A review of special relativity

4.1 Overview of the laws of electromagnetism

4.1.1 Maxwell equations

The laws of electromagnetism can be expressed as follows

- As a function of the electric field ($\overrightarrow{E}$) and the magnetic field ($\overrightarrow{B}$).
- As a function of the vector potential ($\overrightarrow{A}$) and scalar potential (ϕ).

Maxwell expressed the laws of electromagnetism in the form of the following four equations:

$$div \cdot \overrightarrow{D} = \rho \tag{4.1}$$

$$div \cdot \overrightarrow{B} = 0 \tag{4.2}$$

$$\overrightarrow{rot}\,\overrightarrow{H} = \overrightarrow{j} + \frac{\partial \overrightarrow{D}}{\partial t} \tag{4.3}$$

$$\overrightarrow{rot}\,\overrightarrow{E} = -\frac{\partial \overrightarrow{B}}{\partial t} \tag{4.4}$$

The equations (4.1), (4.3), (4.4) represent Gauss's law, Maxwell-Ampere's law and Lenz-Faraday's law respectively.

- $\overrightarrow{D}$ is the electrical displacement vector.
- $\overrightarrow{H}$ is the excitation field vector.
- ρ is an electrical charge density.
- $\overrightarrow{j}$ is an electric charge current.

These vectors are related to the electric field $\overrightarrow{E}$ and the magnetic field $\overrightarrow{B}$ by the next equations:

$$\overrightarrow{D} = \varepsilon \overrightarrow{E} \tag{4.5}$$

$$\overrightarrow{B} = \mu \overrightarrow{H} \tag{4.6}$$

Where ε is the dielectric permittivity and μ the magnetic permeability of the medium. In vacuum, the equations (4.5) and (4.6) become

$$\overrightarrow{D} = \varepsilon_o \overrightarrow{E} \tag{4.7}$$

$$\overrightarrow{B} = \mu_o \overrightarrow{H} \Longrightarrow \overrightarrow{H} = \frac{\overrightarrow{B}}{\mu_o} \tag{4.8}$$

Where ε_o and μ_o are two constants given respectively by: $\varepsilon_o = 8.854\,\mathrm{pF\,m^{-1}}$ and $\mu_0 = 4\pi \times 10^{-7}$ Henry/meter.
By introducing the operator $\overrightarrow{\nabla}$, the previous equations become:

$$\overrightarrow{\nabla} \cdot \overrightarrow{D} = \rho \tag{4.9}$$

$$\overrightarrow{\nabla} \cdot \overrightarrow{B} = 0 \tag{4.10}$$

$$\overrightarrow{\nabla} \wedge \overrightarrow{H} = \overrightarrow{j} + \frac{\partial \overrightarrow{D}}{\partial t} \tag{4.11}$$

$$\overrightarrow{\nabla} \wedge \overrightarrow{E} = -\frac{\partial \overrightarrow{B}}{\partial t} \tag{4.12}$$

By replacing the equations (4.7) and (4.8) in the equations (4.9), (4.10), (4.11) and (4.12), we find:

$$\overrightarrow{\nabla} \cdot (\epsilon_o \overrightarrow{E}) = \rho \Longrightarrow \overrightarrow{\nabla} \cdot \overrightarrow{E} = \frac{\rho}{\epsilon_o} \tag{4.13}$$

$$\overrightarrow{\nabla} \cdot \overrightarrow{B} = 0 \tag{4.14}$$

$$\overrightarrow{\nabla} \wedge \left(\frac{\overrightarrow{B}}{\mu_o}\right) = \overrightarrow{j} + \frac{\partial}{\partial t}\left(\overrightarrow{\epsilon_o E}\right) \Longrightarrow \overrightarrow{\nabla} \wedge \overrightarrow{B} = \mu_o \overrightarrow{j} + \mu_o \epsilon_o \frac{\partial \overrightarrow{E}}{\partial t} \tag{4.15}$$

$$\overrightarrow{\nabla} \wedge \overrightarrow{E} = -\frac{\partial \overrightarrow{B}}{\partial t} \tag{4.16}$$

The Lorentz force acting on a particle with charge q and velocity *overrightarrowV* is given by

$$\overrightarrow{F} = q(\overrightarrow{E} + \overrightarrow{V} \wedge \overrightarrow{B}) \tag{4.17}$$

The charge conservation equation is given by,

$$\overrightarrow{\nabla} \cdot \overrightarrow{j} + \frac{\partial \rho}{\partial t} = 0 \tag{4.18}$$

Ramarque:

- Maxwell's equations and the equation for the conservation of electric charge are valid at all points in the medium and at all times. They are, therefore, local equations.

4.1.2 Vector and scalar potentials

The magnetic field $\overrightarrow{B}$ and the electric field $\overrightarrow{E}$ are derived from the Lorentz potentials $\overrightarrow{A}$ and ϕ, where

$$\overrightarrow{E} = -\overrightarrow{grad}\phi - \frac{\partial \overrightarrow{A}}{\partial t} \tag{4.19}$$

$$\overrightarrow{B} = \overrightarrow{rot}\,\overrightarrow{A} \tag{4.20}$$

The latter equations can be rewritten in terms of $\overrightarrow{\nabla}$,

$$\overrightarrow{E} = -\overrightarrow{\nabla} \cdot \phi - \frac{\partial \overrightarrow{A}}{\partial t} \tag{4.21}$$

$$\overrightarrow{B} = \overrightarrow{\nabla} \wedge \overrightarrow{A} \tag{4.22}$$

Ramarque:

In vacuum, the potential vectors $\overrightarrow{A}$ and scalar ϕ satisfy the following equation:

$$div\overrightarrow{A} + \mu_o \epsilon_o \frac{\partial \phi}{\partial t} = 0 \tag{4.23}$$

This equation is known as the Lorentz Gauge. This equation can be written as a function of $\overrightarrow{\nabla}$,

$$\overrightarrow{\nabla} \cdot \overrightarrow{A} + \mu_o \epsilon_o \frac{\partial \phi}{\partial t} = 0 \tag{4.24}$$

We have,

$$\mu_o \epsilon_o c^2 = 1 \Longrightarrow c = 1/\sqrt{\mu_o \epsilon_o} \tag{4.25}$$

Using Maxwell's equations and the Lorentz gauge, we obtain:

$$\Delta \phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial^2 t} = 0 \tag{4.26}$$

$$\Delta \overrightarrow{A} - \frac{1}{c^2} \frac{\partial^2 \overrightarrow{A}}{\partial^2 t} = 0 \tag{4.27}$$

Solving these two equations gives the values of the potentials ϕ and $\overrightarrow{A}$.

4.2 Vector analysis in Minkowski space

The "quadi-nabla" operator is introduced into Minkowski's four-dimensional space and defined as follows:

$$\overrightarrow{\partial} = \left(\overrightarrow{\nabla}, -\frac{1}{c} \frac{\partial}{\partial t} \right) \tag{4.28}$$

of components,

$$\partial_1 = \frac{\partial}{\partial x}, \quad \partial_2 = \frac{\partial}{\partial y}, \quad \partial_3 = \frac{\partial}{\partial z}, \quad \partial_4 = -\frac{1}{c} \frac{\partial}{\partial t} \tag{4.29}$$

4.2.1 Quadri-divergence and quadri-gradient

Let be the quadri-vector $\overrightarrow{A}$, with components:

$$\overrightarrow{A} = (a_x, a_y, a_z, a_4) = (\overrightarrow{a}, a_4) \quad \text{où} \quad \overrightarrow{a} = (a_x, a_y, a_z) \tag{4.30}$$

The metric of the Minkowski space is given by $(+,+,+,-)$. So the scalar product of two quadri-vectors $\overrightarrow{A}$ and $\overrightarrow{B}$ is given by

$$\overrightarrow{A} \cdot \overrightarrow{B} = \begin{pmatrix} a_x \\ a_y \\ a_z \\ a_4 \end{pmatrix} \cdot \begin{pmatrix} b_x \\ b_y \\ b_z \\ b_4 \end{pmatrix} = +a_x b_x + a_y b_y + a_z b_z - a_4 b_4 \tag{4.31}$$

The quadri-divergence of a quadri-vector $\overrightarrow{V}$ is given by

$$\overrightarrow{\partial} \cdot \overrightarrow{V} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \\ -\frac{1}{c}\frac{\partial}{\partial t} \end{pmatrix} \cdot \begin{pmatrix} v_x \\ v_y \\ v_z \\ v_4 \end{pmatrix} = +\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} + \frac{1}{c}\frac{\partial v_4}{\partial t} \tag{4.32}$$

which can also be written as,

$$\overrightarrow{\partial} \cdot \overrightarrow{V} = \left(\overrightarrow{\partial}, -\frac{1}{c}\frac{\partial}{\partial t}\right) \cdot (\overrightarrow{v}, v_4) = \left(\overrightarrow{\partial}\,\overrightarrow{v}, -\frac{1}{c}\frac{\partial v_4}{\partial t}\right) \tag{4.33}$$

In the same way, we define the quadri-gradient of a ϕ scalar function as,

$$\overrightarrow{\partial}\phi = \left(\overrightarrow{\partial}\phi, -\frac{1}{c}\frac{\partial \phi}{\partial t}\right) \tag{4.34}$$

4.2.2 Quad-vector current density

The equation (4.18) expresses the principle of conservation of charge. This equation can be written as

$$\frac{\partial \rho}{\partial t} + \overrightarrow{\nabla} \cdot \overrightarrow{j} = 0 \Longrightarrow \frac{\partial j_x}{\partial x} + \frac{\partial j_y}{\partial y} + \frac{\partial j_z}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t}(c\rho) = 0 \tag{4.35}$$

which can be written as:

$$\frac{\partial j_x}{\partial x} + \frac{\partial j_y}{\partial y} + \frac{\partial j_z}{\partial z} - \left(-\frac{1}{c}\frac{\partial}{\partial t}\right)(c\rho) = 0 \Longrightarrow \left(\overrightarrow{\partial}\,\overrightarrow{j}, -\frac{1}{c}\frac{\partial}{\partial t}(\rho c)\right) = 0 \tag{4.36}$$

This equation appears as the quadri-divergence of a quadri-vector

$$\left(\overrightarrow{\partial}, -\frac{1}{c}\frac{\partial}{\partial t}\right)\left(\overrightarrow{j}, \rho c\right) = 0 \Longrightarrow \overrightarrow{\partial}\,\overrightarrow{J} = 0 \tag{4.37}$$

Equation (4.37) represents the writing of the charge conservation equation in Minkowski space and the current quadri-vector is given by,

$$\overrightarrow{J} = \left(\overrightarrow{j}, \rho c\right) \tag{4.38}$$

4.2.3 Quad-vector potential

The Lorentz gauge given in the equation (**??**) can be rewritten as follows,

$$\overrightarrow{\nabla} \cdot \overrightarrow{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0 \Longrightarrow \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{\phi}{c}\right) = 0 \tag{4.39}$$

which can be written as:

$$\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} - \left(-\frac{1}{c}\frac{\partial}{\partial t}\right)\left(\frac{\phi}{c}\right) = 0 \Longrightarrow \left(\overrightarrow{\partial}\,\overrightarrow{A}, -\frac{1}{c}\frac{\partial}{\partial t}\frac{\phi}{c}\right) = 0 \tag{4.40}$$

This equation appears as the quadri-divergence of a quadri-vector

$$\left(\overrightarrow{\partial}, -\frac{1}{c}\frac{\partial}{\partial t}\right)\left(\overrightarrow{A}, \frac{\phi}{c}\right) = 0 \Longrightarrow \overrightarrow{\partial}\,\overrightarrow{\mathbf{A}} = 0 \tag{4.41}$$

Equation (4.41) represents the writing of the Lorentz gauge in Minkowski space and the potential quadri-vector is given by,

$$\overrightarrow{\mathbf{A}} = \left(\overrightarrow{A}, \frac{\phi}{c}\right) \tag{4.42}$$

4.2.4 Electromagnetic field tensor

The fields $\overrightarrow{E}$ and $\overrightarrow{B}$ are given as functions of the potentials ϕ and $\overrightarrow{A}$ by the two equations (4.21) and (4.22).
Writing the equation (4.21) in three-dimensional space gives:

$$\overrightarrow{E} = -\overrightarrow{\nabla}\phi - \frac{\partial \overrightarrow{A}}{\partial t} \Longrightarrow \tag{4.43}$$

$$E_x = -\frac{\partial \phi}{\partial x} - \frac{\partial A_x}{\partial t} \tag{4.44}$$

$$E_y = -\frac{\partial \phi}{\partial y} - \frac{\partial A_y}{\partial t} \tag{4.45}$$

$$E_z = -\frac{\partial \phi}{\partial z} - \frac{\partial A_z}{\partial t} \tag{4.46}$$

These last equations can be rewritten in the following form:

$$E_x = -c\frac{\partial}{\partial x}\left(\frac{\phi}{c}\right) - c\frac{1}{c}\frac{\partial A_x}{\partial t} \tag{4.47}$$

$$E_y = -c\frac{\partial}{\partial y}\left(\frac{\phi}{c}\right) - c\frac{1}{c}\frac{\partial A_y}{\partial t} \tag{4.48}$$

$$E_z = -c\frac{\partial}{\partial z}\left(\frac{\phi}{c}\right) - c\frac{1}{c}\frac{\partial A_z}{\partial t} \tag{4.49}$$

Now, taking c as a factor, we find:

$$\frac{E_x}{c} = -\frac{\partial}{\partial x}\left(\frac{\phi}{c}\right) - \frac{1}{c}\frac{\partial A_x}{\partial t} \tag{4.50}$$

$$\frac{E_y}{c} = -\frac{\partial}{\partial y}\left(\frac{\phi}{c}\right) - \frac{1}{c}\frac{\partial A_y}{\partial t} \tag{4.51}$$

$$\frac{E_z}{c} = -\frac{\partial}{\partial z}\left(\frac{\phi}{c}\right) - \frac{1}{c}\frac{\partial A_z}{\partial t} \tag{4.52}$$

Writing the equation (4.22) in three-dimensional space gives:

$$\overrightarrow{B} = \overrightarrow{\nabla} \wedge \overrightarrow{A} \Longrightarrow \tag{4.53}$$

$$B_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \tag{4.54}$$

$$B_y = -\left[\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right] \Longrightarrow B_y = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \tag{4.55}$$

$$B_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \tag{4.56}$$

4.2.5 Change of variable

A point in Minkowski space is represented by the quadri-vector position

$$\begin{pmatrix} x \\ y \\ z \\ -ct \end{pmatrix} \tag{4.57}$$

Let's make the following change of variables:

$$\begin{cases} x = x_1 \\ y = x_2 \\ z = x_3 \\ -ct = x_4 \end{cases} \implies \begin{cases} \frac{\partial}{\partial x} = \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial y} = \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial z} = \frac{\partial}{\partial x_3} \\ -\frac{1}{c}\frac{\partial}{\partial t} = \frac{\partial}{\partial x_4} \end{cases} \implies \begin{cases} \frac{\partial}{\partial x_1} = \partial_1 \\ \frac{\partial}{\partial x_2} = \partial_2 \\ \frac{\partial}{\partial x_3} = \partial_3 \\ \frac{\partial}{\partial x_4} = \partial_4 \end{cases} \tag{4.58}$$

$$\begin{cases} A_x = A_1 \\ A_y = A_2 \\ A_z = A_3 \\ \frac{\phi}{c} = A_4 \end{cases} \tag{4.59}$$

The equations (4.50), (4.51), (4.52), (4.54), (4.55) and (4.56) become,

$$\frac{E_x}{c} = -\partial_1 A_4 + \partial_4 A_1 \tag{4.60}$$

$$\frac{E_y}{c} = -\partial_2 A_4 + \partial_4 A_2 \tag{4.61}$$

$$\frac{E_z}{c} = -\partial_3 A_4 + \partial_4 A_3 \tag{4.62}$$

$$B_x = \partial_2 A_3 - \partial_3 A_2 \tag{4.63}$$

$$B_y = \partial_3 A_1 - \partial_1 A_3 \tag{4.64}$$

$$B_z = \partial_1 A_2 - \partial_2 A_1 \tag{4.65}$$

These six equations can be written in the following general form

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \qquad \mu, \nu = 1, 2, 3, 4 \tag{4.66}$$

Where the coefficients $F_{\mu\nu}$ are the matrix elements of a tensor in Minkowski space, called the "electromagnetic field tensor" and given by,

$$F^{\mu\nu} = \begin{pmatrix} F_{11} & F_{12} & F_{13} & F_{14} \\ F_{21} & F_{22} & F_{23} & F_{24} \\ F_{31} & F_{32} & F_{33} & F_{34} \\ F_{41} & F_{42} & F_{43} & F_{44} \end{pmatrix} \tag{4.67}$$

The tensor is antisymmetric $F^{\mu} = -F^{\mu}$ and $F^{\mu} = 0$. Therefore, the matrix elements of the electromagnetic field tensor are given by,

$$\frac{E_x}{c} = -\partial_1 A_4 + \partial_4 A_1 = F_{41} = -F_{14} \tag{4.68}$$

$$\frac{E_y}{c} = -\partial_2 A_4 + \partial_4 A_2 = F_{42} = -F_{24} \tag{4.69}$$

$$\frac{E_z}{c} = -\partial_3 A_4 + \partial_4 A_3 = F_{43} = -F_{34} \tag{4.70}$$

$$B_x = \partial_2 A_3 - \partial_3 A_2 = F_{23} = -F_{32} \tag{4.71}$$

$$B_y = \partial_3 A_1 - \partial_1 A_3 = F_{31} = -F_{13} \tag{4.72}$$

$$B_z = \partial_1 A_2 - \partial_2 A_1 = F_{12} = -F_{21} \tag{4.73}$$

Finally, the electromagnetic field tensor is given by,

$$F^{\mu\nu} = \begin{pmatrix} 0 & B_z & -B_y & -\frac{E_x}{c} \\ -B_z & 0 & B_x & -\frac{E_y}{c} \\ B_y & -B_x & 0 & -\frac{E_z}{c} \\ \frac{E_x}{c} & \frac{E_y}{c} & \frac{E_z}{c} & 0 \end{pmatrix} \tag{4.74}$$