Klein-Gordon equation in the presence of an external electromagnetic field

This equation describes the interaction between a particle with charge q and the external electromagnetic field, which is represented by the four-vector potential $A_\mu = \left(\overrightarrow{A}, i\frac{\phi}{c}\right)$.
To obtain the Klein-Gordon equation in the presence of an external electromagnetic field, the minimal coupling method is employed, which involves substituting the momentum and energy $(\overrightarrow{p}, E)$ with.

$$E \rightarrow E - q\phi \qquad\qquad \overrightarrow{p} \rightarrow \overrightarrow{p} - q\overrightarrow{A} \tag{9.1}$$

In the free Klein-Gordon equation, the transformation presented in equation (9.1) can be reformulated using four-vectors. Its expression is provided by:

$$P_\mu \rightarrow P_\mu - qA_\mu \tag{9.2}$$

<u>Exercice</u> 8 :

- Demonstrate the equivalence of the two transformations provided in equations (9.1) and (9.2).

We have

$$P_\mu = -i\hbar\,\partial_\mu \Longrightarrow P_\mu = -i\,\partial_\mu \quad \text{pour } \hbar = 1 \tag{9.3}$$

The transformation (9.2) becomes,

$$-i\,\partial_\mu \rightarrow -i\,\partial_\mu - qA_\mu \Longrightarrow \partial_\mu \rightarrow \partial_\mu - iqA_\mu \Longrightarrow \partial_\mu \cdot \partial_\mu \rightarrow \left(\partial_\mu - iqA_\mu\right)\left(\partial_\mu - iqA_\mu\right) \tag{9.4}$$

If we replace in the free Klein-Gordon equation, we get

$$\left[\left(\partial_\mu - iqA_\mu\right)\left(\partial_\mu - iqA_\mu\right) - m^2\right]\phi(x_\mu) = 0 \tag{9.5}$$

This equation is known as the Klein-Gordon equation in the presence of an external electromagnetic field A_μ. Introducing $D_\mu = (\partial_\mu - iqA_\mu)$, equation (9.5) can be expressed as

$$\left[D_\mu D_\mu - m^2\right] \phi(x_\mu) = 0 \tag{9.6}$$

The conjugate of the latter equation is provided by

$$\left[D^*_\mu D^*_\mu - m^2\right] \phi^*(x_\mu) = 0 \Longrightarrow \left[(\partial_\mu + iqA_\mu)\,(\partial_\mu + iqA_\mu) - m^2\right] \phi^*(x_\mu) = 0 \tag{9.7}$$

9.1 invariance of the Klein-Gordon equation under the presence of an external electromagnetic field through gauge transformation

Exercice 9 :

In the presence of an external electromagnetic field $A_\mu(\overrightarrow{A}, iV)$, the motion of a particle with mass m, zero spin, and relativistic speed c is characterized by the following Klein-Gordon equation

$$\left[(\partial_\mu - iqA_\mu)(\partial_\mu - iqA_\mu) - m^2\right] \phi(x_\mu) = 0$$

- Demonstrate the invariance of this equation under the following gauge transformation

$$\begin{cases} A_\mu \longrightarrow A'_\mu = A_\mu - \partial_\mu \alpha(x_\mu) \\ \phi(x_\mu) \longrightarrow \phi'(x_\mu) = e^{-iq\alpha(x_\mu)}\phi(x_\mu) \quad , \quad \phi(x_\mu), \alpha(x_\mu) \text{sont deux réels arbitraires.} \end{cases}$$

9.2 Klein-Gordon equation current in the presence of an external electromagnetic field

Exercice 10 :

The Klein-Gordon equation, which governs the dynamics of a relativistic particle with mass m, charge q, and subject to an external electromagnetic-magnetic field $A_\mu(\overrightarrow{A}, i\phi)$, is presented

$$\left[(\partial_\mu - iqA_\mu)(\partial_\mu - iqA_\mu) - m^2\right] \psi(x) = 0$$

Determine the quadri-vector current expression of Klein-Gordon J_μ that solves the equation

$$\partial_\mu J_\mu = 0$$

We give : $(\partial_\mu^* - iqA_\mu^*)(\partial_\mu^* - iqA_\mu^*) = (\partial_\mu + iqA_\mu)(\partial_\mu + iqA_\mu)$