Chapter I: Classical Mechanics Review

1. Introduction

Mechanics is the science that studies the motion of systems. It is divided into two main branches:

1 **Kinematics**: This branch provides a purely descriptive study of motion without considering its causes. Kinematics involves defining time, reference frames, and the position, velocity, and acceleration vectors, as well as the transformation laws of these quantities when changing reference frames.
2 **Dynamics**: This branch deals with the forces that cause motion and establishes the relationship between motion and its causes.

Mechanics can also be subdivided based on the type of systems to which it applies:

1 **Point Mechanics**
2 **Solid Mechanics**
3 **Fluid Mechanics**

2. Kinematics of the Point Particle

2.1 Definitions

The position of a point particle with respect to the origin of a coordinate system at a given time t is given by:

$$\vec{r}(t) = \overrightarrow{OM}$$

To know the movement of the point, it is sufficient to define the time equations:

- The coordinate x(t), the coordinate y(t), and the coordinate z(t) in Cartesian coordinates, where the position vector is written as: $\overrightarrow{OM}(t)=x(t)\ \vec{i}+y(t)\ \vec{j}+z(t)\ \vec{k}$
- r(t), θ(t), and the coordinate z(t) in cylindrical coordinates, where the position vector is: $\overrightarrow{OM}(t)= r(t)\ \vec{e}_r+z(t)\ \vec{k}$
- r(t), θ(t), and ϕ(t) in spherical coordinates, where the position vector is: $\overrightarrow{OM}(t)=r(t)\vec{e}_r$

The velocity vector $\vec{v}$ of the point is defined as: $\vec{v} = \frac{d\overrightarrow{OM}}{dt} = \dot{\overrightarrow{OM}}$

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\overrightarrow{OM}}{dt^2} = \ddot{\overrightarrow{OM}}$$

The acceleration vector $\vec{a}$ of the point is defined as:
We also define the momentum $\vec{p}$ of a point particle with mass mmm and velocity $\vec{v}$ by: $\vec{p}=m\vec{v}$

3. Dynamics of the Point Particle

Chapter I: Classical Mechanics Review

3.1 Newton's First Law: The Principle of Inertia
also known as the Principle of Inertia, states that a point particle will remain at rest or continue to move in a straight line at constant velocity unless acted upon by an external force. This law defines the concept of inertia, which is the tendency of an object to resist changes in its state of motion.

3.2 Newton's Second Law: The Principle of Dynamics
A point particle is rarely isolated and is typically subjected to forces. These forces indicate the presence of an actor (the entity exerting the force) and a receiver (the object experiencing the force). Newton's Second Law establishes that the acceleration of a point particle is directly proportional to the net force acting upon it and inversely proportional to its mass. This relationship can be expressed by the equation:

$$\frac{d\vec{p}}{\mathrm{dt}} = \vec{F}$$

$\vec{F}$ is the cause, and $\frac{d\vec{p}}{\mathrm{dt}}$ is the effect. A force acts in a certain direction (axis or line of the force), with a certain sense, intensity, and at a particular point.

We distinguish the following forces:

- **Distance interaction forces**: gravitational, electromagnetic.
- **Contact forces**: friction and tension.

3.3 Newton's Third Law: The Principle of Reciprocal Actions
This law, commonly known as the principle of action and reaction, states that if body A exerts a force on body B, then body B exerts an equal force on body A. These forces are equal in magnitude and direction but opposite in sense (see Figure-1).

$$\vec{F}_{A/B} = -\vec{F}_{B/A}$$

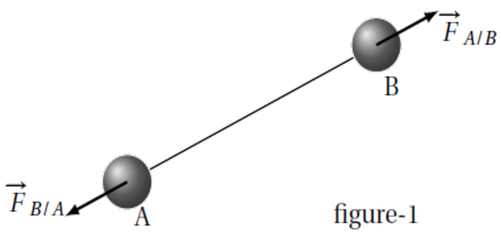


figure-1

3.4 AngularMomentum

Consider a point particle M located in a Galilean reference frame R(O,x,y,z). The angular momentum of point MMM with respect to a fixed point O on the z-axis is defined as the moment of its linear momentum. It is denoted by:

$$\vec{L}_{M/O} = \overrightarrow{OM} \wedge \vec{p} = \overrightarrow{OM} \wedge (m\vec{v})$$

$\vec{L}_{M/O}$ where is the angular momentum, $\vec{r}$ is the position vector of point M relative to O, and $\vec{p}=m\vec{v}$ is the linear momentum of the particle, with m being its mass and $\vec{v}$ its velocity.

3.4.1 Special Case: Curvilinear Motion

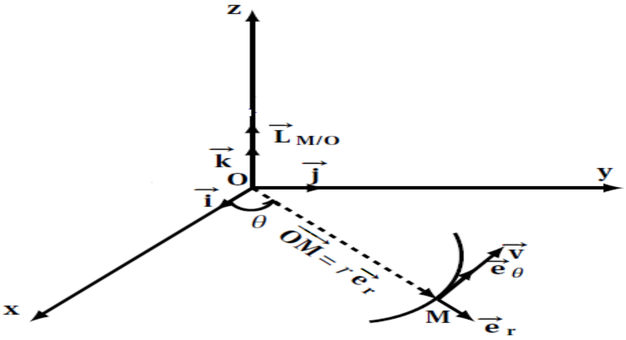
Consider the curvilinear motion of a point particle in the XOY plane, in cylindrical coordinates:

$$\vec{L}_{M/O} = r\vec{e_r} \wedge [m(\dot{r}\vec{e_r} + r\dot{\theta}\vec{e_\theta} + 0\vec{e_z})] = mr^2\dot{\theta}(\vec{e_r} \wedge \vec{e_\theta})$$

Since $\vec{e_r} \wedge \vec{e_\theta} = \vec{e_z}$, we have:

$$\vec{L}_{M/O} = mr^2\dot{\theta}\vec{e_z} = I_{z'}\dot{\theta}\vec{e_z}$$

where $I_{z'z}=mr^2\dot{\theta}$ is by definition the moment of inertia of the point particle M of mass m at a distance r from the axis of rotation z'oz. Angular momentum reflects the tendency of the point particle to follow a curvilinear motion or deviate from a rectilinear trajectory.



3.4.2 Theorem of Angular Momentum

The temporal variation of angular momentum is:

$$\frac{d\vec{L}_{M/O}}{dt} = \frac{d\vec{r}}{dt} \wedge (m\vec{v}) + \vec{r} \wedge \frac{d}{dt}(m\vec{v})$$

The first term is zero since $\frac{d\vec{r}}{dt} = \vec{v}$ is parallel to $\vec{v}$, thus:

$$\frac{d\vec{L}_{M/O}}{dt} = \vec{M}_{\vec{F}/O} = \vec{r} \wedge \vec{F}$$

$$\frac{d\vec{L}_{M/O}}{dt} = \sum \vec{M}_{\vec{F}/O}$$

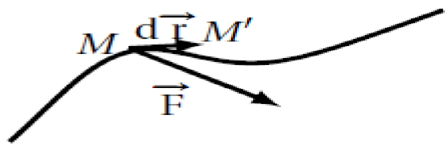
where $\vec{M}_{\vec{F}/O}$ is the moment of the external force relative to point O.

- If the point is in equilibrium (no rotation), then the sum of the moments of the external forces is zero, and the angular momentum is zero.
- If $\vec{r}$ and $\vec{F}$ share the same support (parallel or anti-parallel), then $\vec{L}_{M/O}$=constant. $\vec{F}$ is called a central force.

3.5 Work of a Variable Force

To calculate the work done by a variable force (both in direction and magnitude), the path from point A to point B is divided into a series of infinitesimally small, straight elementary displacements $d\vec{r}$, such that the force $\vec{F}$ remains constant over each segment. On such a straight segment, the elementary work dW is given by::

$$dW_{M\to M'} = \vec{F}\cdot d\vec{r}$$



To obtain the total work of $\vec{F}$, it is sufficient to sum the elementary works between A and B. The summation is continuous, thus: $W_{A\to B}(\vec{F}) = \int_{\vec{r}_a}^{\vec{r}_b} \vec{F}\cdot d\vec{r}$

Theorem

The work of a force during a displacement corresponds to the circulation of the force vector along the path.

Power of a Force

The instantaneous power P(t) is defined by: $P(t) = \frac{dW}{dt} = \vec{F}\cdot\frac{d\vec{r}}{dt} = \vec{F}\cdot\vec{v}$

3.6 Energy

3.6.1 Variation of Kinetic Energy

During an elementary displacement $d\vec{r}$, the work of $\vec{F}$ is:

$$dW = \vec{F}\cdot d\vec{r} = m\frac{d\vec{v}}{dt}\cdot\vec{v}\,dt = m\vec{v}\cdot d\vec{v}$$

thus: $d\left(\frac{1}{2}mv^2\right) = \vec{F}\cdot d\vec{r}$

Therefore, for a material point of mass m moving at a speed $\vec{v}$ in a Galilean reference frame R, we attribute a state function called kinetic energy: $T = \frac{1}{2}mv^2$

A material point of mass m rotating around a fixed axis at a distance r from the material point, rotating at an angular speed $\dot{\theta}$ around the axis, has a linear speed $v=r\dot{\theta}$, thus:

$$T = \frac{1}{2}(mr^2)\dot{\theta}^2$$

which is in the form:

$$T = \frac{1}{2} I \dot{\theta}^2$$

with $I=mr^2$ being the moment of inertia of the material point relative to the axis of rotation.

3.6.2 Kinetic Energy Theorem

In a Galilean reference frame, the change in the kinetic energy of a material point subjected to a force $\vec{F}$ between positions $\vec{r}_a$ and $\vec{r}_b$ is equal to the work done by $\vec{F}$ over this displacement. This is expressed as:

$$T_b - T_a = \int_{\vec{r}_b}^{\vec{r}_a} \vec{F} \cdot d\vec{r}$$

3.6.3 Variation of Potential Energy

It is possible to define a second state function known as **potential energy**. To achieve this, it is essential to distinguish between two types of forces:

- **Non-Conservative Force:**

The work depends on the path taken, such as the solid friction force that constantly opposes the displacement of an object, which is expressed as: $\vec{F} = -c\vec{u}\,(\text{with } |\vec{u}| = 1)$

The work on the first segment is:

$$W_{A\to B}(\vec{F}) = -c\int_{x_a}^{x_b} dx = -c(x_b - x_a) < 0\,(\text{with } x_b > x_a)$$

The work on the return path, which is a portion of a circle, is:

$$W_{B\to A}(\vec{F}) = -c\int_{x_b}^{x_a} dl = -c(x_a - x_b) < 0$$

We note that the work of this force on a closed curve is not zero and is always negative, and depends on the path taken. This force does not derive from a potential energy. Generally, it is not possible to define potential energy in the presence of non-conservative forces.

- **Conservative Force:**

These are forces whose work does not depend on the path taken, but only on the starting and ending points, for example, the work of gravity, spring tension force, gravitational force, and electrostatic force. The work of these forces can be expressed in terms of the variation of potential energy U or Ep. For reasons of mechanical energy conservation, we agree that the variation of U is opposed to the work of the conservative force:

Chapter I: Classical Mechanics Review

$$U_{\vec{r}_b} - U_{\vec{r}_a} = -W_{A\to B}(\vec{F}(c)) = -\int_{\vec{r}_b}^{\vec{r}_a} \vec{F}(c)\cdot d\vec{r}$$

From this last equation, it is possible to deduce the differential of potential energy by showing the elementary work:

$$dU = -dW = -\vec{F}(c)\cdot d\vec{r}$$

Moreover, the differential of the state function Uis written as: $dU = \vec{\nabla} U \cdot d\vec{r}$

with $\vec{\nabla}$ being the gradient operator. From these two equations, we finally arrive at the local definition of the conservative force: $\vec{F}(c) = -\vec{\nabla} U$

Theorem:

A force field is conservative if and only if: $\vec{\nabla}\times\vec{F}(c)=\vec{0}$

Theorem:

A force field is conservative if and only if the circulation of the vector along a closed curve is zero.

Conservation of Mechanical Energy

In the case where the material point of mass m is subjected to a conservative force, we have: $dT = d\left(\frac{1}{2}m\vec{v}^2\right) = dW = -dU = \vec{F}(c)\cdot d\vec{r}$

d(T+U)=0 so T+U is constant between two points A and B, thus:$T_b+U(\vec{r}_b)=T_a+U_a(\vec{r}_a)$

The quantity E_m is called mechanical energy and is defined by: $E_m=T+U$

Theorem:

If a particle is subjected to a conservative force, its mechanical energy is conserved during its motion: $E_m=T+U=\text{constant}$

Variation of Mechanical Energy

In the presence of conservative and non-conservative forces, we have:

$$m\frac{d\vec{v}}{dt}\cdot d\vec{r} = \vec{F}(c)\cdot d\vec{r} + \vec{F}(nc)\cdot d\vec{r}$$

since $d\vec{r}=\vec{v}\,dt$, then: $dT=-dU+\vec{F}(nc)\cdot d\vec{r}$ We obtain: $d(E_m)=\vec{F}(nc)\cdot d\vec{r}$

Therefore, between two points A and B, we have: $E_{m_b} - E_{m_a} = \int_{\vec{r}_b}^{\vec{r}_a} \vec{F}(nc) \cdot d\vec{r}$

4. System of N Material Points

The laws of mechanics applicable to a material point or a system that can be modeled by a material point can be generalized to the case of a system consisting of a set of material points. However, it is necessary to distinguish between two types of systems:

- Discrete system Continuous system

Hereafter, we consider a system of N material points, each material point of mass m_i being placed at point A_i in the space of the system ($\overrightarrow{OA_i}=\vec{r_i}$).

4.1 Center of Mass or Center of Inertia:

The total mass of the system is defined by: $M=\sum_i^N m_i$

Center of Mass (or Barycenter) C: is such that:

$$\sum_{i=1}^{N} m_i \overrightarrow{CA_i} = \vec{0}$$

where:

$$\overrightarrow{CA_i} = \overrightarrow{CO} + \overrightarrow{OA_i} = -\overrightarrow{OC} + \overrightarrow{OA_i}$$

Thus, we have:

$$\left(\sum_{i=1}^{N} m_i\right) \overrightarrow{OC} = \sum_{i=1}^{N} m_i \overrightarrow{OA_i}$$

We finally obtain:

$$\overrightarrow{OC} = \frac{\sum_{i=1}^{N} m_i \overrightarrow{OA_i}}{M}$$

where:

$$\vec{R} = \frac{\sum_{i=1}^{N} m_i \vec{r_i}}{M}$$

4.2 System Kinematics

The kinematics of the system reveals:

- A motion of the center of mass (translation or rotation).
- A rotation of the system around axes passing through the center of mass.

Remark:

If $|\overrightarrow{A_\iota A_{\iota+1}}|$ evolves over time, the system can either expand or contract, indicating a deformable system. For a non-deformable or rigid system: $|\overrightarrow{A_\iota A_{\iota+1}}|$ =constant

4.2.1 Continuous System

A system of material points with a continuous mass distribution (linear, surface, or volume distribution) replaces the sum by an integral (single, double, or triple, depending on the mass distribution). The position vector is:

$$\overrightarrow{R} = \frac{\int \overrightarrow{r}\rho\, d\tau}{\int \rho\, d\tau}$$

with:

$$M = \int \rho\, d\tau$$

and:

$$d\tau = dx\, dy\, dz$$

Remark:

If the system is in a uniform gravitational field, the center of mass coincides with the center of gravity.

4.2.2 Kinetic Elements of a System with N Particles

Consider a Galilean reference frame R(o,ox,oy,oz) associated with the laboratory and a reference frame R*(o,ox′,oy′,oz′) associated with the center of mass of the system, with axes parallel to those of R

Let $\overrightarrow{v_i^*}$ and $\overrightarrow{p_i^*}$ be the velocity and momentum of A_i in R^*. Since:

$$\sum_{i=1}^{N} m_i \overrightarrow{CA_i} = \overrightarrow{0}$$

then:

$$\sum_{i=1}^{N} m_i \frac{d\overrightarrow{CA_i}}{dt} = \overrightarrow{0} \implies \sum_{i=1}^{N} m_i \overrightarrow{v_i^*} = \overrightarrow{0} \implies \sum_{i=1}^{N} \overrightarrow{p_i^*} = \overrightarrow{0}$$

Thus:

$$\overrightarrow{p^*} = \overrightarrow{0}$$

Using the composition law of position vectors:

$$\overrightarrow{OA_i} = \overrightarrow{OC} + \overrightarrow{CA_i}$$

and differentiating with respect to t (the composition law of velocities): $\overrightarrow{v_\iota}=\overrightarrow{V_c}+\overrightarrow{v_\iota^*}$

where $\overrightarrow{V_c}$ is the velocity vector of C with respect to R.

The total momentum of the system in R is:

$$\vec{p} = \sum_{i=1}^{N} m_i \vec{v_i} = \left(\sum_{i=1}^{N} m_i\right) \vec{V_c} + \sum_{i=1}^{N} m_i \vec{v_i^*}$$

Since:

$$\sum_{i=1}^{N} m_i \vec{v_i^*} = \vec{0}$$

then:

$$\vec{p} = M\vec{V_c}$$

Since:

Thus, the total momentum of the system in R is that of a material point (the center of mass) of mass M moving at velocity $\vec{V}_c$.

4.3 Koenig's Theorem

4.3.1 First Theorem of Koenig: Angular Momentum of a System with N Particles

The angular momentum of the system with N particles relative to R is:

$$\overrightarrow{L/o} = \sum_i \overrightarrow{L_{Ai/o}} = \sum_i \overrightarrow{OA_i} \times (m_i\vec{v_i}) = \sum_i \left(\overrightarrow{OC} + \overrightarrow{CA_i}\right) \times [m_i(\vec{V_c} + \vec{v_i^*})]$$

The second term:

$$\overrightarrow{OC} \times \sum_{i=1}^{N} m_i \vec{v_i^*} = \vec{0}$$

(since the total momentum in the center of mass is zero). The third term:

$$\left(\sum_{i=1}^{N} m_i \vec{CA_i}\right) \wedge \vec{V_c} = \vec{0}$$

(by the definition of CdM).

The angular momentum simplifies to:

$$\vec{L}_o = \vec{OC} \wedge \left(\sum_i m_i\right) \vec{V_c} + \sum_i \vec{CA_i} \wedge m_i \vec{v_i}^*$$

. which can be written as:

$$\vec{L}_o = \vec{OC} \times \left(\sum_i m_i\right) \vec{v_c} + \sum_i \vec{CA_i} \times m_i \vec{v_i}^*$$

Which can be written as:

$$\vec{L}_o = \vec{L}_{c/o} + \vec{L}_{A_i/o}$$

with:

$$\vec{L}_{c/o} = \vec{OC} \times (M\vec{v_c}) \quad \text{and} \quad \vec{L}_{A_i/c} = \sum_i \vec{CA_i} \times m_i \vec{v_i}^*$$

4.3.2 Second Koenig's Theorem: Kinetic Energy of a System with N Particles:

The kinetic energy of the system with N material points in R is:

$$T = \sum_i T_i = \sum_i \frac{1}{2}(m_i \vec{v}_i^{\,2}) = \sum_i \frac{1}{2} m_i (\vec{v}_c + \vec{v}_i^{\,*})^2$$

This can be expanded as:

$$T = \frac{1}{2}\left(\sum_i m_i\right)\vec{v}_c^{\,2} + \vec{v}_c \cdot \sum_i m_i \vec{v}_i^{\,*} + \sum_i \frac{1}{2} m_i \vec{v}_i^{\,*2}$$

The second term $\vec{v}_c \cdot \sum_i m_i \vec{v}_i^{\,*} = 0$ ((since the total momentum in the center of mass is zero). Thus, the kinetic energy reduces to:) $T = \frac{1}{2}\sum_i m_i \vec{v}_c^{\,2} + \sum_i \frac{1}{2} m_i \vec{v}_i^{\,*2}$

where: $$T = T_{c/o} + T_{A_i/c}$$

With $$T_{c/o} = \frac{1}{2} M \vec{v}_c^{\,2} \quad \text{and} \quad T_{A_i/c} = \sum_i \frac{1}{2} m_i \vec{v}_i^{\,*2}$$

4.4 External and Internal Forces

The forces acting on the points A_i can be decomposed into internal forces from other points A_j ($j \neq i$) in the system, denoted as $\vec{F}_{ji}^{(int)}$ and external forces, denoted as $\vec{F}_{ji}^{(ext)}$ coming from sources outside the system.

Remark:

If the internal forces derive from potential energy, this energy is called interaction potential energy or internal energy.

5. Fundamental Principle of Dynamics for the System

The equation of motion for the material point i is:

$$\frac{d\vec{p}_i}{dt} = m_i \frac{d\vec{v}_i}{dt} = \vec{F}_i^{(ext)} + \sum_{j=1, j\neq i}^{N} \vec{F}_{ji}^{(int)}$$

The fundamental principle of dynamics for the system is written as:

$$\sum_{i=1}^{N} \frac{d\vec{p}_i}{dt} = \sum_{i=1}^{N} m_i \frac{d\vec{v}_i}{dt} = \sum_{i=1}^{N} \vec{F}_i^{(ext)}$$

Chapter I: Classical Mechanics Review

6. Theorem of Angular Momentum

The time derivative of the angular momentum with respect to ooo in the Galilean reference frame is:

$$\frac{d\vec{L}_o}{dt} = \frac{d}{dt}\sum_{i=1}^{N}(\vec{r}_i \times \vec{p}_i) = \sum_{i=1}^{N}(\vec{r}_i \times \frac{d\vec{p}_i}{dt}) = \sum_{i=1}^{N}(\vec{r}_i \times \vec{F}_i^{(ext)}) + \sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N}(\vec{r}_i \times \vec{F}_{ji}^{(int)})$$

In the case where Newton's third law is valid (principle of action and reaction), the internal forces $\vec{F}_{ji}^{(int)}$ are equal and opposite in pairs and are carried by the lines connecting the particles iii and jjj. The internal forces in this case are central and collinear with $\overrightarrow{r_{ij}}$.

Example:

$$\vec{r}_1 \times \vec{F}_{21}^{(int)} + \vec{r}_2 \times \vec{F}_{12}^{(int)} = (\vec{r}_1 - \vec{r}_2) \times \vec{F}_{21}^{(int)} = \vec{r}_{21} \times \vec{F}_{21}^{(int)} = 0$$

Thus, we obtain:

$$\frac{d\vec{L}_o}{dt} = \sum_{i=1}^{N}(\vec{r}_i \times \vec{F}_i^{(ext)}) = \vec{M}_o^{(ext)}$$

7. Energy of a System of N Material Points

The work done by all forces when the system moves from one configuration to another (each material point in the system moves by d$\vec{r}_i$ is:

$$W = \sum_{i=1}^{N}\int_1^2 \vec{F}_i^{(ext)} \cdot d\vec{r}_i + \sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N}\int_1^2 \vec{F}_{ji}^{(int)} \cdot d\vec{r}_i$$

On the other hand:

$$W = \sum_{i=1}^{N}\int_1^2 m_i\frac{d\vec{v}_i}{dt} \cdot d\vec{r}_i = \sum_{i=1}^{N}\int_1^2 dT_i = \int_1^2 dT = T_2 - T_1 \quad \text{since} \quad dT = \sum_{i=1}^{N} dT_i$$

In the case of conservative forces:

- For external forces:

$$\sum_{i=1}^{N}\int_1^2 \vec{F}_i^{(ext)} \cdot d\vec{r}_i = \sum_{i=1}^{N}\int_1^2 (-\vec{\nabla}_i U_i) \cdot d\vec{r}_i = \sum_{i=1}^{N}\int_1^2 d(-U_i) = -\sum_{i=1}^{N}(U_i|_1^2) = -(U|_1^2)$$

Given that

$\vec{\nabla}_i = \frac{\partial}{\partial \vec{r}_i}$ For internal forces: they obey the law of action and reaction and are central, thus they derive from a potential energy $U_{ij} = U(|\vec{r}_i - \vec{r}_j|)$, we then have:

$$\vec{F}_{ji}^{(\mathrm{int})} = -\vec{\nabla}_i U_{ij}$$
$$\vec{F}_{ij}^{(\mathrm{int})} = -\vec{\nabla}_j U_{ij}$$

And since

$$\vec{F}_{ji}^{(\mathrm{int})} + \vec{F}_{ij}^{(\mathrm{int})} = \vec{0}$$

we have:

$$\sum_{i=1}^{N} \sum_{j=1, j \neq i}^{N} \int_1^2 \vec{F}_{ji}^{(\mathrm{int})} \cdot d\vec{r}_i = -\sum_{i=1}^{N} \sum_{j=1, j \neq i}^{N} \left(\int_1^2 \vec{\nabla}_i U_{ij} \cdot d\vec{r}_i + \vec{\nabla}_j U_{ij} \cdot d\vec{r}_j \right)$$

Since

$$\vec{\nabla}_i U_{ij} = -\vec{\nabla}_j U_{ij}$$

then

$$\vec{\nabla}_i U_{ij} = \vec{\nabla}_{ij} U_{ij}$$

and let $d\vec{r}_{ij} = d\vec{r}_i - d\vec{r}_j$, we then have:

$$\sum_{i=1}^{N} \sum_{j=1, j \neq i}^{N} \int_1^2 \vec{F}_{ji}^{(\mathrm{int})} \cdot d\vec{r}_i = -\sum_{i=1}^{N} \sum_{j=1, j \neq i}^{N} \int_1^2 \vec{\nabla}_{ij} U_{ij} \cdot d\vec{r}_{ij} = -\sum_{i=1}^{N} \sum_{j=1, j \neq i}^{N} \int_1^2 dU_{ij} = -\sum_{i=1}^{N} \sum_{j=1, j \neq i}^{N} U_{ij}$$

In summary, if both internal and external forces derive from potential energy, then:

$$U = \sum_{i=1}^{N} U_i + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} U_{ij}$$

8. Important Case: Rigid Bodies

In the case of a system where relative distances are constant over time, then:

$$\vec{r}_{ij} = \text{constant} \Rightarrow |\vec{r}_{ij}|^2 = \text{constant}^2 \Rightarrow \vec{r}_{ij} \cdot \vec{r}_{ij} = \text{constant} \Rightarrow \vec{r}_{ij} \cdot \frac{d\vec{r}_{ij}}{dt} = \vec{0}$$

then the displacement $d\vec{r}_{ij}$ is always perpendicular to the internal forces which are proportional to it.

So, the displacement $\overrightarrow{r_{ij}}$ is constantly perpendicular to the internal forces, which are proportional to $|\overrightarrow{r_{ij}}|$ and thus the work done is zero. Therefore, the internal energy is a constant.