CHAPTER 4
DETERMINANTS

4.1 Determinant of a Matrix

4.2 Evaluation of a Determinant using Elementary Operations

4.3 Properties of Determinants

4.4 Application of Determinants

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4.1 Determinant of a matrix

- the determinant of a 2 × 2 matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\Rightarrow \det(A) = |A| = a_{11}a_{22} - a_{21}a_{12}$$

- Note:

$$\left|\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

4.1 Determinant of a matrix

- Ex : (The determinant of a matrix of order 2)

$$\begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix} = 2(2) - 1(-3) \quad = 4 + 3 \quad = 7$$

$$\begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} = 2(2) - 4(1) \quad = 4 - 4 \quad = 0$$

$$\begin{vmatrix} 0 & 3 \\ 2 & 4 \end{vmatrix} = 0(4) - 2(3) \quad = 0 - 6 \quad = -6$$

- Note: The determinant of a matrix can be positive, zero, or negative.

4.1 Determinant of a matrix

- Minor of the entry a_{ij} :

The determinant of the matrix determined by deleting the *i*th row and *j*th column of A

$$M_{ij} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1(j-1)} & a_{1(j+1)} & \cdots & a_{1n} \\ \vdots & & & \vdots & \vdots & & \\ a_{(i-1)1} & & \cdots & a_{(i-1)(j-1)} & a_{(i-1)(j+1)} & \cdots & a_{(i-1)n} \\ a_{(i+1)1} & & \cdots & a_{(i+1)(j-1)} & a_{(i+1)(j+1)} & \cdots & a_{(i+1)n} \\ \vdots & & & \vdots & \vdots & & \vdots \\ a_{n1} & & \cdots & a_{n(j-1)} & a_{n(j+1)} & \cdots & a_{nn} \end{vmatrix}$$

- Cofactor of a_{ij} :

$$C_{ij} = (-1)^{i+j} M_{ij}$$

4.1 Determinant of a matrix

- Ex:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\Rightarrow M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} \qquad M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$$

$$\Rightarrow C_{21} = (-1)^{2+1} M_{21} = -M_{21} \qquad C_{22} = (-1)^{2+2} M_{22} = M_{22}$$

4.1 Determinant of a matrix

- Notes: Sign pattern for cofactors

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix} \qquad \begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix} \qquad \begin{bmatrix} + & - & + & - & + & \square \\ - & + & - & + & - & \\ + & - & + & - & & \\ - & + & - & & & \\ + & - & & & & \\ & & & & & \end{bmatrix}$$

3 × 3 matrix   4 × 4 matrix $n \times n$ matrix

- Notes:

Odd positions (where $i+j$ is odd) have negative signs, and even positions (where $i+j$ is even) have positive signs.

4.1 Determinant of a matrix

- Ex: The determinant of a matrix of order 3

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{aligned}
\Rightarrow \det(A) &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\
&= a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} \\
&= a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33} \\
&= a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31} \\
&= a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32} \\
&= a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33}
\end{aligned}$$

4.1 Determinant of a matrix

- Notes:

The row (or column) containing the most zeros is the best choice for expansion by cofactors .

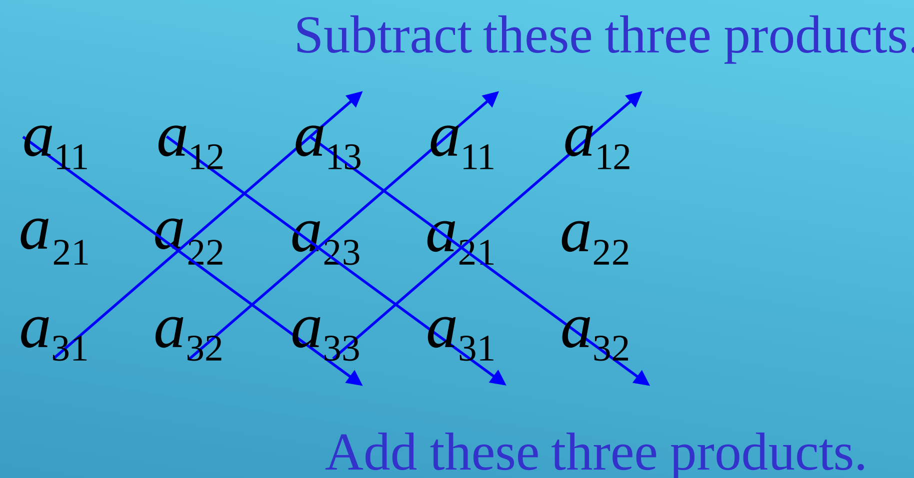
- Ex : (The determinant of a matrix of order 4)

$$A=\begin{bmatrix} 1 & -2 & 3 & 0 \\ -1 & 1 & 0 & 2 \\ 0 & 2 & 0 & 3 \\ 3 & 4 & 0 & -2 \end{bmatrix} \Rightarrow \det(A)=?$$

4.1 Determinant of a matrix

- The determinant of a matrix of order 3:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$



$$\Rightarrow \det(A) = |A| = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}$$

4.1 Determinant of a matrix

- Upper triangular matrix:

All the entries below the main diagonal are zeros.

- Lower triangular matrix:

All the entries above the main diagonal are zeros.

- Diagonal matrix:

All the entries above and below the main diagonal are zeros.

- Note:

A matrix that is both upper and lower triangular is called diagonal.

4.1 Determinant of a matrix

- Ex:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \qquad \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$$

upper triangular | lower triangular | diagonal

4.1 Determinant of a matrix

- Thm : (Determinant of a Triangular Matrix)

If A is an $n \times n$ triangular matrix (upper triangular, lower triangular, or diagonal), then its determinant is the product of the entries on the main diagonal. That is

$$\det(A) = |A| = a_{11}a_{22}a_{33} \ldots a_{nn}$$

4.1 Determinant of a matrix

- Ex : Find the determinants of the following triangular matrices.

$$(a)\quad A=\begin{bmatrix} 2 & 0 & 0 & 0 \\ 4 & -2 & 0 & 0 \\ -5 & 6 & 1 & 0 \\ 1 & 5 & 3 & 3 \end{bmatrix} \qquad (b)\quad \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix}$$

Sol:

$$(a)\quad |A| = (2)(-2)(1)(3) = -12$$

$$(b)\quad |B| = (-1)(3)(2)(4)(-2) = 48$$

- Thm : (Elementary row operations and determinants)

Let A and B be square matrices.

$$(a)\quad B = r_{ij}(A) \quad \Rightarrow \quad \det(B) = -\det(A) \qquad (\text{i.e.} \left|r_{ij}(A)\right| = -\left|A\right|)$$

$$(b)\quad B = r_i^{(k)}(A) \quad \Rightarrow \quad \det(B) = k\det(A) \qquad (\text{i.e.} \left|r_i^{(k)}(A)\right| = k\left|A\right|)$$

$$(c)\quad B = r_{ij}^{(k)}(A) \quad \Rightarrow \quad \det(B) = \det(A) \qquad (\text{i.e.} \left|r_{ij}^{(k)}(A)\right| = \left|A\right|)$$

- Ex:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 1 & 2 & 1 \end{bmatrix} \Rightarrow \det(A) = -2$$

$$A_1 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 1 & 2 & 1 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} A_3 = \begin{bmatrix} 1 & 2 & 3 \\ -2 & -3 & -2 \\ 1 & 2 & 1 \end{bmatrix}$$

$$A_1 = r_1^{(4)}(A) \quad \Rightarrow \det(A_1) = \det(r_1^{(4)}(A)) = 4\det(A) = (4)(-2) = -8$$

$$A_2 = r_{12}(A) \quad \Rightarrow \det(A_2) = \det(r_{12}(A)) = -\det(A) = -(-2) = 2$$

$$A_3 = r_{12}^{(-2)}(A) \quad \Rightarrow \det(A_3) = \det(r_{12}^{(-2)}(A)) = \det(A) = -2$$

- Notes:

$$\det(r_{ij}(A)) = -\det(A) \quad \Rightarrow \quad \det(A) = -\det(r_{ij}(A))$$

$$\det(r_i^{(k)}(A)) = k\det(A) \quad \Rightarrow \quad \det(A) = 1/k\det(r_i^{(k)}(A))$$

$$\det(r_{ij}^{(k)}(A)) = \det(A) \quad \Rightarrow \quad \det(A) = \det(r_{ij}^{(k)}(A))$$

Note:

A row-echelon form of a square matrix is always upper triangular.

- Ex : (Evaluation a determinant using elementary row operations)

$$A = \begin{bmatrix} 2 & -3 & 10 \\ 1 & 2 & -2 \\ 0 & 1 & -3 \end{bmatrix} \Rightarrow \det(A) = ?$$

Sol:

$$\det(A) = -7$$

- Notes:

$$|EA| = |E||A|$$

(1) $E = R_{ij}$ $\Rightarrow |E| = \left|R_{ij}\right| = -1$

$$\Rightarrow |EA| = \left|r_{ij}(A)\right| = -|A| = \left|R_{ij}\right||A| = |E||A|$$

(2) $E = R_i^{(k)}$ $\Rightarrow |E| = \left|R_i^{(k)}\right| = k$

$$\Rightarrow |EA| = \left|r_i^{(k)}(A)\right| = k|A| = \left|R_i^{(k)}\right||A| = |E||A|$$

(3) $E = R_{ij}^{(k)}$ $\Rightarrow |E| = \left|R_{ij}^{(k)}\right| = 1$

$$\Rightarrow |EA| = \left|r_{ij}^{(k)}(A)\right| = 1|A| = \left|R_{ij}^{(k)}\right||A| = |E||A|$$

- Determinants and elementary column operations

- Thm: (Elementary column operations and determinants)

Let A and B be square matrices.

$$(a)\quad B = c_{ij}(A) \quad \Rightarrow \quad \det(B) = -\det(A) \qquad (\text{i.e. } |c_{ij}(A)| = -|A|)$$

$$(b)\quad B = c_i^{(k)}(A) \quad \Rightarrow \quad \det(B) = k\det(A) \qquad (\text{i.e. } |c_i^{(k)}(A)| = k|A|)$$

$$(c)\quad B = c_{ij}^{(k)}(A) \quad \Rightarrow \quad \det(B) = \det(A) \qquad (\text{i.e. } |c_{ij}^{(k)}(A)| = |A|)$$

- Thm : (Conditions that yield a zero determinant)

If A is a square matrix and any of the following conditions is true, then det (A) = 0.

(a) An entire row (or an entire column) consists of zeros.

(b) Two rows (or two columns) are equal.

(c) One row (or column) is a multiple of another row (or column).

- Note:

Order n	Cofactor Expansion		Row Reduction	
	Additions	Multiplications	Additions	Multiplications
3	5	9	5	10
5	119	205	30	45
10	3,628,799	6,235,300	285	339

4.3 Properties of determinants

- Thm : (Determinant of a matrix product)

$$\det(AB) = \det(A)\det(B)$$

- Notes:

(1) $\det(EA) = \det(E)\det(A)$

(2) $\det(A+B) \neq \det(A) + \det(B)$

(3) $$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{22}+b_{22} & a_{22}+b_{22} & a_{23}+b_{23} \\ A_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ b_{21} & b_{22} & b_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

4.3 Properties of determinants

- Ex : (The determinant of a matrix product)

$$A = \begin{bmatrix} 1 & -2 & 2 \\ 0 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 0 & 1 \\ 0 & -1 & -2 \\ 3 & 1 & -2 \end{bmatrix}$$

Find $|A|$, $|B|$, and $|AB|$

Sol:

$$|A| = \begin{vmatrix} 1 & -2 & 2 \\ 0 & 3 & 2 \\ 1 & 0 & 1 \end{vmatrix} = -7 \qquad |B| = \begin{vmatrix} 2 & 0 & 1 \\ 0 & -1 & -2 \\ 3 & 1 & -2 \end{vmatrix} = 11$$

4.3 Properties of determinants

$$\Rightarrow |AB| = -77$$

- Check:

$$|AB| = |A|\,|B|$$

- Thm : (Determinant of a scalar multiple of a matrix)

If A is an $n \times n$ matrix and c is a scalar, then $\det(cA) = c^n \det(A)$

4.3 Properties of determinants

- Thm : (Determinant of an invertible matrix)

  A square matrix A is invertible (nonsingular) if and only if

  $$\det(A) \neq 0$$

- Thm : (Determinant of an inverse matrix)

  $$\text{If } A \text{ is invertible, then } \det(A^{-1}) = \frac{1}{\det(A)}.$$

- Thm : (Determinant of a transpose)

  $$\text{If } A \text{ is a square matrix, then } \det(A^{T}) = \det(A).$$

4.3 Properties of determinants

- Equivalent conditions for a nonsingular matrix:

If A is an $n \times n$ matrix, then the following statements are equivalent.

(1) A is invertible.

(2) $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $n \times 1$ matrix $\mathbf{b}$.

(3) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(4) A is row-equivalent to I_n

(5) A can be written as the product of elementary matrices.

(6) $\det(A) \neq 0$

4.3 Properties of determinants

- Ex : Which of the following system has a unique solution?

$$(a)\quad \begin{aligned} 2x_2 - x_3 &= -1 \\ 3x_1 - 2x_2 + x_3 &= 4 \\ 3x_1 + 2x_2 - x_3 &= -4 \end{aligned}$$

$$(b)\quad \begin{aligned} 2x_2 - x_3 &= -1 \\ 3x_1 - 2x_2 + x_3 &= 4 \\ 3x_1 + 2x_2 + x_3 &= -4 \end{aligned}$$

4.3 Properties of determinants

Sol:

(a) $A\mathbf{x} = b$

 $|A| = 0$

This system does not have a unique solution.

(b) $B\mathbf{x} = b$

 $|B| = -12 \neq 0$

This system has a unique solution.

4.4 Applications of Determinants

- Matrix of cofactors of *A*:

$$[C_{ij}] = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & & \vdots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix} \qquad C_{ij} = (-1)^{i+j} M_{ij}$$

- Adjoint matrix of *A*:

$$adj(A) = [C_{ij}]^T = \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

4.4 Applications of Determinants

- Thm : (The inverse of a matrix given by its adjoint)

  If A is an $n \times n$ invertible matrix, then

$$A^{-1} = \frac{1}{\det(A)} adj(A)$$

- Ex:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\Rightarrow \det(A) = ad - bc$$

$$adj(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{\det(A)} adj(A)$$

$$= \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

4.4 Applications of Determinants

- Thm : (Cramer's Rule)

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$

$$\vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$$

$$A\mathbf{x} = \mathbf{b} \qquad A = \left[a_{ij}\right]_{n\times n} = \left[A^{(1)}, A^{(2)}, \cdots, A^{(n)}\right] \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \neq 0$$

(this system has a unique solution)

4.4 Applications of Determinants

$$A_J = \left[A^{(1)}, A^{(2)}, \cdots, A^{(j-1)}, b, A^{(j+1)}, \cdots, A^{(n)}\right]$$

$$= \begin{bmatrix} a_{11} & \cdots & a_{1(j-1)} & b_1 & a_{1(j+1)} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2(j-1)} & b_2 & a_{2(j+1)} & \cdots & a_{2n} \\ \vdots & & & \ddots & & & \vdots \\ a_{n1} & \cdots & a_{n(j-1)} & b_n & a_{n(j+1)} & \cdots & a_{nn} \end{bmatrix}$$

$$(\text{ i.e. } \det(A_j) = b_1 C_{1j} + b_2 C_{2j} + \ldots + b_n C_{nj})$$

$$\Rightarrow x_j = \frac{\det(A_j)}{\det(A)}, \qquad j = 1, 2, \ldots, n$$

4.4 Applications of Determinants

- Ex : Use Cramer's rule to solve the system of linear equations.

$$\begin{aligned} -x + 2y - 3z &= 1 \\ 2x + z &= 0 \\ 3x - 4y + 4z &= 2 \end{aligned}$$

Sol :

$$x = \frac{4}{5},\ y = \frac{-3}{2},\ z = \frac{-8}{5}$$