CHAPTER 1
VECTOR SPACES

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1.1 Vectors in R^n

- An ordered n-tuple:

  a sequence of n real number $(x_1, x_2, \cdots, x_n)$

- n-space: R^n

  the set of all ordered n-tuple

- Ex:

$n = 1$ R^1 = 1-space
= set of all real number

$n = 3$ R^3 = 3-space

= set of all ordered triple of real numbers (x_1, x_2, x_3)

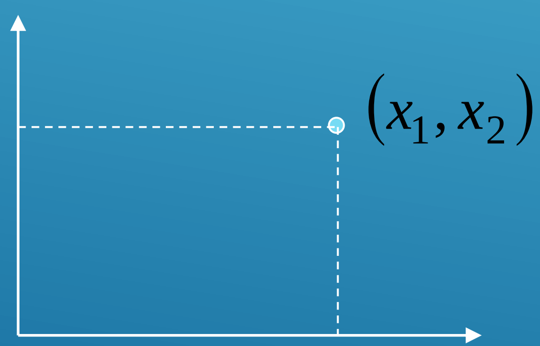
1.1 Vectors in R^n

- Notes:

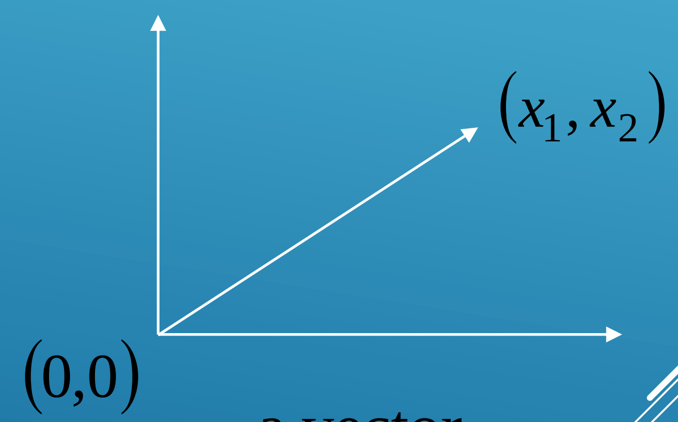
  (1) An n-tuple $(x_1, x_2, \cdots, x_n)$ can be viewed as <u>a point</u> in R^n with the x_i's as its coordinates.

  (2) An n-tuple $(x_1, x_2, \cdots, x_n)$ can be viewed as <u>a vector</u> $x = (x_1, x_2, \cdots, x_n)$ in R^n with the x_i's as its components.

- Ex:



a point



a vector

1.2 Vector Spaces

- Vector spaces:

Let ***V*** be a set on which two operations (*vector addition and scalar multiplication*) are defined. If the following axioms are satisfied for every **u**, **v**, and **w** in *V* and every scalar (real number) *c* and *d*, then *V* is called a **vector space**.

Addition:

(1) **u** + **v** is in *V too* (***closed*** *under addition)*

(2) **u** + **v** = **v** + **u**

(3) **u** + (**v** + **w**) = (**u** + **v**) + **w**

(4) *V* has a zero vector **0** such that for every **u** in *V*, **u**+**0** = **u**

(5) For every **u** in *V*, there is a vector in *V* denoted by –**u** such that **u** + (–**u**) = **0**

Scalar multiplication:

(6) $c\mathbf{u}$ is in V *too* (***closed*** *under multiplication by a scalar*).

(7) $c(\mathbf{u}+\mathbf{v}) = c\mathbf{u} + c\mathbf{v}$

(8) $(c+d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$

(9) $c(d\mathbf{u}) = (cd)\mathbf{u}$

(10) $1(\mathbf{u}) = \mathbf{u}$

1.2 Vector Spaces

- Note:

(1) A vector space consists of **four entities**:

a set of vectors, a set of scalars, and two operations

$$\left\{\begin{array}{lll} V & : & \text{nonempty set} \\ c & : & \text{scalar} \\ + & (\mathbf{u}, \mathbf{v}) = \mathbf{u} + \mathbf{v} & \text{vector addition} \\ \bullet & (c, \mathbf{u}) = c\mathbf{u} & \text{scalar multiplication} \end{array}\right.$$

$(V, \ +, \ \bullet)$ **is then called a vector space**

(2) $V = \{\mathbf{0}\}$: zero vector space

1.2 Vector Spaces

- Examples of vector spaces:

(1) n-tuple space: $\boldsymbol{R^n}$

$$(u_1, u_2, \cdots, u_n) + (v_1, v_2, \cdots, v_n) = (u_1 + v_1, u_2 + v_2, \cdots, u_n + v_n) \text{ vector addition}$$

$$k(u_1, u_2, \cdots, u_n) = (ku_1, ku_2, \cdots, ku_n) \qquad \text{scalar multiplication}$$

(2) n-th degree polynomial space: $V = P_n(x)$

(the set of all real polynomials of degree n or less)

$$p(x) + q(x) = (a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_n + b_n)x^n$$

$$kp(x) = ka_0 + ka_1 x + \cdots + ka_n x^n$$

1.2 Vector Spaces

- Properties of scalar multiplication

Let $\mathbf{v}$ be any element of a vector space V, and let c be any scalar. Then the following properties are true.

(1) $0\mathbf{v} = \mathbf{0}$

(2) $c\mathbf{0} = \mathbf{0}$

(3) If $c\mathbf{v} = \mathbf{0}$, then $c = 0$ or $\mathbf{v} = \mathbf{0}$

(4) $(-1)\mathbf{v} = -\mathbf{v}$

1.2 Vector Spaces

- Ex 1:

$V=R^2$=the set of all ordered pairs of real numbers **defined as:**

- vector addition: $(u_1,u_2)+(v_1,v_2)=(u_1+v_1,u_2+v_2)$
- scalar multiplication: $c(u_1,u_2)=(cu_1,0)$

Verify V is not a vector space.

Sol:

$\because 1(1,1)=(1,0)\neq(1,1)$ Condition (10) is not satisfied

the set (together with the two given operations) **is not** a vector space

1.3 Vector Subspaces

- Subspace:

If $(V,+,\bullet)$: a vector space

$\left.\begin{array}{l} W \neq \phi \\ W \subseteq V \end{array}\right\}$: a nonempty subset

$(W,+,\bullet)$: a vector space (under the operations of addition and scalar multiplication defined in V)

$\Rightarrow$ *Then* $\boldsymbol{W}$ **is a subspace** of $\boldsymbol{V}$

- Trivial subspace:

Every vector space V has at least two subspaces.

(1) Zero vector space $\{\mathbf{0}\}$ is a subspace of V.

(2) V is a subspace of V.

1.3 Vector Subspaces

- Test for a subspace

If W is a **<u>nonempty subset</u>** of a vector space V, then W is a subspace of V **if and only if** the following conditions hold.

(1) If $\mathbf{u}$ and $\mathbf{v}$ are in W, then $\mathbf{u}+\mathbf{v}$ is in W (*closed under addition*).

(2) If $\mathbf{u}$ is in W and c is any scalar, then $c\mathbf{u}$ is in W (*closed under multiplication by a scalar*).

1.3 Vector Subspaces

- Ex1: Subspace of R^2

  (1) $\{\mathbf{0}\}$ $\quad \mathbf{0}=(0,0)$

  (2) Lines through the origin

  (3) R^2

- Ex2: Show that $W=\{(x_1,x_2): x_1 \geq 0 \text{ and } x_2 \geq 0\}$, with the standard operations, is not a subspace of R^2.

  Let $\mathbf{u}=(1,1) \in W$

  $\because (-1)\mathbf{u}=(-1)(1,1)=(-1,-1) \notin W$

  $\therefore W$ is not a subspace of R^2

1.4 Linear Independence

- **Linear combination:**

A vector $\mathbf{v}$ in a vector space V is called a linear combination of the vectors $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k$ in V if $\mathbf{v}$ can be written in the form

$$\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \ldots + c_k\mathbf{u}_k \qquad c_1, c_2, \cdots, c_k : \text{scalars}$$

- Ex1: (Finding a linear combination)

$$\mathbf{v}_1 = (1,2,3) \quad \mathbf{v}_2 = (0,1,2) \quad \mathbf{v}_3 = (-1,0,1)$$

Prove (a) $\mathbf{w} = (1,1,1)$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$

(b) $\mathbf{w} = (1,-2,2)$ is not a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$

Sol:

(a) $\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = 2\mathbf{v}_1 - 3\mathbf{v}_2 + \mathbf{v}_3$

(b) $\mathbf{w} \neq c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$

1.4 Linear Independence

- the span of a set: span (S)

If $S=\{\mathbf{v}_1, \mathbf{v}_2,\ldots, \mathbf{v}_k\}$ is a set of vectors in a vector space V, then **the span of S** is the set of all linear combinations of the vectors in S,

$$span(S)=\{c_1\mathbf{v}_1+c_2\mathbf{v}_2+\cdots+c_k\mathbf{v}_k \mid \forall c_i \in R\}$$

(the set of all linear combinations of vectors in S)

- a spanning set of a vector space:

If **every vector** in a given vector space can be written as a linear combination of vectors in a given set S, then S is called **a spanning set** of the vector space.

1.4 Linear Independence

- Notes:

$$\text{span}(S) = V$$

$$\Rightarrow \; S \text{ spans (generates) } V$$

$$V \text{ is spanned (generated) by } S$$

$$S \text{ is a spanning set of } V$$

- Notes:

(1) $span(\phi) = \{\mathbf{0}\}$

(2) $S \subseteq span(S)$

(3) $S_1, S_2 \subseteq V$

$$S_1 \subseteq S_2 \Rightarrow span(S_1) \subseteq span(S_2)$$

1.4 Linear Independence

- Linear Independence **(L.I.)** and Linear Dependence (**L.D.**):

$S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\}$: a set of vectors in a vector space V

$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$

(1) If the equation has only the trivial solution $(c_1 = c_2 = \cdots = c_k = 0)$

then S is called linearly independent.

(2) If the equation has a nontrivial solution (i.e., not all zeros),

then S is called linearly dependent.

1.4 Linear Independence

- Notes:

(1) ϕ is linearly independent

(2) $\mathbf{0} \in S \Rightarrow S$ is linearly dependent.

(3) $\mathbf{v} \neq \mathbf{0} \Rightarrow \{\mathbf{v}\}$ is linearly independent

(4) $S_1 \subseteq S_2$

S_1 is linearly dependent $\Rightarrow S_2$ is linearly dependent

S_2 is linearly independent $\Rightarrow S_1$ is linearly independent

1.4 Linear Independence

- Ex1: (Testing for linearly independent)

Determine whether the following set of vectors in R^3 is L.I. or L.D.

$$S = \{(1, 2, 3), (0, 1, 2), (-2, 0, 1)\}$$

$$\mathbf{v}_1 \qquad \mathbf{v}_2 \qquad \mathbf{v}_3$$

Sol:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0} \quad \Rightarrow \quad \begin{aligned} c_1 \qquad\quad - 2c_3 &= 0 \\ 2c_1 + \ c_2 + \qquad\ &= 0 \\ 3c_1 + 2c_2 + \ c_3 &= 0 \end{aligned}$$

$$\Rightarrow c_1 = c_2 = c_3 = 0 \quad (\text{only the trivial solution})$$

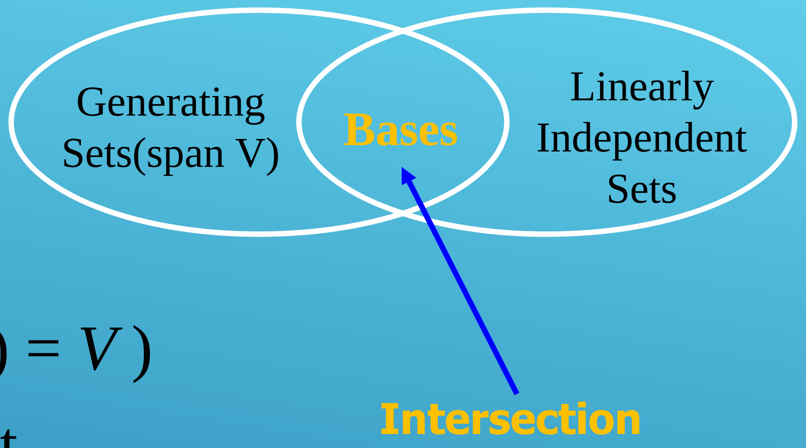
$$\Rightarrow S \text{ is linearly independent}$$

1.5 Basis and Dimension

- Basis:

V : a vector space

$S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} \subseteq V$



<u>If</u> $\begin{cases} (a) \ S \text{ spans } V \ \ (\text{i.e., } \ span(S) = V\,) \\ (b) \ S \text{ is linearly independent} \end{cases}$

$\Rightarrow$ *Then* S is called a **basis** for V

- Notes:

(1) Ø is a basis for $\{\mathbf{0}\}$

(2) the standard basis for R^3:

$\{i, j, k\} \quad i = (1, 0, 0),\ j = (0, 1, 0),\ k = (0, 0, 1)$

(3) the standard basis for $\boldsymbol{R}^{n}$:

$\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ $\mathbf{e}_1=(1,0,\ldots,0)$, $\mathbf{e}_2=(0,1,\ldots,0)$, $\mathbf{e}_n=(0,0,\ldots,1)$

Ex: R^4 $\{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\}$

(4) the standard basis for $\boldsymbol{P_n(x)}$: $\{1, x, x^2, \ldots, x^n\}$

Ex: $P_3(x)$ $\{1, x, x^2, x^3\}$

1.5 Basis and Dimension

- Uniqueness of basis representation

  If $S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ is a basis for a vector space *V*, then every vector in *V* can be written in one and only one way as a linear combination of vectors in *S*.

- Basis and linear dependence

  If $S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ is a basis for a vector space *V*, then every set containing more than *n* vectors in *V* is linearly dependent.

- Number of vectors in a basis

  If a vector space *V* has one basis with ***n* vectors**, then every basis for *V* has *n* vectors. (*All bases for a finite-dimensional vector space has the same number of vectors*.)

1.5 Basis and Dimension

- Finite dimensional:

  A vector space V is called **finite dimensional**,

  if it has a basis consisting of a finite number of elements.

- Infinite dimensional:

  If a vector space V is not finite dimensional,

  then it is called **infinite dimensional**.

- Dimension:

  The **dimension** of a finite dimensional vector space V is defined to be **<u>the number of vectors in a basis</u>** for V.

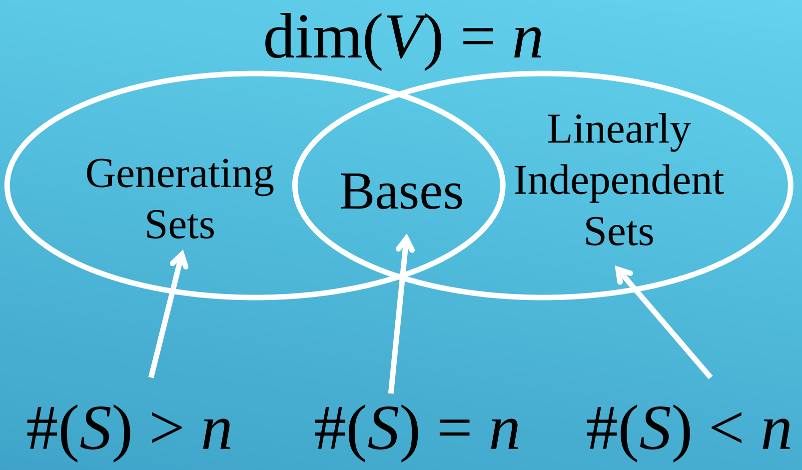
  V: a vector space S: a basis for V

  $\Rightarrow$symbol: $\mathbf{dim}(V) = \#(S)$ (the number of vectors in S)

1.5 Basis and Dimension

- Notes:

(1) $\dim(\{\mathbf{0}\}) = 0 = \#(\varnothing)$



(2) $\dim(V) = n\ ,\ S \subseteq V$

S : a generating set $\Rightarrow \#(S) \geq n$

S : a L.I. set $\Rightarrow \#(S) \leq n$

S : a basis $\Rightarrow \#(S) = n$

(3) $\dim(V) = n\ ,\ W$ is a subspace of $V \Rightarrow \dim(W) \leq n$

1.5 Basis and Dimension

- Exp:

(1) Vector space R^n $\Rightarrow$ basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$

$\Rightarrow \dim(R^n) = n$

(2) Vector space $P_n(x)$ $\Rightarrow$ basis $\{1, x, x^2, \ldots, x^n\}$

$\Rightarrow \dim(P_n(x)) = n+1$

(3) Vector space $P(x)$ $\Rightarrow$ basis $\{1, x, x^2, \ldots\}$

$\Rightarrow \dim(P(x)) = \infty$

1.5 Basis and Dimension

- Ex1:

(a) $W=\{(d, c-d, c):$ c and d are real numbers$\}$

(b) $W=\{(2b, b, 0):$ b is a real number$\}$

Sol:

(a) $(d, c-d, c) = c(0, 1, 1) + d(1, -1, 0)$

$\Rightarrow S = \{(0, 1, 1), (1, -1, 0)\}$ (S is L.I. and S *spans* W)

$\Rightarrow S$ is a basis for W

$\Rightarrow \dim(W) = \#(S) = 2$

(b) $(2b, b, 0) = b(2, 1, 0)$

$\Rightarrow S = \{(2, 1, 0)\}$ spans W and S is L.I.

$\Rightarrow S$ is a basis for W

$\Rightarrow \dim(W) = \#(S) = 1$