A review of quantum mechanics

2.1 Introduction

Several attempts were necessary before arriving at the current formulation of quantum mechanics. Specifically, in the mid-1920s, there were two competing approaches to model quantum phenomena: that of Heisenberg, Born, Jordan, and Dirac, called the matrix mechanics, and that of Schrödinger, called wave mechanics.
Before detailing these two theories, let us recall the essential points of classical mechanics (analytical mechanics). The latter is based on the Lagrangian formalism.

2.2 Wave modeling

Due to the wave-like nature of matter, we need to take a closer look at what a wave is and the appropriate method to use to mathematically model its movement in spacetime. From a mathematical point of view, the dynamic of a wave can be described by solving the following wave equation:

$$\Box\phi = 0 \tag{2.1}$$

where the d'Alembertian operator is given by the expression

$$\Box := -\partial_{tt} + \Delta$$

A wave will then be modeled by a function ϕ, which is a solution of the equation (2.1). An obvious solution to the equation (2.1) is the function

$$\phi(x,t) = \phi_o e^{i(k\cdot x - \omega t)} \tag{2.2}$$

where x represents the position vector, t the time, k the wave vector (i.e., the wave propagation vector), ω is the wave frequency, and $x \cdot k$ is the dot product.

2.3 Schrödinger equation

The idea here is to model particles in the same way as waves, namely by a function ψ. The probability of finding the particle at time t is equal to

$$\int |\psi(x,t)|^2 dx. \tag{2.3}$$

This implies that

$$\int_{R^3} |\psi(x,t)|^2 dx = 1. \tag{2.4}$$

The fundamental principle of wave mechanics is stated as follows
The wave function ψ of a particle with mass m moving in vacuum and subjected to no interactions satisfies the Schrödinger equation

$$i\hbar\partial_t\psi = -\frac{\hbar^2}{2m}\Delta\psi \tag{2.5}$$

$\hbar$ is a universal constant known as the Planck constant, and Δ is a spatial Laplacian, with the following sign convention:

$$\Delta = \partial_{11} + \partial_{22} + \partial_{33}.$$

The Planck constant, denoted as $\hbar$, has dimensions of energy multiplied by time, or equivalently, momentum multiplied by length. Its value is expressed in Joule-seconds:

$$\hbar = 1,054571628 \times 10^{-34} \text{ J.s}$$

The wave function ψ of a particle placed in a potential $V(x,t)$ satisfies:

$$i\hbar\partial_t\psi = -\frac{\hbar^2}{2m}\Delta\psi + V\psi. \tag{2.6}$$

2.4 Harmonic oscillator

This section will be addressed as an exercise (see exercise 3).

2.5 Pauli equation

This section will be addressed as an exercise (see exercise 4).