CHAPTER 2
LINEAR APPLICATIONS

2.1 Definition of a Linear Application

2.2 Kernel and Image of a Linear Application

2.3 Rank Theorem

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2.1 Definition of a Linear Application

- Definition:

A linear application $f: E \longrightarrow F$ between two vector spaces E and F is a function that preserves vector addition and scalar multiplication. Specifically, for all $u, v \in E$ and $\alpha \in R$, the following properties are satisfied:

$$f(u+v) = f(u) + f(v) \qquad \text{(Additivity)}$$

$$f(\alpha u) = \alpha f(u) \qquad \text{(Homogeneity)}$$

Examples:

- $f: R^2 \rightarrow R^2$ defined by $f(x,y)=(2x,3y)$, f is linear.
- $g: P_n(R) \rightarrow P_{n-1}(R)$ where g is the differentiation operator $g(p(x))=p'(x)$ g is linear.
- $h: R^2 \rightarrow R^2$ defined by $h(x,y)= x^2 - y^2$, h is not linear.

2.2 Kernel and Image of a Linear Application

- Kernel (Ker):

The kernel of f is the set of vectors in E that are mapped to the zero vector in F.

$$\mathrm{Ker}(f)=\{v \in V \,/\, f(v) = 0\}$$

Examples:

(1) For $f(x,y)=(2x,3y)$, $\mathrm{Ker}(f)=\{(0, 0)\}$.

(2) For the differentiation operator $f(p(x))=p'(x)$, determine the kernel.

2.2 Kernel and Image of a Linear Application

- Image (Im):

The image of f is the set of vectors in F that are obtained as images of vectors from E.

$$\mathrm{Im}(f)=\{f(v) / v \in V\}$$

Examples:

(1) For $f(x,y)=(2x,3y)$, $\mathrm{Im}(f) = R^2$.

(2) For the differentiation operator $f(p(x))=p'(x)$, determine the Image.

2.3 Rank Theorem

- Statement:

For a linear application f :E⟶F, the dimension of E is equal to the sum of the dimension of the kernel of f and the dimension of the image of f:

$$\dim(E) = \dim(\mathrm{Ker}(f)) + \dim(\mathrm{Im}(f))$$

- Interpretation:

This theorem connects the internal structure of E to how f acts on E.

Examples:

Apply the rank theorem to specific cases, such as $f : R^3 \to R^2$.