CHAPTER 5
SYSTEMS OF LINEAR EQUATIONS

5.1 Introduction to Systems of Linear Equations

5.2 Gaussian Elimination and Gauss-Jordan Elimination

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5.1 Introduction to Systems of Linear Equations

- a linear equation in *n* variables:

$$a_1x_1+a_2x_2+a_3x_3+\cdots+a_nx_n=b$$

$a_1, a_2, a_3, \ldots, a_n$, b: real number

a_1: leading coefficient

x_1: leading variable

- Notes:

(1) Linear equations have no products or roots of variables and no variables involved in trigonometric, exponential, or logarithmic functions.

(2) Variables appear only to the first power.

5.1 Introduction to Systems of Linear Equations

- Ex : (Linear or Nonlinear)

Linear $(a)\ 3x+2y=7$

$(b)\ \frac{1}{2}x+y-\pi z=\sqrt{2}$ Linear

Linear $(c)\ x_1-2x_2+10x_3+x_4=0$

$(d)\ (\sin\frac{\pi}{2})x_1-4x_2=e^2$ Linear

Nonlinear $(e)\ xy+z=2$

not the first power

$(f)\ e^x-2y=4$ Nonlinear

Exponential

Nonlinear $(g)\ \sin x_1+2x_2-3x_3=0$

trigonometric functions

$(h)\ \frac{1}{x}+\frac{1}{y}=4$ Nonlinear

not the first power

5.1 Introduction to Systems of Linear Equations

- a solution of a linear equation in *n* variables:

$$a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = b$$

$$x_1 = s_1,\ x_2 = s_2,\ x_3 = s_3,\ \cdots,\ x_n = s_n$$

such that $$a_1s_1 + a_2s_2 + a_3s_3 + \cdots + a_ns_n = b$$

- Solution set:

the set of <u>all solutions</u> of a linear equation

5.1 Introduction to Systems of Linear Equations

- Ex： (Parametric representation of a solution set)

$$x_1 + 2x_2 = 4$$

a solution: $(x_1, x_2) = (2, 1)$,

If you solve for x_1 in terms of x_2, you obtain

$$x_1 = 4 - 2x_2,$$

By letting $x_2 = t$ you can represent the solution set as

$$x_1 = 4 - 2t$$

And the solutions are $\{(4-2t, t) \mid t \in R\}$ or $\{(s, 2-\frac{1}{2}s) \mid s \in R\}$

5.1 Introduction to Systems of Linear Equations

- a system of m linear equations in n variables:

$$\begin{array}{ccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & + & \cdot & + & a_{1n}x_n & = & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & + & \cdot & + & a_{2n}x_n & = & b_2 \\
a_{31}x_1 & + & a_{32}x_2 & + & a_{33}x_3 & + & \cdot & + & a_{3n}x_n & = & b_3 \\
 & & & \vdots & & & & & & & \\
a_{m1}x_1 & + & a_{m2}x_2 & + & a_{m3}x_3 & + & \cdot & + & a_{mn}x_n & = & b_m
\end{array}$$

- Consistent:

A system of linear equations has at least one solution.

- Inconsistent:

A system of linear equations has no solution.

5.1 Introduction to Systems of Linear Equations

- Notes:

Every system of linear equations has either

(1) exactly one solution,

(2) infinitely many solutions, or

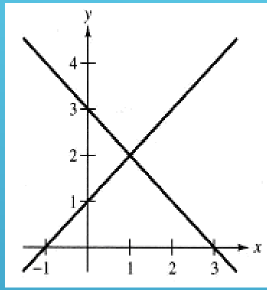
(3) no solution.

5.1 Introduction to Systems of Linear Equations

- Ex : (Solution of a system of linear equations)

(1)
$$\begin{aligned} x + y &= 3 \\ x - y &= -1 \end{aligned}$$

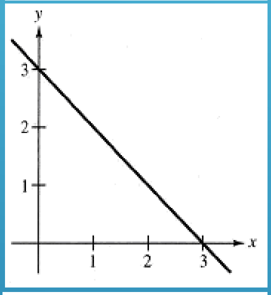
two intersecting lines



exactly one solution

(2)
$$\begin{aligned} x + y &= 3 \\ 2x + 2y &= 6 \end{aligned}$$

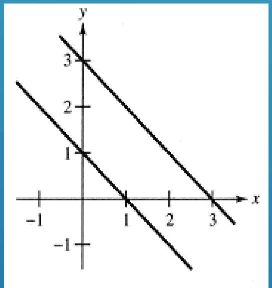
two coincident lines



inifinite number

(3)
$$\begin{aligned} x + y &= 3 \\ x + y &= 1 \end{aligned}$$

two parallel lines



no solution

5.1 Introduction to Systems of Linear Equations

- Ex : (Using back substitution to solve a system in row echelon form)

$$\begin{aligned} x - 2y &= 5 \qquad (1) \\ y &= -2 \qquad (2) \end{aligned}$$

Sol: By substituting $y = -2$ into (1), you obtain

$$\begin{aligned} x - 2(-2) &= 5 \\ x &= 1 \end{aligned}$$

The system has exactly one solution: $x = 1,\ y = -2$

5.1 Introduction to Systems of Linear Equations

- Ex : (Using back substitution to solve a system in row echelon form)

$$\begin{aligned} x - 2y + 3z &= 9 \quad (1) \\ y + 3z &= 5 \quad (2) \\ z &= 2 \quad (3) \end{aligned}$$

Sol: Substitute $z = 2$ into (2)

$$\begin{aligned} y + 3(2) &= 5 \\ y &= -1 \end{aligned}$$

and substitute $y = -1$ and $z = 2$ into (1)

$$\begin{aligned} x - 2(-1) + 3(2) &= 9 \\ x &= 1 \end{aligned}$$

The system has exactly one solution:

$x = 1, y = -1, z = 2$

5.1 Introduction to Systems of Linear Equations

- Equivalent:

  Two systems of linear equations are called **equivalent** if they have precisely the same solution set.

- Notes:

  Each of the following operations on a system of linear equations produces an equivalent system.

  (1) Interchange two equations.

  (2) Multiply an equation by a nonzero constant.

  (3) Add a multiple of an equation to another equation.

5.1 Introduction to Systems of Linear Equations

- Ex : Solve a system of linear equations (consistent system)

$$\begin{aligned} x - 2y + 3z &= 9 \quad (1) \\ -x + 3y &= -4 \quad (2) \\ 2x - 5y + 5z &= 17 \quad (3) \end{aligned}$$

Sol: $(1)+(2) \rightarrow (2)$

$$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \quad (4) \\ 2x - 5y + 5z &= 17 \end{aligned}$$

$(1)\times(-2)+(3) \rightarrow (3)$

$$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ -y - z &= -1 \quad (5) \end{aligned}$$

5.1 Introduction to Systems of Linear Equations

$(4) + (5) \rightarrow (5)$

$$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ 2z &= 4 \qquad (6) \end{aligned}$$

$(6) \times \frac{1}{2} \rightarrow (6)$

$$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ z &= 2 \end{aligned}$$

So the solution is $x = 1$, $y = -1$, $z = 2$ (only one solution)

5.1 Introduction to Systems of Linear Equations

- Ex : Solve a system of linear equations (inconsistent system)

$$\begin{aligned} x_1 - 3x_2 + x_3 &= 1 \quad (1) \\ 2x_1 - x_2 - 2x_3 &= 2 \quad (2) \\ x_1 + 2x_2 - 3x_3 &= -1 \quad (3) \end{aligned}$$

Sol: $(1)\times(-2)+(2)\rightarrow(2)$

$(1)\times(-1)+(3)\rightarrow(3)$

$$\begin{aligned} x_1 - 3x_2 + x_3 &= 1 \\ 5x_2 - 4x_3 &= 0 \quad (4) \\ 5x_2 - 4x_3 &= -2 \quad (5) \end{aligned}$$

5.1 Introduction to Systems of Linear Equations

$(4)\times(-1)+(5)\rightarrow(5)$

$$\begin{aligned} x_1 - 3x_2 + x_3 &= 1 \\ 5x_2 - 4x_3 &= 0 \\ 0 &= -2 \end{aligned}$$

(a false statement)

So the system has no solution (an inconsistent system).

5.2 Gaussian Elimination and Gauss-Jordan Elimination

- $m \times n$ matrix:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdot & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdot & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdot & a_{3n} \\ & \vdots & & & \\ a_{m1} & a_{m2} & a_{m3} & \cdot & a_{mn} \end{bmatrix} \quad m \text{ rows}$$

n columns

- Notes:

(1) Every entry a_{ij} in a matrix is a number.

(2) A matrix with m rows and n columns is said to be of size $m \times n$.

(3) If $m = n$, then the matrix is called square of order n.

(4) For a square matrix, the entries $a_{11}, a_{22}, \ldots, a_{nn}$ are called the main diagonal entries.

5.2 Gaussian Elimination and Gauss-Jordan Elimination

- Ex :

Matrix	Size
$[2]$	1×1
$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$	2×2
$\begin{bmatrix} 1 & -3 & 0 & \frac{1}{2} \end{bmatrix}$	1×4
$\begin{bmatrix} e & \pi \\ 2 & \sqrt{2} \\ -7 & 4 \end{bmatrix}$	3×2

- Note:

One very common use of matrices is to represent a system of linear equations.

- a system of *m* equations in *n* variables:

$$\begin{array}{ccccccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & + & \dots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & + & \dots & + & a_{2n}x_n & = & b_2 \\ a_{31}x_1 & + & a_{32}x_2 & + & a_{33}x_3 & + & \dots & + & a_{3n}x_n & = & b_3 \\ & & & & \vdots & & & & & & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & a_{m3}x_3 & + & \dots & + & a_{mn}x_n & = & b_m \end{array}$$

Matrix form: $Ax = b$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdot & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdot & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdot & a_{3n} \\ & \vdots & & & \\ a_{m1} & a_{m2} & a_{m3} & \cdot & a_{mn} \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

5.2 Gaussian Elimination and Gauss-Jordan Elimination

- Augmented matrix:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdot & a_{1n} & | & b_1 \\ a_{21} & a_{22} & a_{23} & \cdot & a_{2n} & | & b_2 \\ a_{31} & a_{32} & a_{33} & \cdot & a_{3n} & | & b_3 \\ & \vdots & & & & | & \\ a_{m1} & a_{m2} & a_{m3} & \cdot & a_{mn} & | & b_m \end{bmatrix} = [A \mid b]$$

- Coefficient matrix:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdot & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdot & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdot & a_{3n} \\ & \vdots & & & \\ a_{m1} & a_{m2} & a_{m3} & \cdot & a_{mn} \end{bmatrix} = A$$

- Elementary row operation:

(1) Interchange two rows. $r_{ij} : R_i \leftrightarrow R_j$

(2) Multiply a row by a nonzero constant. $r_i^{(k)} : (k)R_i \rightarrow R_i$

(3) Add a multiple of a row to another row. $r_{ij}^{(k)} : (k)R_i + R_j \rightarrow R_j$

- Row equivalent:

Two matrices are said to be **row equivalent** if one can be obtained from the other by a finite sequence of **elementary row operation**.

- Ex : (Elementary row operation)

$$\begin{bmatrix} 0 & 1 & 3 & 4 \\ -1 & 2 & 0 & 3 \\ 2 & -3 & 4 & 1 \end{bmatrix} \xrightarrow{r_{12}} \begin{bmatrix} -1 & 2 & 0 & 3 \\ 0 & 1 & 3 & 4 \\ 2 & -3 & 4 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -4 & 6 & -2 \\ 1 & 3 & -3 & 0 \\ 5 & -2 & 1 & 2 \end{bmatrix} \xrightarrow{r_1^{(\frac{1}{2})}} \begin{bmatrix} 1 & -2 & 3 & -1 \\ 1 & 3 & -3 & 0 \\ 5 & -2 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -4 & 3 \\ 0 & 3 & -2 & -1 \\ 2 & 1 & 5 & -2 \end{bmatrix} \xrightarrow{r_{13}^{(-2)}} \begin{bmatrix} 1 & 2 & -4 & 3 \\ 0 & 3 & -2 & -1 \\ 0 & -3 & 13 & -8 \end{bmatrix}$$

5.2 Gaussian Elimination and Gauss-Jordan Elimination

- Row-echelon form: (1, 2, 3)
- Reduced row-echelon form: (1, 2, 3, 4)

(1) All row consisting entirely of zeros occur at the bottom of the matrix.

(2) For each row that does not consist entirely of zeros, the first nonzero entry is 1 (called **a leading 1**).

(3) For two successive (nonzero) rows, <u>the leading 1 in the higher row</u> is farther to the left than <u>the leading 1 in the lower row</u>.

(4) Every column that has a leading 1 has zeros in every position above and below its leading 1.

5.2 Gaussian Elimination and Gauss-Jordan Elimination

- Ex : (Row-echelon form or reduced row-echelon form)

$$\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$ (row-echelon form)

$$\begin{bmatrix} 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ (reduced row-echelon form)

$$\begin{bmatrix} 1 & -5 & 2 & -1 & 3 \\ 0 & 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$ (row-echelon form)

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ (reduced row-echelon form)

$$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 1 & -3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & -4 \end{bmatrix}$$

5.2 Gaussian Elimination and Gauss-Jordan Elimination

- Gaussian elimination:

  The procedure for reducing a matrix to a row-echelon form.

- Gauss-Jordan elimination:

  The procedure for reducing a matrix to a reduced row-echelon form.

- Notes:

  (1) Every matrix has an unique reduced row echelon form.

  (2) A row-echelon form of a given matrix is not unique.

  (Different sequences of row operations can produce different row-echelon forms.)

5.2 Gaussian Elimination and Gauss-Jordan Elimination

- Ex: (Procedure of Gaussian elimination and Gauss-Jordan elimination)

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 8 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & 4 \end{bmatrix} \xrightarrow{r_{12}} \begin{bmatrix} 2 & 4 & -10 & 6 & 12 & 28 \\ 0 & 0 & -2 & 0 & 8 & 12 \\ 2 & 4 & -5 & 6 & -5 & 4 \end{bmatrix}$$

Produce leading 1

The first nonzero column

$$\xrightarrow{r_1^{(\frac{1}{2})}} \begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 8 & 12 \\ 2 & 4 & -5 & 6 & -5 & 4 \end{bmatrix} \xrightarrow{r_{13}^{(-2)}} \begin{bmatrix} 1 & 4 & -3 & 2 & 6 & 14 \\ 0 & 0 & -2 & 0 & 8 & 12 \\ 0 & 0 & 5 & 0 & -17 & -24 \end{bmatrix}$$

leading 1

Zeros elements below leading 1

Produce leading 1

The first nonzero column

Submatrix

5.2 Gaussian Elimination and Gauss-Jordan Elimination

$$\xrightarrow{r_2^{(-\frac{1}{2})}} \begin{bmatrix} 1 & 4 & -3 & 2 & 6 & 14 \\ 0 & 0 & 1 & 0 & -4 & -6 \\ 0 & 0 & 5 & 0 & -17 & -24 \end{bmatrix} \qquad \xrightarrow{r_{23}^{(-5)}} \begin{bmatrix} 1 & 4 & -3 & 2 & 6 & 14 \\ 0 & 0 & 1 & 0 & -4 & -6 \\ 0 & 0 & 0 & 0 & 3 & 6 \end{bmatrix}$$

leading 1

Zeros elements below leading 1

Submatrix

Produce leading 1

$$\xrightarrow{r_3^{(\frac{1}{3})}} \begin{bmatrix} 1 & 4 & -3 & 2 & 6 & 14 \\ 0 & 0 & 1 & 0 & -4 & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \qquad \xrightarrow{r_{31}^{(-6)}} \begin{bmatrix} 1 & 4 & -3 & 2 & 0 & 2 \\ 0 & 0 & 1 & 0 & -4 & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

Zeros elsewhere

leading 1

(row-echelon form) (row-echelon form)

$$\xrightarrow{r_{32}^{(4)}} \begin{bmatrix} 1 & 4 & -3 & 2 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \qquad \xrightarrow{r_{21}^{(3)}} \begin{bmatrix} 1 & 4 & 0 & 2 & 0 & 8 \\ 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

(row-echelon form) (reduced row-echelon form)

5.2 Gaussian Elimination and Gauss-Jordan Elimination

- Ex : Solve a system by Gauss-Jordan elimination method (only one solution)

$$\begin{array}{rcrcrcr} x & - & 2y & + & 3z & = & 9 \\ -x & + & 3y & & & = & -4 \\ 2x & - & 5y & + & 5z & = & 17 \end{array}$$

Sol:

augmented matrix

$$\begin{bmatrix} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{bmatrix} \xrightarrow{r_{12}^{(1)},\, r_{13}^{(-2)}} \begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & -1 & -1 & -1 \end{bmatrix} \xrightarrow{r_{23}^{(1)}} \begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

$$\xrightarrow{r_3^{(\frac{1}{2})}} \begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{r_{21}^{(2)},\, r_{32}^{(-3)},\, r_{31}^{(-9)}} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \longrightarrow \begin{array}{rcr} x & = & 1 \\ y & = & -1 \\ z & = & 2 \end{array}$$

(row-echelon form) (reduced row-echelon form)

- Ex：Solve a system by Gauss-Jordan elimination method (infinitely many solutions)

$$\begin{aligned} 2x_1 + 4x_2 - 2x_3 &= 0 \\ 3x_1 + 5x_2 \quad\quad &= 1 \end{aligned}$$

Sol: augmented matrix

$$\begin{bmatrix} 2 & 4 & -2 & 0 \\ 3 & 5 & 0 & 1 \end{bmatrix} \xrightarrow{r_1^{(\frac{1}{2})},\, r_{12}^{(-3)},\, r_2^{(-1)},\, r_{21}^{(-2)}} \begin{bmatrix} 1 & 0 & 5 & 2 \\ 0 & 1 & -3 & -1 \end{bmatrix}$$ (reduced row - echelon form)

the corresponding system of equations is

$$\begin{aligned} x_1 \quad + 5x_3 &= 2 \\ x_2 - 3x_3 &= -1 \end{aligned}$$

leading variable : x_1, x_2

free variable : x_3

$$\begin{aligned} x_1 &= 2 - 5x_3 \\ x_2 &= -1 + 3x_3 \end{aligned}$$

Let $x_3 = t$

$$\begin{aligned} x_1 &= 2 - 5t, \\ x_2 &= -1 + 3t, \qquad t \in R \\ x_3 &= t, \end{aligned}$$

So this system has infinitely many solutions.