrace{\vec{\nabla}.(\vec{\nabla}\wedge\vec{E})}_{=0,\forall\vec{E}} = \vec{\nabla}.\left(-\frac{\partial\vec{B}}{\partial t}\right) = \underbrace{-\frac{\partial}{\partial t}(\vec{\nabla}.\vec{B})}_{=0\ (III)} = 0$$

Now, when applying the same action on equation IV:

$$\underbrace{\vec{\nabla}.(\vec{\nabla}\wedge\vec{B})}_{=0,\forall\vec{B}} = \vec{\nabla}.(\mu_0\vec{j}) = \mu_0\vec{\nabla}.\vec{j} = \underbrace{\mu_0\vec{\nabla}.\vec{j}}_{?}$$

In fact, the quantity $\vec{\nabla}.\vec{j}$ does not vanish for all $\vec{j}$, only for special cases corresponding to $\partial\rho/\partial t = 0$, according to charge continuity equation.

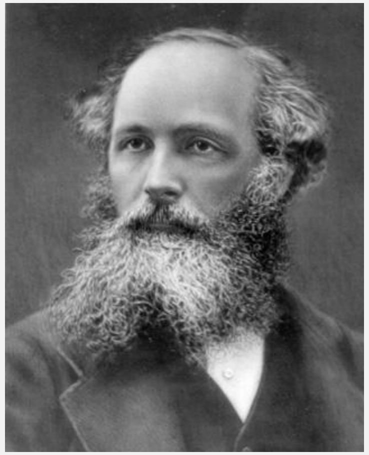
To prevent this, Maxwell proposed to add a term which could cancel the divergent of the current density:

$\vec{J'} = \vec{J} + \vec{G}$ ***in such a way that:***

$$\vec{\nabla}.\vec{J'} = \vec{\nabla}.\vec{J} + \vec{\nabla}.\vec{G} = 0$$

This will give the following result:

$$\vec{\nabla}.\vec{G} = -\vec{\nabla}.\vec{j} = \frac{\partial\rho}{\partial t}$$



James C. Maxwell
1831- 1897, UK

V. Maxwell's Equations

Maxwell correction

5. Maxwell correction of Ampere law:

This new term, will ingeniously ensure the complete relationship (in both senses) between electric and magnetic fields.

According to Gauss's law: $\vec{\nabla}.\vec{E} = \frac{\rho}{\varepsilon_0}$ (Eq. I), the Maxwell condition could be rewritten as:

$$\vec{\nabla}.\vec{G} = -\vec{\nabla}.\vec{J} = \frac{\partial \rho}{\partial t} = \frac{\partial \vec{D}}{\partial t} = \frac{\partial\left(\varepsilon_0 \vec{\nabla}.\vec{E}\right)}{\partial t} = \vec{\nabla}.\left(\varepsilon_0 \frac{\partial \vec{E}}{\partial t}\right)$$

This implies that:

$$\vec{G} = \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Finally, we get:

$$\vec{J'} = \vec{J} + \vec{G} = \vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Consequently, the new version of eq. IV:

$$\vec{\nabla} \wedge \vec{B} = \mu_0 \vec{J'} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

And, when applying the divergent :

$$\underbrace{\vec{\nabla}.\left(\vec{\nabla} \wedge \vec{B}\right)}_{=0,\forall \vec{B}} = \vec{\nabla}.\left(\mu_0 \vec{J'}\right) = \mu_0 \vec{\nabla}.\vec{J'} = \underbrace{\mu_0 \vec{\nabla}.\vec{J} + \mu_0 \varepsilon_0 \vec{\nabla}.\frac{\partial \vec{E}}{\partial t}}_{=0\ (charge\ continuity)}$$

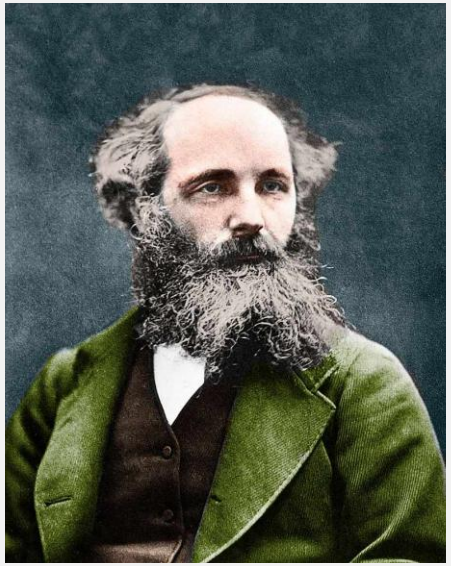
The term $\vec{G}$, *known also as "Maxwell correction", is called the "displacement current". The reduced form of the equation IV, using both* $\vec{H}$ and $\vec{D}$ fields :

$$\vec{\nabla} \wedge \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

The new set of electrodynamics equations could be now completed and finalized, as Maxwell's Equations.

V. Maxwell's Equations

Heaviside formulation

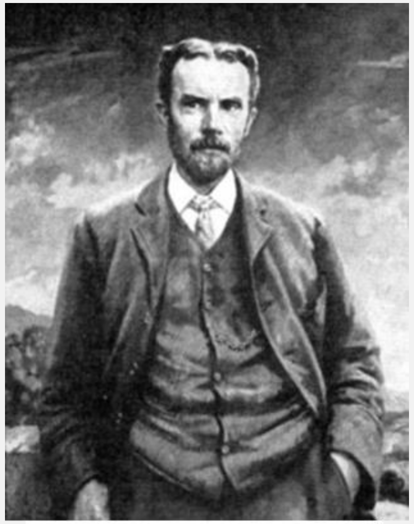


J.C. Maxwell
(1831-1879, UK)

- **Maxwell's equations (1865):**

$e+\frac{df}{dx}+\frac{dg}{dy}+\frac{dh}{dz}=0$	(1)	Gauss' Law
$\mu\alpha=\frac{dH}{dy}-\frac{dG}{dz}$ $\mu\beta=\frac{dF}{dz}-\frac{dH}{dx}$ $\mu\gamma=\frac{dG}{dx}-\frac{dF}{dy}$	(2)	Equivalent to Gauss' Law for magnetism
$P=\mu\left(\gamma\frac{dy}{dt}-\beta\frac{dz}{dt}\right)-\frac{dF}{dt}-\frac{d\Psi}{dx}$ $Q=\mu\left(\alpha\frac{dz}{dt}-\gamma\frac{dx}{dt}\right)-\frac{dG}{dt}-\frac{d\Psi}{dy}$ $R=\mu\left(\beta\frac{dx}{dt}-\alpha\frac{dy}{dt}\right)-\frac{dH}{dt}-\frac{d\Psi}{dz}$	(3)	Faraday's Law (with the Lorentz Force and Poisson's Law)
$\frac{d\gamma}{dy}-\frac{d\beta}{dz}=4\pi p'$ $\quad p'=p+\frac{df}{dt}$ $\frac{d\alpha}{dz}-\frac{d\gamma}{dx}=4\pi q'$ $\quad q'=q+\frac{dg}{dt}$ $\frac{d\beta}{dx}-\frac{d\alpha}{dy}=4\pi r'$ $\quad r'=r+\frac{dh}{dt}$	(4)	Ampère-Maxwell Law
$P=-\xi p \quad Q=-\xi q \quad R=-\xi r$		Ohm's Law
$P=kf \quad Q=kg \quad R=kh$		The electric elasticity equation (**E** = **D**/ε)
$\frac{de}{dt}+\frac{dp}{dx}+\frac{dq}{dy}+\frac{dr}{dz}=0$		Continuity of charge

The present days vector version of Maxwell's equations were elaborated by his fellow citizen, the physicist O. Heaviside in 1884.



Oliver HEAVISIDE
(1850-1925,UK)

(1) $\vec{\nabla}.\vec{E}=\frac{\rho}{\varepsilon_0}$ **Maxwell-Gauss**

(2) $\vec{\nabla}\wedge\vec{E}=-\frac{\partial\vec{B}}{\partial t}$ **Maxwell-Faraday**

(3) $\vec{\nabla}.\vec{B}=0$ **Maxwell-Thomson**

(4) $\vec{\nabla}\wedge\vec{B}=\mu_0\vec{J}+\mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$ **Maxwell-Ampère**

V. Maxwell's Equations

Maxwell's equations:

The electromagnetism now are well described by the set of Maxwell's equations:

$$\begin{cases} \vec{\nabla}.\vec{E} = \dfrac{\rho}{\varepsilon_0} & I(Maxwell - Gauss\ law) \\ \vec{\nabla} \wedge \vec{E} = -\dfrac{\partial \vec{B}}{\partial t} & II(Maxwell - Faraday\ Law) \\ \vec{\nabla}.\vec{B} = 0 & III(Gauss\ Law\ for\ magnetism) \\ \vec{\nabla} \wedge \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \dfrac{\partial \vec{E}}{\partial t} & IV(Maxwell - Ampere\ Law) \end{cases}$$

These equations are also known as Maxwell equations for time-varying fields $\vec{E}(t)$ and $\vec{B}(t)$.

V. Maxwell's Equations

Maxwell's equations:

The compact form without electromagnetic constants, by introducing density current $\vec{D}$ and magnetic field $\vec{H}$:

$$\begin{cases} \vec{\nabla}.\vec{D} = \rho & I(Maxwell - Gauss\ law) \\ \vec{\nabla} \wedge \vec{E} = -\dfrac{\partial \vec{B}}{\partial t} & II(Maxwell - Faraday\ Law) \\ \vec{\nabla}.\vec{B} = 0 & III(Gauss\ Law\ for\ magnetism) \\ \vec{\nabla} \wedge \vec{H} = \vec{J} + \dfrac{\partial \vec{D}}{\partial t} & IV(Maxwell - Ampere\ Law) \end{cases}$$

The integral forms of previous equations of Maxwell are given in compact expressions:

$$\begin{cases} \oint_S \vec{D}.d\vec{S} = Q & (I) \\ \oint_C \vec{E}.d\vec{l} = -\int_S \dfrac{\partial \vec{B}}{\partial t}.d\vec{S} & (II) \\ \oint_S \vec{B}.d\vec{S} = 0 & (III) \\ \oint_C \vec{H}.d\vec{l} = \int_S \left(\vec{J} + \dfrac{\partial \vec{D}}{\partial t}\right).d\vec{S} & (IV) \end{cases}$$

V. Maxwell's Equations

Maxwell's equations:

The compact form without electromagnetic constants, by introducing density current $\vec{D}$ and magnetic field $\vec{H}$:

$$\begin{cases} \vec{\nabla}.\vec{D} = \rho & I(Maxwell - Gauss\ law) \\ \vec{\nabla} \wedge \vec{E} = -\dfrac{\partial \vec{B}}{\partial t} & II(Maxwell - Faraday\ Law) \\ \vec{\nabla}.\vec{B} = 0 & III(Gauss\ Law\ for\ magnetism) \\ \vec{\nabla} \wedge \vec{H} = \vec{J} + \dfrac{\partial \vec{D}}{\partial t} & IV(Maxwell - Ampere\ Law) \end{cases}$$

Maxwell's equations on a plaque on his statue in Edinburgh