Chapter 4

Functions of several variables

"Nature does nothing in vain, and more is in vain when less will serve; for Nature is pleased for simplicity, and affects not the pomp of superfluous causes".——Sir Isaac Newton (1642–1727), Principia.

Functions of several variables are needed to describe complex processes. For example the temperature changes during the day needs four variables : three coordinates of place and one coordinate of the time. The mathematical description of complex systems e.g. the motion of gas or fluids, may need millions variables.

Definition 4.0.1 $\mathbb{R}^n$ *denotes the set of ordered n-tuples of real numbers, that is*

$$\mathbb{R}^n = \{(x_1, x_2, x_3, \ldots, x_n) : x_1, x_2, \ldots, x_n \in \mathbb{R}\}.$$

The points of $\mathbb{R}^n$ are sometimes called n-dimensional vectors.

$x \in \mathbb{R}^n \Leftrightarrow x = (x_1, x_2, \ldots, x_n) = x_1(1, 0, \ldots, 0) + x_2(0, 1, 0, \ldots, 0) + \ldots + x_n(0, 0, \ldots, 1).$

We observe that

$$x = (x_1, x_2, \ldots, x_n) = x_1(1, 0, \ldots, 0)+x_2(0, 1, 0, \ldots, 0)+\ldots+x_n(0, 0, \ldots, 1) = x_1e_1+x_2e_2+\ldots+x_ne_n,$$

where

$$e_1 = (1, 0, 0, \ldots, 0),\ e_2 = (0, 1, 0, \ldots, 0), \ldots, e_n = (0, 0, 0, \ldots, 1).$$

Example 4.0.1 *Note that when $n = 1$, we get the real line*

$$\mathbb{R} = \{x_1 : x_1 \in \mathbb{R}\}.$$

When $n = 2$, we get the plane

$$\mathbb{R}^2 = \{(x_1, x_2) : x_1, x_2 \in \mathbb{R}\}.$$

Note that if $x \in \mathbb{R}^2$, then

$$x = (x_1, x_2) = x_1(1, 0) + x_2(0, 1) = x_1i + x_2j,$$

where $i = (1,0)$ and $j = (0,1)$.
When $n = 3$, we get the 3-dimensional space

$$\mathbb{R}^3 = \{(x_1, x_2, x_3) : \ x_1, x_2, x_3 \in \mathbb{R}\}.$$

$$x \in \mathbb{R}^3 \Longleftrightarrow x = x_1 \overrightarrow{i} + x_2 \overrightarrow{j} + x_3 \overrightarrow{k}, \quad \overrightarrow{i} = (1,0,0), \ \overrightarrow{j} = (0,1,0), \ \overrightarrow{k} = (0,0,1).$$

The 4-dimensional space

$$\mathbb{R}^4 = \{(x_1, x_2, x_3, x_4) : \ x_1, x_2, x_3, x_4 \in \mathbb{R}\},$$

has many important applications in physics. For example we can describe a position of a point by 3 coordinates and the fourth coordinate as time.

We know that the sum of vectors $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ is the vector

$$x + y = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n),$$

and the product of a vector $x = (x_1, x_2, \ldots, x_n)$ and a real number c is the vector

$$cx = (cx_1, cx_2, \ldots, cx_n).$$

4.1 Inner product (scalar product) (dot product)

Definition 4.1.1 *Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ be two vectors. We define the inner product of x and y as*

$$\langle x, y\rangle = x.y = \sum_{i=1}^{n} x_i y_i = x_1 y_1 + x_2 y_2 + \ldots + x_n y_n.$$

Note that, when $x = y$ we get

$$\langle x, x\rangle = x_1^2 + x_2^2 + \ldots + x_n^2.$$

Example 4.1.1 *The inner product of $x = (1, 3, -2)$ and $y = (0, 1, 2)$ is*

$$\langle x, y\rangle = 1 \times 0 + 3 \times 1 + 2 \times -2 = -1.$$

From the definition of the inner product, it is easy to see that

1. $\langle x, y\rangle = \langle y, x\rangle$ for any $x, y \in \mathbb{R}^n$.
2. $\langle x, y + z\rangle = \langle x, y\rangle + \langle x, z\rangle$ for any $x, y, z \in \mathbb{R}^n$.
3. If $\lambda \in \mathbb{R}$, then
$$\langle \lambda x, y\rangle = \lambda\langle x, y\rangle, \quad \forall x, y \in \mathbb{R}^n.$$

4. If $x = (0, 0, \ldots, 0)$, then
$$\langle x, y\rangle = 0, \quad \forall y \in \mathbb{R}^n.$$

Definition 4.1.2 *We say that two vectors x and y are orthogonal, if*
$$\langle x, y\rangle = 0.$$

Example 4.1.2 *The vectors $x = (1, 0, 0)$ and $y = (0, 1, 0)$ are orthogonal, since*
$$\langle x, y\rangle = 1 \times 0 + 1 \times 0 + 0 \times 0 = 0.$$

Definition 4.1.3 *The distance between the vector $x = (x_1, x_2, \ldots, x_n)$ the origin $0_{\mathbb{R}^n} = (0, 0, \ldots, 0)$ is*
$$||x|| = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2} = \sqrt{\langle x, x\rangle}.$$
The distance between two vectors $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ is
$$||x - y|| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \ldots + (x_n - y_n)^2} = \sqrt{\langle x - y, x - y\rangle}.$$

Example 4.1.3 *Find the distance between the vectors $x = (1, 2, 3)$ and $y = (0, 2, 1)$. By definition we have*
$$||x - y|| = \sqrt{1^2 + 0^2 + 2^2} = \sqrt{5}.$$

Proposition 4.1.4 *The inner product of two vectors x and y is*
$$\langle x, y\rangle = ||x||||y|| \cos(\theta), \quad \theta \in [0, \pi].$$
θ is the angle between the vectors x and y.

Proof. Apply the law of cosines gives
$$||x - y||^2 = ||x||^2 + ||y||^2 - 2||x||||y|| \cos(\theta) \tag{4.1}$$
Using the properties of the inner product, we have
$$||x - y||^2 = ||x||^2 + ||y||^2 - 2\langle x, y\rangle.$$
Substituting this into (4.1) yields
$$\langle x, y\rangle = ||x||||y|| \cos(\theta).$$

Example 4.1.5 *Find the angle between $x = (2, 1, 0)$ and $y = (-2, 2, 1)$. Note that $||x|| = \sqrt{5}$, $||y|| = 3$, and $\langle x, y\rangle = -2$. Hence, we get*
$$\cos\theta = \frac{-2}{3\sqrt{5}} \Rightarrow \theta = \arccos\left(\frac{-2}{3\sqrt{5}}\right).$$

4.2 The cross product

Definition 4.2.1 *Let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$. Then, the cross product $x \times y$ is*

$$x \times y = det \begin{pmatrix} i & j & k \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3. \end{pmatrix} = (x_2y_3 - x_3y_2)i + (x_3y_1 - x_1y_3)j + (x_1y_2 - x_2y_1)k.$$

Example 4.2.1 *Let $x = (-1, 2, 5)$ and $y = (4, 0, 3)$. Find $x \times y$.*

4.3 Functions of two variables

Definition 4.3.1 *Let D be a subset of $\mathbb{R}^2$ ($D \subseteq \mathbb{R}^2$). A function of two variables $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ maps each ordered pair $(x, y) \in D$ to a unique real number z, that is*

$$D \ni (x, y) \mapsto f(x, y) = z.$$

- *The set D is called the domain of the function f.*
- *The ragne of f is*
$$R(f) = \{f(x, y) : \quad (x, y) \in D\} \subseteq \mathbb{R}.$$

Example 4.3.1 *Find the domain and the range of each of the following functions*

1. $f(x, y) = 3x + 5y + 1$
2. $f(x, y) = \sqrt{9 - x^2 - y^2}$.

Solution. We see that for any $(x, y) \in \mathbb{R}^2$ the function $f(x, y) = 3x + 5y + 1$ is defined. To determine the range, we pick $z \in \mathbb{R}$ and find a solution for the equation

$$f(x, y) = z \Longleftrightarrow 3x + 5y + 1 = z.$$

It is easy to see that $((z - 1)/3, 0)$ gives a solution to this equation. Hence, $R(f) = \mathbb{R}$. The second function is defined when $9 - x^2 - y^2 \geq 0$, that is

$$D = \left\{(x, y) \in \mathbb{R}^2 : \; x^2 + y^2 \leq 9\right\}.$$

The graph of this set of points can be described by as disk of radius 3 centred at the origin. To find the range we observe that

$$0 \leq z = f(x, y) \leq 3 \Rightarrow R(f) \subset [0, 3].$$

On the other hand, for any $z \in [0, 3]$, there exists a solution (x, y) for the equation

$$9 - z^2 = x^2 + y^2.$$

Therefore, $R(f) = [0, 3]$.

4.4 Graphing functions of two variables

We know that the graph of the function $y = f(x)$ is a curve in $\mathbb{R}^2$ given by

$$graph\, f = \{(x, y) : x \in [a, b]\ y = f(x)\}.$$

The graph of a function of two variables $z = f(x, y)$ is a surface in $\mathbb{R}^3$ given by

$$graph\, f = \{(x, y, z) : \ (x, y) \in D, \ \ z = f(x, y)\}.$$

Example 4.4.1 *Create a graph of each of the following functions :*

$$f(x, y) = \sqrt{9 - x^2 - y^2} \quad \text{and} \quad f(x, y) = x^2 + y^2.$$

4.5 Functions of more than two variables

Similarly, we define functions of more than two variables as $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$.

Example 4.5.1 *As examples of more than two variables, we give*

$$g(x, y, z) = x^2 + y^2 + z^2,$$

and

$$g(x, y, z, t) = \sqrt{x^2 + y^2 + z^2 + t^2}.$$

Example 4.5.2 *Find the domain of definition of each of the following functions :*

1. $f(x, y, z) = \frac{3x-4y+2z}{\sqrt{9-x^2-y^2-z^2}}$.
2. $g(x, y, t) = \frac{\sqrt{2t-4}}{x^2-y^2}$.

4.6 Limits and Continuity

Definition 4.6.1 *Let f be a function of two variables x and y.*

$$\lim_{(x,y)\to(a,b)} f(x, y) = L \Leftrightarrow \forall \epsilon > 0, \quad \exists \delta > 0 \ \ s.t \ \ 0 < ||(x-a, y-b)|| < \delta \Longrightarrow |f(x, y)-L| < \epsilon.$$

Example 4.6.1 *Use the definition of the limit to show that*

$$\lim_{(x,y)\to(0,0)} x^2 + y^2 = 0.$$

For any $\epsilon > 0$ can we find $\delta > 0$ such that if $||(x, y)|| < \delta$, then

$$|f(x, y)| < \epsilon.$$

We have

$$|f(x, y)| = x^2 + y^2 = ||(x, y)||^2 < \delta^2.$$

Thus, taking $\delta = \sqrt{\epsilon}$ yields the result.

4.6.1 Limits law for functions of two variables

Let $f(x,y)$ and $g(x,y)$ be defined for all $(x,y) \neq (a,b)$. Assume that L and M are real numbers, and let c be a constant. Then each of the following statements :

1. Constant law
$$\lim_{(x,y)\to(a,b)} c = c.$$
2. Identity law
$$\lim_{(x,y)\to(a,b)} x = a.$$
$$\lim_{(x,y)\to(a,b)} y = a.$$
3. Sum law
$$\lim_{(x,y)\to(a,b)} (f(x,y) \pm g(x,y)) = L \pm M.$$
4. Product law
$$\lim_{(x,y)\to(a,b)} (f(x,y)g(x,y)) = LM.$$
5. Quotient law
$$\lim_{(x,y)\to(a,b)} \frac{f(x,y)}{g(x,y)} = \frac{L}{M}, \quad M \neq 0.$$
6. Power law
$$\lim_{(x,y)\to(a,b)} (f(x,y))^n = L^n.$$
7. Root law
$$\lim_{(x,y)\to(a,b)} \sqrt[n]{f(x,y)} = \sqrt[n]{L},$$
for $L \in \mathbb{R}$ if n is odd positive, and for all $L \geq 0$ if n is even and positive.

Example 4.6.2 *Find each of the following limits :*

1. $\lim_{(x,y)\to(2,-1)} (x^2 - 2xy + 3y^2 - 4x + 3y - 6)$.

2. $\lim_{(x,y)\to(2,-1)} \frac{2x+3y}{4x-3y}$.

4.6.2 Limits fail to exist

Definition 4.6.2 *We say* $\lim_{(x,y)\to(a,b)} f(x,y)$ *does not exist, if the function* $f(x,y)$ *takes two differents limits along different curves passing through* (a,b).

Example 4.6.3 *Show that neither of the following limits exist :*

1. $\lim_{(x,y)\to(0,0)} \frac{2xy}{3x^2+y^2}$.
2. $\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+3y^4}$.

Solution. For the first example we see that $f(x,0) = 0$ and $f(x,x) = 1/2$. Thus, $\lim_{(x,y)\to(0,0)} \frac{2xy}{3x^2+y^2}$ does not exist. For the second, we observe that $f(x,0) = 0 \neq f(y^2,y) = 1$. Hence, $\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+3y^4}$ does not exist.

4.7 Continuity of functions of two variables

Definition 4.7.1 *Let* $f(x,y)$ *be a function of two variables defined on* D. *We say that* f *is continuous at* $(a,b) \in D$ *if*

1. $f(a,b)$ *exists*
2. $\lim_{(x,y)\to(a,b)} f(x,y)$ *exists*
3. $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$.

Example 4.7.1 *Let* $f(x,y) = \frac{xy}{(x^2+y^2)^\alpha}$ *if* $(x,y) \neq (0,0)$, *and* $f(0,0) = 0$. *For what values of* α f *will be continuous at the origin.*

Solution. Note that $D(f) = \mathbb{R}^2$. Assume first, $\alpha \geq 1$. Then, we have

$$\lim_{(x,x)\to(0,0)} \frac{1}{x^{2(\alpha-1)}} = +\infty.$$

Thus, f is discontinuous at $(0,0)$ when $\alpha \geq 1$. Assume $\alpha \in (0,1)$, we observe that

$$-(x^2+y^2)^{1-\alpha} \leq \frac{xy}{(x^2+y^2)^\alpha} \leq (x^2+y^2)^{1-\alpha}$$

Here we used the fact that

$$-(x^2+y^2) \leq xy \leq (x^2+y^2), \quad \forall x,y \in \mathbb{R}.$$

Thus, by the squeeze theorem we get

$$\lim_{(x,y)\to(0,0)} \frac{xy}{(x^2+y^2)^\alpha} = \lim_{(x,x)\to(0,0)} (x^2+y^2)^{1-\alpha} = \lim_{(x,x)\to(0,0)} -(x^2+y^2)^{1-\alpha} = 0 = f(0,0).$$

Therefore, f is continuous at $(0,0)$ when $\alpha \in (0,1)$. Assume $\alpha \leq 0$, then it follows that

$$\lim_{(x,x)\to(0,0)} \frac{xy}{(x^2+y^2)^\alpha} = \lim_{(x,x)\to(0,0)} xy(x^2+y^2)^{-\alpha} = 0 = f(0,0).$$

Consequently, f is continuous at $(0,0)$ when $\alpha < 1$.

4.8 Properties of continuous functions of two variables

Let f and g are continuous functions at (a,b). Then

- The sum of continuous function $f(x,y)+g(x,y)$ is continuous at (a,b).
- The product of continuous functions $f(x,y)g(x,y)$ is continuous at (a,b).

4.9 Continuity of functions of more than two variables

Definition 4.9.1 *Let $f(x,y,z)$ be a function of three variables defined on D. We say that f is continuous at $(a,b,c)\in D$ if*

1. *$f(a,b,c)$ exists*
2. *$\lim_{(x,y,z)\to(a,b,c)} f(x,y,z)$ exists*
3. *$\lim_{(x,y,z)\to(a,b,c)} f(x,y) = f(a,b,c)$.*

4.10 Partial derivative of a function of two variables

Definition 4.10.1 *Let $f(x,y)$ be a function of two variables. Then the partial derivative of f with respect to x, written as $\frac{\partial f}{\partial x}$ or f_x is defined by*

$$\frac{\partial f}{\partial x}(x,y) = \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}.$$

The partial derivative of f with respect to y, written as $\frac{\partial f}{\partial y}$ or f_y is defined by

$$\frac{\partial f}{\partial y}(x,y) = \lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h}.$$

Definition 4.10.2 *Use the definition of the partial derivative to calculute $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ where*

$$f(x,y) = x^2(y+1).$$

Solution. By the definition of the partial derivative, we have

$$\frac{\partial f}{\partial x} = \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h} = \lim_{h\to 0}\frac{(x+h)^2(y+1)-x^2(y+1)}{h} = 2x(y+1),$$

and
$$\frac{\partial f}{\partial y} = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h} = \lim_{h\to 0} \frac{x^2(y+h+1) - x^2(y+1)}{h} = x^2.$$

From this we observe that we get $\frac{\partial f}{\partial x}$ by fixing y and differentiating f with respect to x. Similarly, we get $\frac{\partial f}{\partial y}$ by fixing x and differentiating f with respect to y.

Example 4.10.1 *Calculate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ of*

$$f(x,y) = xy(x^2 + y^2 + 1).$$

We have
$$\frac{\partial f}{\partial x} = y(x^2 + y^2 + 1) + 2x^2 y,$$
and
$$\frac{\partial f}{\partial y} = x(x^2 + y^2 + 1) + 2y^2 x.$$

Example 4.10.2 *Solve exercise 4 in recitation 3.*

4.11 Higher order partial derivatives

Definition 4.11.1 *Let $f(x,y)$ be a function of two variables. The second order partial derivatives of $f(x,y)$ are defined by*

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = f_{xx}; \quad \frac{\partial^2 f}{\partial xy} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{xy}; \quad \frac{\partial^2 f}{\partial yx} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{yx}; \quad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = f_{yy}$$

Example 4.11.1 *Calculute $\frac{\partial^2 f}{\partial x^2}$ and $\frac{\partial^2 f}{\partial xy}$ of*

$$f(x,y) = x^2 + y^2.$$

We have
$$\frac{\partial f}{\partial x} = 2x$$
and
$$\frac{\partial f}{\partial y} = 2y.$$
Then,
$$\frac{\partial^2 f}{\partial x^2} = 2, \quad \frac{\partial^2 f}{\partial xy} = 0.$$

Example 4.11.2 *Solve exercise 5 in recitation 3.*

Remark 5 *In general we do not have* $f_{xy}(x,y) = f_{yx}(x,y)$*. For example, consider* $f(x,y) = xy(x^2 - y^2)/(x^2 + y^2)$ *for* $(x,y) \neq (0,0)$ *and* $f(0,0) = 0$*. Through direct calculations we get* $f_{xy}(0,0) = -1$ *and* $f_{yx}(0,0) = 1$*.*

Theorem 4.11.3 *(Schwarz's theorem) Suppose that* f *is defined on* D*. If* f_{xy} *and* f_{yx} *are continuous on* D *then* $f_{xy}(x,y) = f_{yx}(x,y)$*.*

4.12 The chain rule

Suppose $x = g(u,v)$ and $y = h(u,v)$ are differentiable functions of u and v, and let $z = f(x,y)$ is a differentiable function of x and y. Then

$$z = f(g(u,v), h(u,v))$$

is differentiable of u and v and

$$\frac{\partial z}{\partial u} = \frac{\partial g}{\partial u}\frac{\partial f}{\partial x} + \frac{\partial h}{\partial u}\frac{\partial f}{\partial y},$$

$$\frac{\partial z}{\partial v} = \frac{\partial g}{\partial v}\frac{\partial f}{\partial x} + \frac{\partial h}{\partial v}\frac{\partial f}{\partial y}.$$

Example 4.12.1 *Solve exercise 7 in recitation 3.*