CHAPTER 3
BASICS OF MATRICES CONCEPT

3.1 Operations with Matrices

3.2 Properties of Matrix Operations

3.3 The Inverse of a Matrix

3.4 Elementary Matrices

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3.1 Operations with Matrices

- Matrix:

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m\times n} \in M_{m\times n}$$

(*i*, *j*)-th entry: a_{ij}

row: m

column: n

size: $m \times n$

3.1 Operations with Matrices

- *i*-th row vector

$$R_i = \begin{bmatrix} a_{i1} & a_{i2} & \dots & a_{in} \end{bmatrix}$$

row matrix

- *j*-th column vector

$$c_j = \begin{bmatrix} c_{1j} \\ c_{2j} \\ \vdots \\ c_{mj} \end{bmatrix}$$

column matrix

- Square matrix: $m = n$

3.1 Operations with Matrices

- Diagonal matrix:

$$A = diag(d_1, d_2, \dots, d_n) = \begin{bmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \square & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \square & & d_n \end{bmatrix} \in M_{n\times n}$$

- Trace:

If $A = [a_{ij}]_{n\times n}$

Then $Tr(A) = a_{11} + a_{22} + \dots + a_{nn}$

3.1 Operations with Matrices

- Ex:

  - $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$

    $\Rightarrow \quad r_1 = [1 \quad 2 \quad 3], \; r_2 = [4 \quad 5 \quad 6]$

  - $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = [c_1 \quad c_2 \quad c_3]$

    $\Rightarrow \quad c_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \quad c_2 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \quad c_3 = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$

3.1 Operations with Matrices

- Equal matrix:

If $A = [a_{ij}]_{m \times n}$, $B = [b_{ij}]_{m \times n}$

Then $A = B$ if and only if $a_{ij} = b_{ij} \ \forall 1 \le i \le m, \ 1 \le j \le n$

- Ex :

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

If $A = B$

Then $a = 1, \ b = 2, \ c = 3, \ d = 4$

3.1 Operations with Matrices

- Matrix addition:

If $A=[a_{ij}]_{m\times n},\ \ B=[b_{ij}]_{m\times n}$

Then $A+B=[a_{ij}]_{m\times n}+[b_{ij}]_{m\times n}=[a_{ij}+b_{ij}]_{m\times n}$

- Ex :

$$\begin{bmatrix}-1 & 2\\ 3 & 5\end{bmatrix}+\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}=\begin{bmatrix}-1+1 & 2+2\\ 3+3 & 5+4\end{bmatrix}=\begin{bmatrix}0 & 4\\ 6 & 9\end{bmatrix}$$

$$\begin{bmatrix}-1\\ 3\\ 2\end{bmatrix}+\begin{bmatrix}1\\ -3\\ -2\end{bmatrix}=\begin{bmatrix}-1+1\\ 3-3\\ 2-2\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

3.1 Operations with Matrices

- Scalar multiplication:

  If $A = [a_{ij}]_{m\times n}$, c : scalar Then $cA = [ca_{ij}]_{m\times n}$

- Matrix subtraction:

  $$A - B = A + (-1)B$$

- Ex :

  $$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & 3 & 0 \\ 1 & 6 & 5 \end{bmatrix}$$

  Find (*a*) $3A$, (*b*) $-B$, (*c*) $3A - B$

3.1 Operations with Matrices

- Matrix multiplication:

IF $A=[a_{ij}]_{m\times n}, B=[b_{ij}]_{n\times p}$

Then $AB=[a_{ij}]_{m\times n}[b_{ij}]_{n\times p}=[c_{ij}]_{m\times p}$

Size of AB

where $c_{ij}=\sum_{k=1}^{n}a_{ik}b_{kj}=a_{i1}b_{1j}+a_{i2}b_{2j}+\ldots+a_{in}b_{nj}$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} + \begin{bmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1n} \\ b_{21} & \vdots & b_{2j} & \cdots & b_{2n} \\ \vdots & \vdots & \vdots & \vdots & \\ b_{n1} & \cdots & b_{nj} & \cdots & b_{nn} \end{bmatrix} = \begin{bmatrix} & & & & & \\ c_{i1} & c_{i2} & \ldots & c_{ij} & \ldots & c_{in} \\ & & & & & \end{bmatrix}$$

- Notes: (1) $A+B=B+A$, (2) $AB\neq BA$

3.1 Operations with Matrices

- Ex : (Find AB)

$$A=\begin{bmatrix}-1 & 3\\ 4 & -2\\ 5 & 0\end{bmatrix} \qquad B=\begin{bmatrix}-3 & 2\\ -4 & 1\end{bmatrix}$$

Sol:

$$AB=\begin{bmatrix}(-1)(-3)+(3)(-4) & (-1)(2)+(3)(1)\\ (4)(-3)+(-2)(-4) & (4)(2)+(-2)(1)\\ (5)(-3)+(0)(-4) & (5)(2)+(0)(1)\end{bmatrix}$$

$$=\begin{bmatrix}-9 & 1\\ -4 & 6\\ -15 & 10\end{bmatrix}$$

3.1 Operations with Matrices

- Matrix form of a system of linear equations:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m \end{cases}$$

m linear equations

$$\Downarrow$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$\underset{A}{\parallel} \qquad \underset{x}{\parallel} \qquad \underset{b}{\parallel}$$

Single matrix equation

$$\underset{m\times n}{A}\,\underset{n\times 1}{x} = \underset{m\times 1}{b}$$

3.1 Operations with Matrices

- Partitioned matrices:

Submatrix

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}=\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}=\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}$$

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}=\begin{bmatrix} c_1 & c_2 & c_3 & c_4 \end{bmatrix}$$

3.1 Operations with Matrices

- Three basic matrix operators:

  (1) matrix addition

  (2) scalar multiplication

  (3) matrix multiplication

- Zero matrix: $0_{m\times n}$

- Identity matrix of order *n*: I_n

3.2 Properties of Matrix Operations

- Properties of matrix addition and scalar multiplication:

If $A, B, C \in M_{m\times n}$, c, d : scalar Then

(1) $A+B = B + A$

(2) $A + (B + C) = (A + B) + C$

(3) $(cd)A = c(dA)$

(4) $1A = A$

(5) $c(A+B) = cA + cB$

(6) $(c+d)A = cA + dA$

3.2 Properties of Matrix Operations

- Properties of zero matrices:

If $A \in M_{m\times n}, c : scalar$ Then

$$(1)\ A + 0_{m\times n} = A$$
$$(2)\ A + (-A) = 0_{m\times n}$$
$$(3)\ cA = 0_{m\times n} \Rightarrow \quad c = 0 \ or\ A = 0_{m\times n}$$

- Notes:

(1) $0_{m\times n}$: **the additive identity** for the set of all $m \times n$ matrices

(2) $-A$: **the additive inverse** of A

3.2 Properties of Matrix Operations

- Transpose of a matrix:

If $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \in M_{m\times n}$

Then $A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix} \in M_{n\times m}$

3.2 Properties of Matrix Operations

- Properties of transposes:

$$(1)\ (A^T)^T = A$$

$$(2)\ (A+B)^T = A^T + B^T$$

$$(3)\ (cA)^T = c(A^T)$$

$$(4)\ (AB)^T = B^T A^T$$

3.2 Properties of Matrix Operations

- Symmetric matrix:

  A square matrix A is **symmetric** if $A = A^T$

- Skew-symmetric matrix:

  A square matrix A is **skew-symmetric** if $A^T = -A$

- Ex:

  If $A = \begin{bmatrix} 1 & 2 & 3 \\ a & 4 & 5 \\ b & c & 6 \end{bmatrix}$ is symmetric, find a, b, c?

3.2 Properties of Matrix Operations

- Ex:

$$\text{If } A = \begin{bmatrix} 0 & 1 & 2 \\ a & 0 & 3 \\ b & c & 0 \end{bmatrix} \text{ is a skew-symmetric, find } a, b, c?$$

- Note: AA^T is symmetric

Pf: $(AA^T)^T = (A^T)^T A^T = AA^T$

$\therefore AA^T$ is symmetric

3.2 Properties of Matrix Operations

- Real number:

  $AB = BA$ (Commutative law for multiplication)

- Matrix:

  $\underset{m\times n}{A}\,\underset{n\times p}{B} \neq BA$

  Three situations:

  (1) If $m \neq p$, then AB is defined, BA is undefined.

  (2) If $m = p, m \neq n$, then $AB \in M_{m\times m}$, $BA \in M_{n\times n}$ (Sizes are not the same)

  (3) If $m = p = n$, then $AB \in M_{m\times m}$, $BA \in M_{m\times m}$

  (Sizes are the same, but matrices are not equal)

3.2 Properties of Matrix Operations

- Real number:

$$ac = bc,\ \ c \neq 0$$

$$\Rightarrow\ a = b \qquad \text{(Cancellation law)}$$

- Matrix:

$$AC = BC \qquad C \neq 0$$

(1) If C is invertible, then $A = B$

(2) If C is not invertible, then $A \neq B$ (Cancellation is not valid)

3.3 The Inverse of a Matrix

- Inverse matrix:

  Consider $A \in M_{n\times n}$

  If there exists a matrix $B \in M_{n\times n}$ such that $AB = BA = I_n$,

  Then (1) A is **invertible** (or **nonsingular**)

  (2) B is **the inverse** of A

- Note:

  A matrix that does not have an inverse is called **noninvertible** (or **singular**).

3.3 The Inverse of a Matrix

- Thm: (The inverse of a matrix is unique)

  If B and C are both inverses of the matrix A, then $B = C$.

  Pf:

  - $AB = I \Rightarrow C(AB) = CI \Rightarrow (CA)B = C$
  - $IB = C$
  - $B = C$
  - Consquently, the inverse of a matrix is unique.

- Notes:

  (1) *The inverse of A is denoted by* A^{-1}

  (2) $AA^{-1} = A^{-1}A = I$

3.3 The Inverse of a Matrix

- Thm ： (Properties of inverse matrices)

If A is an invertible matrix, k is a positive integer, and c is a scalar not equal to zero, then

$$(1)\ A^{-1} \text{ is invertible and } (A^{-1})^{-1} = A$$

$$(2)\ cA \text{ is invertible and } (cA)^{-1} = cA^{-1}, c \neq 0$$

$$(3)\ A^{T} \text{ is invertible and } (A^{T})^{-1} = (A^{-1})^{T}$$

3.3 The Inverse of a Matrix

- Thm : (The inverse of a product)

If A and B are invertible matrices of size n, then AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}$$

Pf:

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = A(I)A^{-1} = (AI)A^{-1} = AA^{-1} = I$$

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}(I)B = B^{-1}(IB) = B^{-1}B = I$$

If AB is invertible, then its inverse is unique. So $(AB)^{-1} = B^{-1}A^{-1}$

3.3 The Inverse of a Matrix

- Thm : (Systems of equations with unique solutions)

If A is an invertible matrix, then the system of linear equations $Ax = b$ has a unique solution given by

$$x = A^{-1}b$$

Pf:

$$Ax = b$$

$$A^{-1}Ax = A^{-1}b \quad (A \text{ is nonsingular})$$

$$Ix = A^{-1}b$$

$$x = A^{-1}b$$

If x_1 and x_2 were two solutions of equation $Ax = b$.

then $Ax_1 = b = Ax_2 \Rightarrow x_1 = x_2$ (Left cancellation property)

This solution is unique.

3.4 Elementary Matrices

- Row elementary matrix:

  An $n \times n$ matrix is called an elementary matrix if it can be obtained from the identity matrix I_n by a single elementary operation.

- Three row elementary matrices:

  (1) $R_{ij} = r_{ij}(I)$ Interchange two rows.

  (2) $R_i^{(k)} = r_i^{(k)}(I)$ $(k \neq 0)$ Multiply a row by a nonzero constant.

  (3) $R_{ij}^{(k)} = r_{ij}^{(k)}(I)$ Add a multiple of a row to another row.

- Note:

  Only do a single elementary row operation.

3.4 Elementary Matrices

- Thm : (Representing elementary row operations)

Let E be the elementary matrix obtained by performing an elementary row operation on I_m. If that same elementary row operation is performed on an $m \times n$ matrix A, then the resulting matrix is given by the product EA.

$$r(I) = E$$
$$r(A) = EA$$